35.6 Dot Product

$\vec{u} = \langle a_1, b_1 \rangle \qquad \vec{v} = \langle a_2, b_2 \rangle$

$\vec{u} \cdot \vec{v} = a_1 a_2 + b_1 b_2$

Ex $\vec{u} = 2\hat{i} + \hat{j} \qquad \vec{v} = 5\hat{i} - 6\hat{j}$

$u \cdot v = (2)(5) + (1)(-6) = 4$

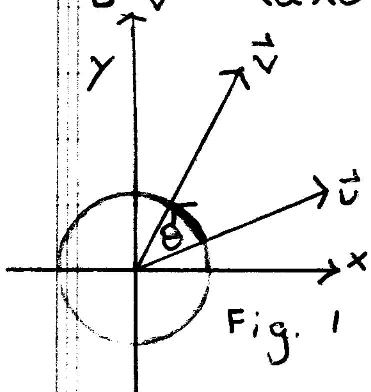


Fig. 1

PROPERTIES OF THE DOT PRODUCT

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$$

$$(a\mathbf{u}) \cdot \mathbf{v} = a(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (a\mathbf{v})$$

$$(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$$

$$|\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{u}$$

■ **Proof** We prove only the last property. The proofs of the others are left as exercises. Let $\mathbf{u} = \langle a\ b \rangle$. Then

$$\mathbf{u} \cdot \mathbf{u} = \langle a, b \rangle \cdot \langle a, b \rangle = a^2 + b^2 = |\mathbf{u}|^2$$ □

Let **u** and **v** be vectors and sketch them with initial points at the origin. We define the **angle θ between u and v** to be the smaller of the angles formed by these representations of **u** and **v** (see Figure 1). Thus, $0 \leqslant \theta \leqslant \pi$. The next theorem relates the angle between two vectors to their dot product.

THE DOT PRODUCT THEOREM

If θ is the angle between two nonzero vectors **u** and **v**, then

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta$$

■ **Proof** The proof is a nice application of the Law of Cosines. Applying the Law of Cosines to triangle AOB in Figure 2 gives

$$|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}|\,|\mathbf{v}| \cos\theta$$

Using the properties of the dot product, we write the left-hand side as follows:

$$\begin{aligned} |\mathbf{u} - \mathbf{v}|^2 &= (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} \\ &= |\mathbf{u}|^2 - 2(\mathbf{u} \cdot \mathbf{v}) + |\mathbf{v}|^2 \end{aligned}$$

Equating the right-hand sides of the displayed equations, we get

$$\begin{aligned} |\mathbf{u}|^2 - 2(\mathbf{u} \cdot \mathbf{v}) + |\mathbf{v}|^2 &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}|\,|\mathbf{v}| \cos\theta \\ -2(\mathbf{u} \cdot \mathbf{v}) &= -2|\mathbf{u}|\,|\mathbf{v}| \cos\theta \\ \mathbf{u} \cdot \mathbf{v} &= |\mathbf{u}|\,|\mathbf{v}| \cos\theta \end{aligned}$$

