§4.3 Double-Angle, Half-Angle, Product-Sum Formulae

$\sin 2x = \sin(x+x) = 2\sin x\cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

$= 1 - 2\sin^2 x$

$= 2\cos^2 x - 1$

$\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

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$$\frac{\sin x + \sin 3x + \sin 5x}{\cos x + \cos 3x + \cos 5x} = \tan 3x$$

$$\frac{\sin(3x-2x) + \sin 3x + \sin(3x+2x)}{\cos(3x-2x) + \cos 3x + \cancel{\sin}\cos(3x+2x)} =$$

$$\frac{\sin 3x\cos 2x - \sin 2x\cos 3x + \sin 3x + \sin 3x\cos 2x + \sin 2x\cos(3x)}{\cos 3x\cos 2x + \sin 3x\sin 2x + \cos 3x + \cos 3x\cos 2x - \sin 3x\sin 2x}$$

$$= \frac{2\sin 3x\cos 2x + \sin 3x}{2\cos 3x\cos 2x + \cos 3x} = \frac{\sin 3x(1+2\cos 2x)}{\cos 3x(1+2\cos 2x)} = \tan 3x$$

$\sin^2 x = \dfrac{1-\cos 2x}{2}$ Note $\cos 2x = \cos^2 x - \sin^2 x$

$\cos^2 x = \dfrac{1+\cos 2x}{2}$

$\tan^2 x = \dfrac{1-\cos 2x}{1+\cos 2x}$

$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}}$; choice of +/- depends on which quadrant $u/2$ lies

Proof:

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

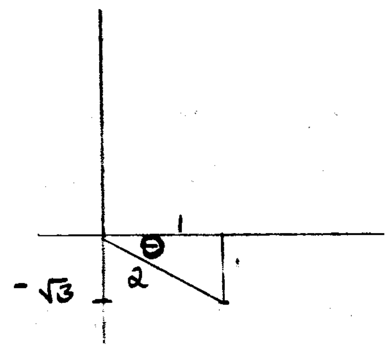
Let $x = \frac{u}{2}$:

$$\sin^2\frac{u}{2} = \frac{1-\cos u}{2}$$

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}}$$

$$\tan\frac{u}{2} \pm \sqrt{\frac{1-\cos u}{1+\cos u}} = \frac{1-\cos u}{\sin u} = \frac{\sin u}{1+\cos u}$$

$\sec x = 2$, $x \in$ Q IV



$$\sin 2x = 2\sin x\cos x$$

$$\sin x = -\sqrt{3}/2$$

$$\cos x = \frac{1}{2}$$

$$\sin 2x = -\sqrt{3}/2$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$= \frac{1}{4} - \frac{3}{4} = -\frac{1}{2}$$

$$\tan 2x = \sqrt{3}$$

$$\cos^4 x\sin^2 x = \frac{1+\cos 2x}{2}\cdot\frac{1+\cos 2x}{2}\cdot\frac{1-\cos 2x}{2}$$

$$= \frac{1-\cos^2 2x}{4}\cdot\frac{1+\cos 2x}{2}$$

$$= \frac{\sin^2 2x}{4}\cdot\frac{1+\cos 2x}{2}$$

$$= \frac{1-\cos 4x}{8}\cdot\frac{1+\cos 2x}{2}$$

$$\sin 75^\circ = \sin \frac{150}{2} = +\sqrt{\frac{1-\cos 150}{2}} = \sqrt{\frac{2+\sqrt{3}}{4}}$$

$$\cos 150 = \cos(180-30) = \cos 180 \cos 30 + \sin 180 \sin 30$$
$$= -1 \cdot \frac{\sqrt{3}}{2}$$

$$\frac{2\tan 7^\circ}{1-\tan^2 7^\circ} = 2\tan 7 / \sec^2 7 = 2\sin 7 \cos 7 = \sin 14^\circ$$

