I. Angle Measure

1. Radian Measure

- $180^\circ = \pi \text{ rad}$

angles in standard position

- Coterminal angles
  - $30^\circ + 360^\circ = 390^\circ$
  - $30^\circ + 720^\circ = 750^\circ$

Find negative angles coterminal with $\Theta$, subtract any multiple of $360^\circ$:
- $30^\circ - 360^\circ = -330^\circ$
- $30^\circ - 720^\circ = -690^\circ$

Find positive angles coterminal with $\Theta$, add any multiple of $2\pi$:
- $\frac{\pi}{3} + 2\pi = \frac{7\pi}{3}$

To find positive angles that are coterminal, add any multiple of $2\pi$; likewise, to find negative angles that are coterminal with $\Theta$, subtract any multiple of $2\pi$.

Ex. Angle coterminal with $1290^\circ$: $1290^\circ \% 360^\circ = 210^\circ$

2. Length of a circular arc:

\[ \ell = r\theta \]

units of measure: same as those of $r$

Area of circular sector:

\[ A = \frac{1}{2} r^2 \theta \]
Circular Motion

A point moves along a circle of radius \( r \) and the ray from the center of the circle to the point traverses \( \theta \) rad in time \( t \). Let \( s = r \theta \) be the distance the point travels in time \( t \).

\[
\text{Speed:}
\begin{align*}
& a) \text{ angular speed: } \omega = \frac{\theta}{t} \quad v = r \omega \\
& b) \text{ linear speed: } v = \frac{s}{t}
\end{align*}
\]

Example

A potter's wheel with radius 8 in. spins at 150 rpm. Find the angular and linear speeds of a point on the rim of the wheel.

\[
\begin{align*}
\omega &= \frac{150 \pi}{60} = 7.854 \text{ rad/min} \\
v &= r \frac{\omega}{t} = 8 \times 7.854 = 40 \pi \text{ in/s} \approx 120 \text{ in/s}
\end{align*}
\]

In an automobile transmission a gear ratio \( g \) is the ratio \( g = \frac{\text{angular speed of engine}}{\text{angular speed of wheels}} \).

The angular speed of the engine is shown on the tachometer in rpm. A sports car has wheels with radius 11". Gear ratios shown in table. The car is in fourth gear and the tach reads 3500 rpm.

a) What is the angular speed of the engine?

b) Find angular speed of the wheels.

c) How fast in mph is the car traveling?

\[
\begin{align*}
g &= \frac{3500 \pi}{7000 \pi} = 0.9 \quad \omega = 2 \pi \text{ rad/min} \\
v &= \omega r = 7.7778 \text{ rad/min} \times 11 \text{ in} \times \frac{1 \text{ mi}}{63360 \text{ in}} \approx 0.26 \text{ mph}
\end{align*}
\]

<table>
<thead>
<tr>
<th>Gear</th>
<th>Ratio</th>
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<tr>
<td>1st</td>
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<tr>
<td>2nd</td>
<td>3.0</td>
</tr>
<tr>
<td>3rd</td>
<td>1.6</td>
</tr>
<tr>
<td>4th</td>
<td>0.9</td>
</tr>
<tr>
<td>5th</td>
<td>0.7</td>
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</table>
II Trigonometry of Right Triangles

\[
\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} \\
\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} \\
\tan \theta = \frac{\sin \theta}{\cos \theta} \\
\cot \theta = \frac{1}{\tan \theta} \\
\sec \theta = \frac{1}{\cos \theta} \\
\csc \theta = \frac{1}{\sin \theta}
\]

<table>
<thead>
<tr>
<th>\theta</th>
<th>\operatorname{rad}</th>
<th>\sin \theta</th>
<th>\cos \theta</th>
<th>\tan \theta</th>
<th>\cot \theta</th>
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<td>1</td>
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<td>60°</td>
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<td>\frac{\sqrt{3}}{2}</td>
<td>\frac{1}{2}</td>
<td>\frac{1}{\sqrt{3}}</td>
<td>\sqrt{3}</td>
<td>\frac{2}{\sqrt{3}}</td>
<td></td>
</tr>
</tbody>
</table>

Ex A tree casts a shadow 532 ft long. Find the height of the tree if the angle of elevation of the sun is 25.7°.

Ans \[ \frac{h}{532} = \tan 25.7° \quad h \approx 256' \]
Trigonometric Functions of Angles

1. Reference angle \( \overline{\Theta} \) associated with \( \Theta \) is the acute angle formed by the terminal side of \( \Theta \) and the x-axis.

2. Eval Trig Functions at any Angle
   a) Find reference angle \( \overline{\Theta} \) associated with the angle \( \Theta \).
   b) Determine the sign of the trig function of \( \Theta \).
   c) The value of the trig function of \( \Theta \) is the same, except possibly for the sign, as the value of the trig function of \( \overline{\Theta} \).