This proves the theorem. □

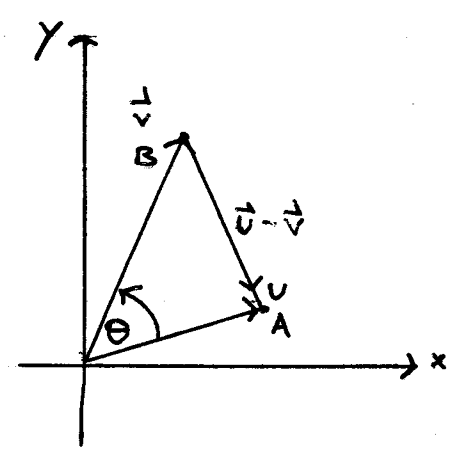


Fig 2

* Angle Between Two Vectors

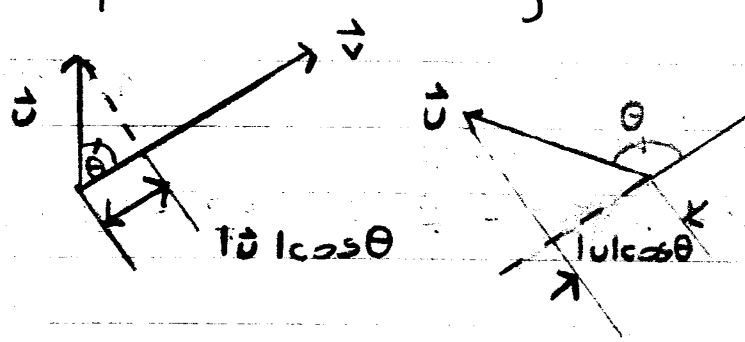
$\vec{u} \neq 0 \wedge \vec{v} \neq 0$: $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$

Show angle between vectors $\vec{u} = \langle 2, 5\rangle \wedge \vec{v} = \langle 4, -3\rangle$ is $\theta = \cos^{-1}\left(\frac{-7}{5\sqrt{29}}\right) \approx 105.1$

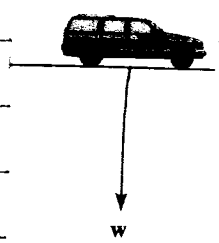
* Orthogonal Vectors

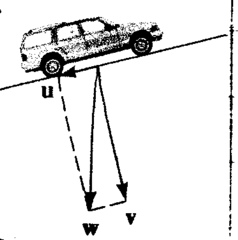
$\vec{u} \perp \vec{v}$ if $\vec{u}\cdot\vec{v} = 0$

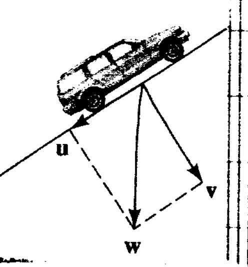
* Component of $\vec{u}$ along $\vec{v}$: $|\vec{u}|\cos\theta$

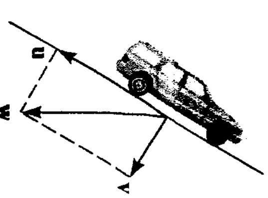


* Ex: The weight of a car $\vec{w}$ is a vector $\vec{w}$ that points directly downward









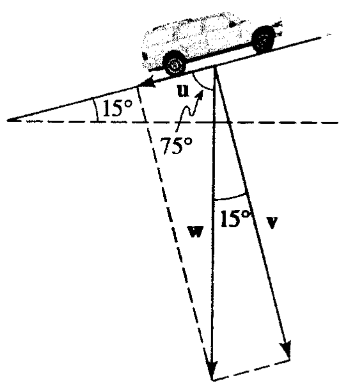


FIGURE 5

EXAMPLE 4* ■ *Resolving a Force into Components

A car weighing 3000 lb is parked on a driveway that is inclined 15° to the horizontal, as shown in Figure 5.

(a) Find the magnitude of the force required to prevent the car from rolling dow the driveway.

(b) Find the magnitude of the force experienced by the driveway due to the wei of the car.

SOLUTION

The car exerts a force **w** of 3000 lb directly downward. We resolve **w** into the s of two vectors **u** and **v**, one parallel to the surface of the driveway and the other perpendicular to it, as shown in Figure 5.

(a) The magnitude of the part of the force **w** that causes the car to roll down the driveway is

$$|\mathbf{u}| = \text{component of } \mathbf{w} \text{ along } \mathbf{u} = 3000\cos 75° \approx 776$$

Thus, the force needed to prevent the car from rolling down the driveway is about 776 lb.

(b) The magnitude of the force exerted by the car on the driveway is

$$|\mathbf{v}| = \text{component of } \mathbf{w} \text{ along } \mathbf{v} = 3000\cos 15° \approx 2898$$

The force experienced by the driveway is about 2898 lb.