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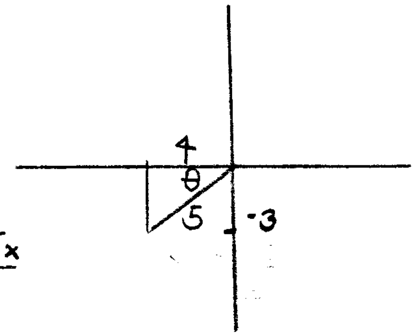
$$\cos x = -\frac{4}{5} \quad 180 < x < 270$$

$\sin \frac{x}{2}$, $\cos \frac{x}{2}$, $\tan \frac{x}{2}$



$$\sin \frac{x}{2} = -\sqrt{\frac{1-\cos x}{2}}$$
$$= -\sqrt{9/10}$$

$$\cos \frac{x}{2} = -\sqrt{\frac{1+\cos x}{2}}$$
$$= -\sqrt{1/10}$$

$$\tan \frac{x}{2} = 3$$

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Product-to-Sum Formulae

$$\sin(u+v) = \sin u \cos v + \cos u \sin v$$
$$\sin(u-v) = \sin u \cos v - \cos u \sin v$$
$$\sin(u+v) + \sin(u-v) = 2\sin u \cos v$$

$$\sin u \cos v = \frac{1}{2}\left[\sin(u+v) + \sin(u-v)\right]$$

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Exercises

1 $\cos u \sin v = \frac{1}{2}\left[\sin(u+v) - \sin(u-v)\right]$

2 $\cos u \cos v = \frac{1}{2}\left[\cos(u+v) + \cos(u-v)\right]$

3 $\sin u \sin v = \frac{1}{2}\left[\cos(u-v) - \cos(u+v)\right]$

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$$\cos 5x \cos 3x = \frac{1}{2}\left[\cos 8x + \cos 2x\right]$$

Sum-to-Product Formulae

$u = \frac{x+y}{2}$ $\quad v = \frac{x-y}{2}$

$\sin \frac{x+y}{2} \cos \frac{x-y}{2} = \frac{1}{2}\left[\sin x + \sin y\right]$

$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$

Derive

$\sin x - \sin y = 2\cos \frac{x+y}{2} \sin \frac{x-y}{2}$

$\cos x + \cos y = 2\cos \frac{x+y}{2} \cos \frac{x-y}{2}$

$\cos x - \cos y = -2\sin \frac{x+y}{2} \sin \frac{x-y}{2}$

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$\frac{\sin 3x + \sin 7x}{\cos 3x - \cos 7x} = \frac{2\sin 5x \cos(-2x)}{-2\sin 5x \sin(-2x)} = \cot 2x$

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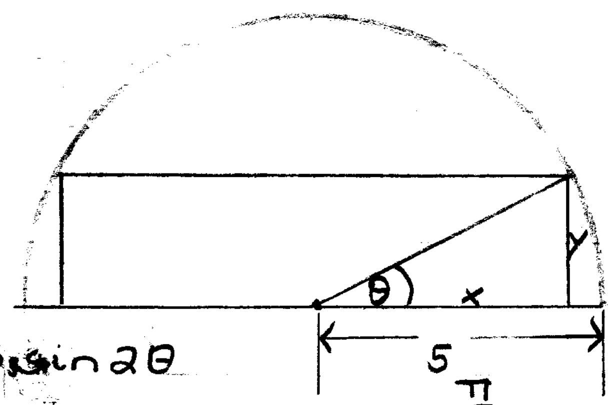


$A = 2xy$

$x = 5\cos\theta$

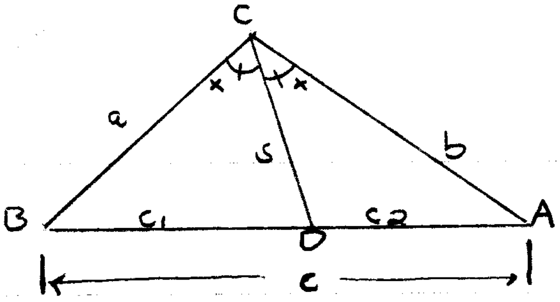
$y = 5\sin\theta$

$A = 2 \cdot 25 \cos\theta \sin\theta = 25 \sin 2\theta$

max area when $\sin 2\theta = 1 \Rightarrow 25$, $\theta = \frac{\pi}{4}$

$x = \frac{5\sqrt{2}}{2} = y$

In $\triangle ABC$, line segment s bisects $\angle C$. Find length of s.