   Example:
   \[ \sin 240^\circ = -\sin 60^\circ = -\frac{\sqrt{3}}{2} \]
   \[ \sin \frac{16\pi}{3} = \sin \frac{4\pi}{3} = -\sin \frac{\pi}{3} = -\frac{\sqrt{3}}{2} \]
   \[ \sec(-\frac{\pi}{4}) = \sec \frac{\pi}{4} = \sqrt{2} \]

3. Identities
   a) \( \sin^2 \Theta + \cos^2 \Theta = 1 \)
   \[ \left( \frac{y}{r} \right)^2 + \left( \frac{x}{r} \right)^2 = \frac{y^2}{r^2} + \frac{x^2}{r^2} = 1 \]
   b) \( \tan \Theta + 1 = \sec \Theta \)
   c) \( 1 + \cot^2 \Theta = \csc^2 \Theta \)

4. Convert one trig function to another
   \( \sin^2 \Theta + \cos^2 \Theta = 1 \)
   \[ \sin \Theta = \sqrt{1 - \cos^2 \Theta}, \text{ } \Theta \text{ in QI, QII} \]
   \[ = -\sqrt{1 - \cos^2 \Theta}, \text{ } \Theta \text{ in QIII, QIV} \]
   \[ \tan \Theta = \cos \Theta / \sqrt{1 - \sin^2 \Theta}, \text{ } \Theta \text{ in QII} \]

If \( \sec \Theta = 2 \) and \( \Theta \) in QIV, find other trig functions of \( \Theta \).
Since $\theta$ in QIV, $\sec \theta = -\sec \theta$
\[\sin \theta = -\frac{1}{2}, \cos \theta = \frac{1}{2}, \tan \theta = -\sqrt{3}\n\csc \theta = -\frac{2}{\sqrt{3}}, \sec \theta = 2, \cot \theta = -\frac{\sqrt{3}}{2}\]

If $\tan \theta = \frac{2}{3}$ and $\theta$ in QIII

find $\cos \theta$.

\[
\tan^2 \theta + 1 = \sec^2 \theta, \quad \sec \theta = -\sqrt{\tan^2 \theta + 1} = -\sqrt{\left(\frac{2}{3}\right)^2 + 1} = -\sqrt{\frac{13}{9}}
\]

\[
\therefore \cos \theta = -\frac{3}{\sqrt{13}}
\]

5. Areas of Triangles

\[A = \frac{1}{2} \text{ base} \cdot \text{height} = \frac{1}{2} ab \sin \theta\]

From a point $A$ on the ground, the angle of elevation to the top of a tall building is 24.1°. From a point $B$, which is 600 ft closer to the building, the angle of elevation is measured to be 30.2°. Find the height of the building.

Let $h$ represent the height of the building in feet.

\[x = \text{horizontal distance from the building to point } B\]

\[
\tan 24.1 = \frac{h}{600 + x}; \quad \tan 30.2 = \frac{h}{x} \Rightarrow x = h \cot 30.2
\]

\[
\tan 24.1 = \frac{h}{(h \cot 30.2 + 600)} \Rightarrow h = \tan 24.1(h \cot 30.2 + 600)
\]

\[
h = \frac{600 \tan 24.1}{(1 - \tan 24.1 \cot 30.2)^2} \approx 1160\]
The pistons in a car engine move up and down repeatedly to turn the crankshaft, as shown. Find the height of the point \( P \) above the center \( O \) of the crankshaft in terms of the angle \( \theta \).

As the crankshaft moves in its circular pattern, point \( Q \) is determined by the angle \( \theta \). \( Q(2\cos \theta, 2\sin \theta) \). Split triangle into two right triangles \( \triangle OQR \) and \( \triangle PQR \). \( h \) = height of piston.

**Case I: \( 0^\circ \leq \theta < 180^\circ \)**

\[
h = OR + RP = \sqrt{8^2 - (2\cos \theta)^2} + 2\sin \theta
\]

\[
h = \sqrt{64 - 4\cos^2 \theta} + 2\sin \theta
\]

**Case II: \( 180^\circ \leq \theta < 360^\circ \)**

\[
h = RP - RO = \sqrt{64 - 4\cos^2 \theta} - 2\sin \theta
\]

\[
h = \sqrt{64 - 4\cos^2 \theta} + 2\sin \theta
\]
Law of Sines.

\[
\sin A = \frac{\sin B}{b} = \frac{\sin C}{c}
\]

\[
a \Delta ABC = \frac{1}{2} ab \sin C = \frac{1}{2} ac \sin B = \frac{1}{2} bc \sin A
\]

\[
\left[ \frac{1}{2} abc \sin A = \frac{1}{2} ac \sin B = \frac{1}{2} ab \sin C \right] \cdot \frac{2}{abc}
\]

Solve for Triangles

a) Unique Solution ASA or SAA

\[\angle B = 180^\circ - (20^\circ + 25^\circ) = 135^\circ\]
\[\sin A = \frac{\sin B}{b} \]
\[a = \frac{c \sin A}{\sin C} \approx 65.1\]
\[b = \frac{c \sin B}{\sin C} \approx 134.5\]

b) Ambiguous Case SSA

(i) Solve \(\Delta ABC\) if \(\angle A = 45^\circ\), \(a = 7\sqrt{2}\), \(b = 7\)
\[\sin A = \frac{\sin B}{b}, \quad \sin B = \frac{bsinA}{a} = \left(\frac{7\sqrt{2}}{7}\right) \sin 45^\circ = \frac{1}{2}\]
Thus \(B = 30^\circ\) or \(150^\circ\)