The component of $\vec{u}$ along $\vec{v}$ is $\vec{u}\cdot\vec{v}/|\vec{v}|$

The Projection of $\vec{u}$ onto $\vec{v}$

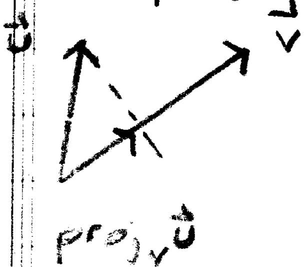
The projection of $\vec{u}$ onto $\vec{v}$, $\text{proj}_v\vec{u}$ is the vector whose direction is the same as $\vec{v}$ and whose length is the component of $\vec{u}$ along $\vec{v}$

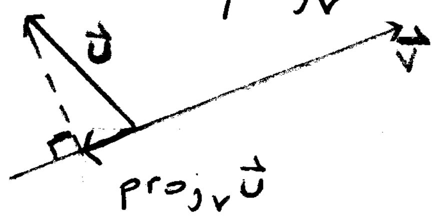
$$\text{proj}_v\vec{u} = (\text{component of } \vec{u} \text{ along } \vec{v})(\text{unit vector in direction } \vec{v})$$

$$= \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|} \cdot \frac{\vec{v}}{|v|} = \left[\frac{\vec{u}\cdot\vec{v}}{|v|^2}\right]\vec{v}$$

If $\vec{u}$ is resolved into $\vec{u_1}$ and $\vec{u_2}$, $\vec{u_1} \parallel \vec{v} \wedge \vec{u_2} \perp \vec{v}$,

$\vec{u_1} = \text{proj}_v\vec{u} \wedge \vec{u_2} = \vec{u} - \text{proj}_v\vec{u}$





$\vec{u} = \langle -2, 9\rangle$, $\vec{v} = \langle -1, 2\rangle$

$$\text{proj}_v\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|v|^2}\cdot\vec{v} = \left(\frac{\langle -2,9\rangle\cdot\langle -1,2\rangle}{(-1)^2+2^2}\right)\langle -1,2\rangle = \langle -4, 8\rangle$$

$\vec{u} = \vec{u_1} + \vec{u_2}$: $\vec{u_1} = \text{proj}_v\vec{u} = \langle -4, 8\rangle$

$\vec{u_2} = \vec{u} - \text{proj}_v\vec{u} = \langle -2, 9\rangle - \langle -4, 8\rangle = \langle 2, 1\rangle$

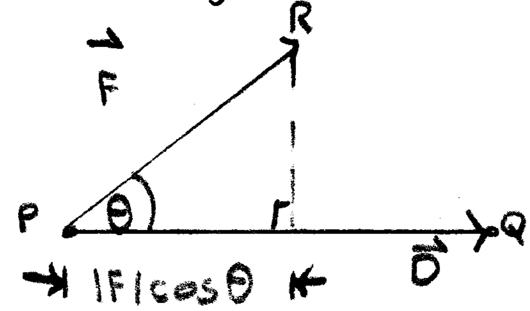
Work

work: constant force of magnitude F that moves an object through a distance d along a straight line.

W = force × distance

$\vec{F}$ moves an object from P to Q.

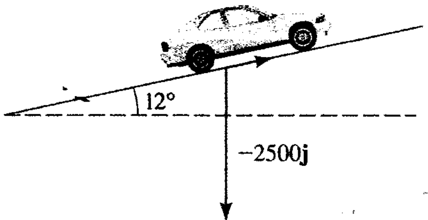
In this case, only the component of the force in the direction of $D = \vec{PQ}$ affects the object.



component of $\vec{F}$ along $\vec{D} = |\vec{F}| \cos\theta$

$W = (|F| \cos\theta)|D| = |\vec{F}||\vec{D}| \cos\theta = \vec{F} \cdot \vec{D}$

36. A car drives 500 ft on a road that is inclined 12° to the horizontal, as shown in the figure. The car weighs 2500 lb. Thus, gravity acts straight down on the car with a constant force $\mathbf{F} = -2500\mathbf{j}$. Find the work done by the car in overcoming gravity.