Use Law of Sines

$$\frac{c}{\sin 2x} = \frac{b}{\sin B}, \quad c = \frac{b\sin 2x}{\sin B}$$

For $\triangle BCD$: $\frac{s}{\sin B} = \frac{c_1}{\sin x}$ $\quad c_1 = \frac{s\sin x}{\sin B}$

For $\triangle ACD$: $\frac{s}{\sin A} = \frac{c_2}{\sin x}$ $\quad c_2 = \frac{s\sin x}{\sin A}$

$c = c_1 + c_2$: $\frac{b\sin 2x}{\sin B} = \frac{s\sin x}{\sin B} + \frac{s\sin x}{\sin A} = s\sin x\left[\frac{1}{\sin B} + \frac{1}{\sin A}\right]$

For $\triangle ABC$: $\frac{b}{\sin B} = \frac{a}{\sin A} \Leftrightarrow \frac{1}{\sin A} = \frac{b}{a\sin B}$

$$\frac{b\sin 2x}{\sin B} = s\sin x\left[\frac{1}{\sin B} + \frac{b}{a\sin B}\right] = s\sin x \cdot \frac{1}{\sin B}\left[1 + \frac{b}{a}\right]$$

$$b\sin 2x = s(\sin x)\left[\frac{b+a}{a}\right] \Leftrightarrow 2b\sin x\cos x = s\sin x\left[\frac{b+a}{a}\right]$$

$$2b\cos x = s\left[\frac{b+a}{a}\right]$$

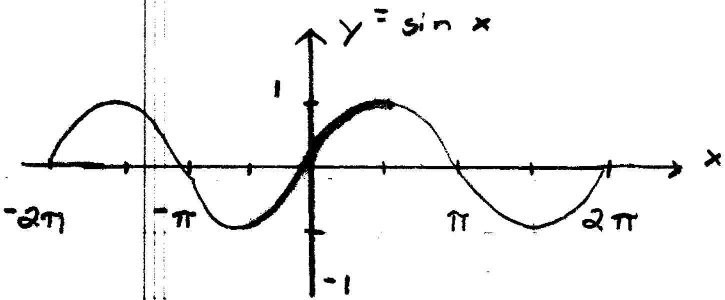
$$\frac{2ab\cos x}{b+a} = s$$

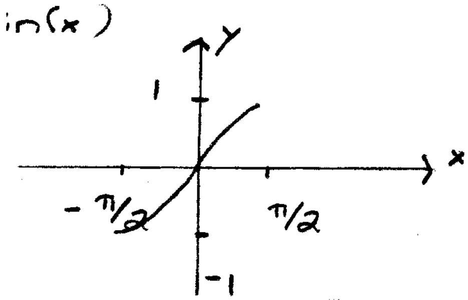
§ 4.4 Inverse Trig Functions

$f^{-1}(x) = y \Leftrightarrow f(y) = x$

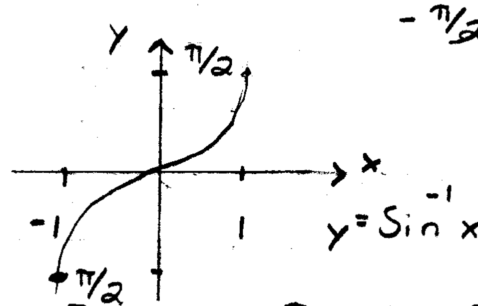
Inverse functions must be 1-1. Trig functions are not. Construct "principal parts" of trig functions. These parts are 1-1 ^ are written as $\mathrm{Sin}(x)$, $\mathrm{Cos}(x)$, $\mathrm{Tan}(x)$, $\mathrm{Csc}(x)$, $\mathrm{Cot}(x)$, $\mathrm{Sec}(x)$