Since \(\angle A = 45^\circ\), \(\angle B = 30^\circ\)
\[\angle C = 180^\circ - (45^\circ + 30^\circ) = 105^\circ\]
\[\sin B = \frac{\sin C}{c} = \frac{bsinC}{c} = \frac{7\sin 105^\circ}{\sin 30^\circ} \approx 13.5\]

(ii) Solve \(\Delta ABC\) if \(\angle A = 43.1^\circ\), \(a = 186.2\), \(b = 248.6\)
\[\sin B = \frac{bsinA}{a} = \frac{248.6 \sin 43.1^\circ}{186.2} \approx 0.91225\]
\[\angle B = 65.8^\circ\] or \(180^\circ - 65.8^\circ = 114.2^\circ = \angle C\)
\[\angle C = 180^\circ - (43.1^\circ + 65.8^\circ) = 71.1^\circ\]
\[c = \frac{a \sin C}{\sin A} \approx 257.8\]
Alternatively,

\[ \angle c_2 = 180^\circ - (43.1^\circ + 114.2^\circ) = 22.7^\circ \]

\[ c_2 = \frac{a_2 \sin c_2}{\sin A_2} \approx 105.2 \]

(iii) Solve \( \triangle ABC \) where \( \angle A = 42^\circ \), \( a = 70 \), \( b = 122 \)

\[ \sin \frac{a}{A} = \sin \frac{b}{B} \]

\[ c = \frac{122 \sin 42^\circ}{\sin 68.9^\circ} \approx 120.08 \]

\[ d = b \sin A = 120.08 \sin 42.3^\circ \approx 80.8 \text{ m} \]

A and \( B \) are 120 mi apart on the shore, as shown. It is found that \( \angle A = 42.3^\circ \) and \( \angle B = 68.9^\circ \). Find the shortest distance from the boat to the shore.
1. Law of Cosines
   For any \( \Delta ABC \),
   \[
   \begin{align*}
   a &= b^2 + c^2 - 2bc \cos A \\
   b &= a^2 + c^2 - 2ac \cos B \\
   c &= a^2 + b^2 - 2ab \cos C
   \end{align*}
   \]
   distance formula
   \[
   a^2 = (bc \cos A - c)^2 + (bsinA - 0)^2 = b^2 \cos A - 2bc \cos A + c + b^2 \sin A
   \]

2. SSS, Law of Cosines
   \[
   \cos A = \frac{b^2 + c^2 - a^2}{2bc}
   \]
   \[
   \frac{abc}{\sin A} = 0.953125 \implies A \approx 18^\circ
   \]
   Use the Law of Sines to complete
   In any triangle, the longest side is opposite the largest angle. The smallest side is opposite the smallest angle.

3. Navigation: Heading and Bearing
   Bearing: acute angle measured from due north or south
A pilot sets out from an airport in the direction N 20° E at 200 mph. After one hour, makes a course correction and heads N 40° E. Half an hour later, lands at point C.

Find distance and bearing for AC when he makes his course correction, angle between two legs of trip: 180° - 20° - 160°, or 20° + 50° + 90°

\[ b = 200 + 100 - 2 \cdot 200 \cdot 100 \cdot \cos 160° \]

\[ b \approx 295.95 \]

\[ \sin A : \sin 160° \quad \sin A = 0.11557 \quad \angle A \approx 6.636° \]

\[ \theta \quad 295.95 \]

4. Area of Triangle

\[ s = \frac{1}{2} (a + b + c) \]

\[ a = \frac{1}{2} \cdot a \cdot b \cdot \sin C \]

\[ A = \frac{1}{4} \cdot a \cdot b \cdot \sin C \]

\[ = \frac{1}{4} \cdot a \cdot b \cdot (1 - \cos C) \]

\[ = \frac{1}{4} \cdot \frac{a^2 \cdot b^2}{(1 + \cos C)(1 - \cos C)} \cdot \cos C = \frac{a \cdot b \cdot (a + b + c)}{(a + b + c)} \]

\[ \frac{1}{4} \cdot \frac{a^2 \cdot b^2}{a + b + c} \]

\[ A = \frac{1}{4} \cdot a \cdot b \cdot \frac{a + b + c}{2} = \frac{1}{4} \cdot a \cdot b \cdot \frac{(a + b + c)(a + b + c)(c + a - b)(c - a + b)}{2} \]

\[ = s \cdot (s - a) \cdot (s - b) \cdot (s - c) \]
Find the distance between points A and B on opposite sides of a lake from the information shown.

Two ships leave a port at the same time. One travels at 20 mi/h in a direction N 32° E, and the other travels at 28 mi/h in a direction S 42° E (see the figure). How far apart are the two ships after 2 h?