$\vec{W} = \vec{F} \cdot \vec{d}$

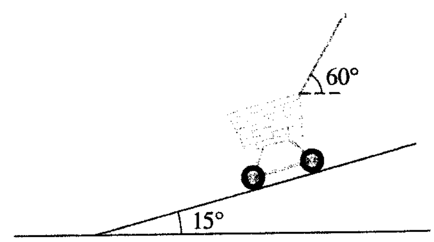
$F = \langle 0, -2500 \rangle$

$d = \langle 500 \cos 12^\circ, 500 \sin 12^\circ \rangle$

$W = \langle 0, -2500 \rangle \cdot \langle 500 \cos 12^\circ, 500 \sin 12^\circ \rangle$

$\approx 0 - (2500)(104) = -260000$ ft-lb

40. A cart weighing 40 lb is placed on a ramp inclined at 15° to the horizontal. The cart is held in place by a rope inclined at 60° to the horizontal, as shown in the figure. Find the force that the rope must exert on the cart to keep it from rolling down the ramp.



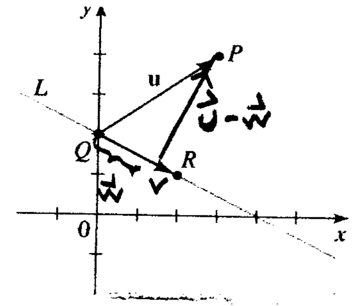
$\vec{R}$ represents the rope

$\vec{d}$ represents the force that causes the cart to roll up the ramp.

Gravity acting on the cart exerts a force $\vec{w}$ of 40 lb downward.

The magnitude of the part of force $\vec{w}$ that causes the cart to roll down the ramp is $|d| = 40 \cos 15$. The angle between $\vec{R}$ & $\vec{d}$ is 45°, so the magnitude of the force that holds the cart is $|\vec{R}| \cos 45°$

$|\vec{R}| \cos 45^\circ = 40 \cos 15^\circ$ so $|\vec{R}| = \dfrac{40 \cos 15^\circ}{\cos 45^\circ} \approx 54.6$ lb



$L = : 2x + 4y = 8 \qquad P(3,4)$

1. $Q(0,2) \in L \ \wedge \ R(2,1) \in L$

2 $\vec{u} = \overrightarrow{QP} \ , \ \vec{v} = \overrightarrow{QR}$

$\vec{w} = proj_v \vec{u}$ Find $\vec{w}$

$\vec{u} = \overrightarrow{QP} = \langle 0,2\rangle - \langle 3,4\rangle = \langle -3,-2\rangle$

$\vec{v} = \overrightarrow{QR} = \langle 0,2\rangle - \langle 2,1\rangle = \langle -2, 1\rangle$

$\vec{w} = proj_v \vec{u} = \left(\dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v} = \left[\dfrac{\langle -3,-2\rangle\cdot\langle 2,1\rangle}{(-2)^2 + (1)^2}\right]\langle -2,1\rangle = -\frac{8}{5}\langle -2,1\rangle$

$\vec{u} - \vec{w} \perp \vec{v}$ $\therefore$ distance from P to L is $|\vec{u} - \vec{w}|$

#24 $\vec{u} = \langle 7,-4\rangle, \ \vec{v} = \langle 2,1\rangle$

$proj_v \vec{u}$:

$\vec{u}_1 = proj_{\vec{v}} \vec{u} = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2} \cdot \vec{v} = \dfrac{\langle 7,-4\rangle\cdot\langle 2,1\rangle}{2^2 + 1^2} \langle 2,1\rangle = \langle 4,2\rangle$

$\vec{u}_2 = \vec{u} - \vec{u}_1 = \langle 7,-4\rangle - \langle 4,2\rangle = \langle 3,-6\rangle$

#30 $\vec{F} = 400\hat{i} + 50\hat{j}$, $P(-1,1), \ Q(200,1)$

$\vec{W} = \langle 400, 50\rangle \cdot \langle 201, 0\rangle = 80{,}400$