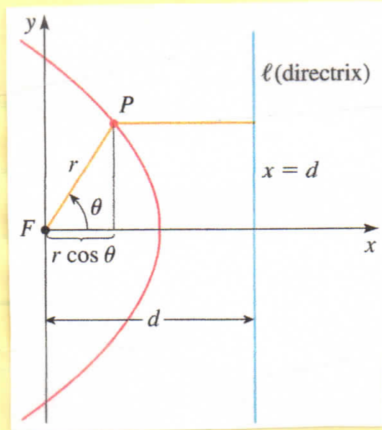
§6.6 Polar Equations of Conics

EQUIVALENT DESCRIPTION OF CONICS

Let F be a fixed point (the **focus**), ℓ a fixed line (the **directrix**), and e a fixed positive number (the **eccentricity**). The set of all points P such that the ratio of the distance from P to F to the distance from P to ℓ is the constant e is a conic. That is, the set of all points P such that

$$\frac{d(P, F)}{d(P, \ell)} = e$$

is a conic. The conic is a parabola if $e = 1$, an ellipse if $e < 1$, or a hyperbola if $e > 1$.



Proof:

If $e=1$, then $d(P,F)=d(P,\ell)$ and this is the definition of a parabola.

Suppose $e \neq 1$. Place the focus at the origin and the directrix parallel to the y-axis and d units to the right. Thus the directrix has equation $x=\ell$ and is $\perp$ to the polar axis.

If P has polar coordinates (r, θ), $d(P,F)=r$ and $d(P,\ell) = d - r\cos\theta$. Thus $\frac{d(P,F)}{d(P,\ell)} = e$ and

$r = e(d - r\cos\theta)$. Square both sides, convert to rect. coords

$$x^2 + y^2 = e^2(d-x)^2$$

$$(1-e^2)x^2 + 2de^2x + y^2 = e^2d^2$$

$$\left(x + \frac{e^2d}{1-e^2}\right)^2 + \frac{y^2}{1-e^2} = \frac{e^2d^2}{(1-e^2)^2}$$

If $e<1$, divide by $e^2d^2/(1-e^2)^2$

$$\frac{(x-h)^2}{a^2} + \frac{y^2}{b^2} = 1, \quad h = \frac{-e^2d}{1-e^2} \quad a^2 = \frac{e^2d^2}{(1-e^2)^2} \quad b^2 = \frac{e^2d^2}{1-e^2}$$

$$c^2 = a^2 - b^2 = \frac{e^4d^2}{(1-e^2)^2} \Rightarrow c = -h$$

$e = \frac{c}{a}$, eqn. of ellipse

$$r = \frac{ed}{1+e\cos\theta}$$

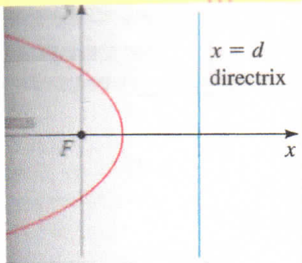
POLAR EQUATIONS OF CONICS

A polar equation of the form

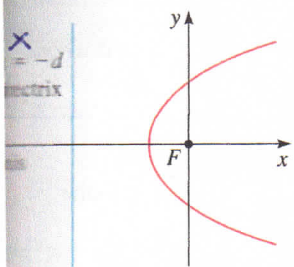
$$r = \frac{ed}{1 \pm e\cos\theta} \quad \text{or} \quad r = \frac{ed}{1 \pm e\sin\theta}$$

represents a conic with one focus at the origin and with eccentricity e. The conic is

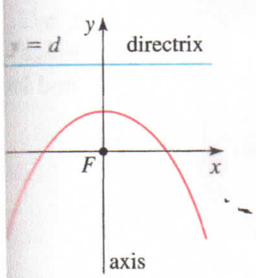
1. a parabola if $e = 1$.
2. an ellipse if $0 < e < 1$.
3. a hyperbola if $e > 1$.



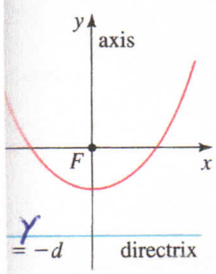
(a) $r = \dfrac{ed}{1 + e\cos\theta}$



(b) $r = \dfrac{ed}{1 - e\cos\theta}$



(c) $r = \dfrac{ed}{1 + e\sin\theta}$



(d) $r = \dfrac{ed}{1 - e\sin\theta}$

Polar equation for parabola has focus at origin and directrix $y = -6$

$e = 1$, $d = 6$, use part (d) : $r = \dfrac{6}{1 - \sin\theta}$

$$r = \frac{10}{3 - 2\cos\theta} = \frac{10}{3} \Big/ \left(1 - \frac{2}{3}\cos\theta\right)$$

$e = \frac{2}{3} < 1 \Rightarrow$ ellipse

major axis: horizontal

$V_1(10, 0)$, $V_2(2, \pi)$

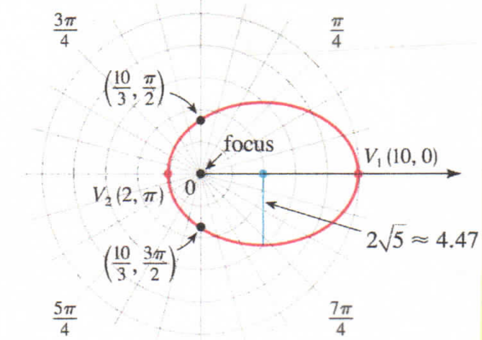
$C(4, 0)$

$d(V_1 V_2) = 12$

$2a = 12$, $a = 6$

$b^2 = a^2 - c^2 = 6^2 - 4^2 = 20 \quad b = 4\sqrt{5}$

θ	r
0	10
$\pi/2$	$\frac{10}{3}$
π	2
$3\pi/2$	$\frac{10}{3}$



$$r = \frac{12}{2 + 4\sin\theta} = \frac{6}{1 + 2\sin\theta}$$

$e = 2 > 1 \Rightarrow$ hyperbola

Vertical transverse axis

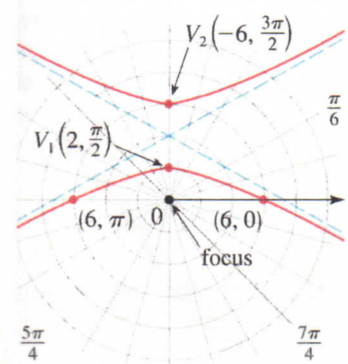
$V_1\left(2, \frac{\pi}{2}\right)$, $V_2\left(-6, \frac{3\pi}{2}\right)$

$C\left(4, \frac{\pi}{2}\right)$

$2a = 4 \quad a = 2$

$b^2 = \underbrace{c^2}_{= ae} - a^2 = 12$

θ	r
0	6
$\pi/2$	2
π	6
$3\pi/2$	-6



exercises 2-22