Ranges ~~Domains~~ of each function are same as ranges ~~domains~~ of their counterparts. Their domains are intervals that map x-values in a 1-1 manner to their ranges





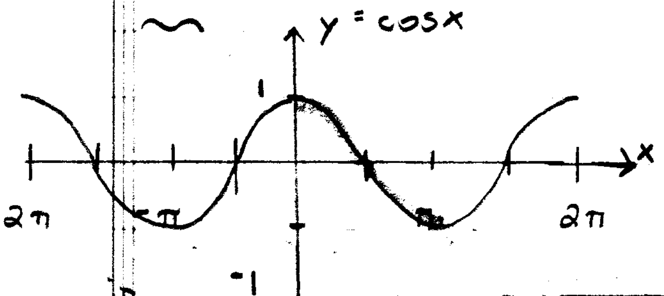
arcsin(x)

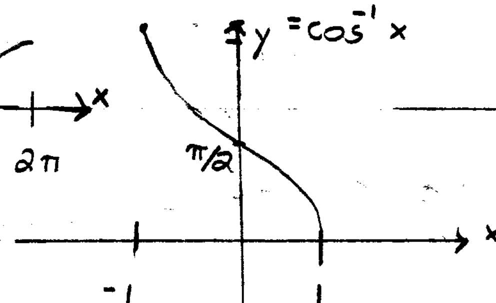


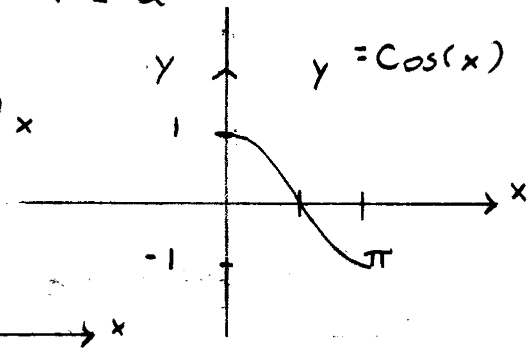
$\sin^{-1} x$: D $[-1, 1]$ R $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$\sin(\sin^{-1} x) = x$ for $-1 \leq x \leq 1$

$\sin^{-1}(\sin x) = x$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$



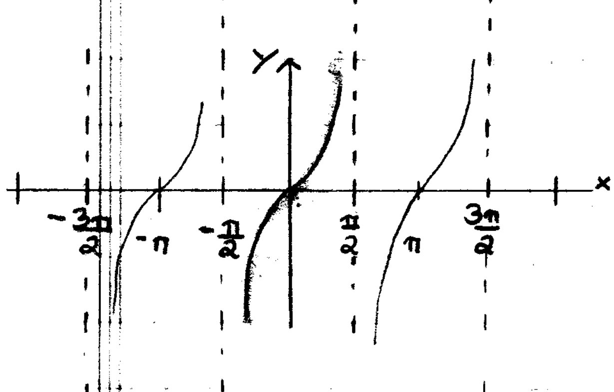


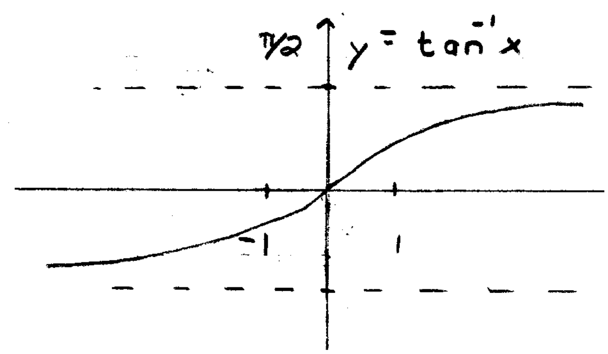


arccos(x) $\cos^{-1} x$: D $[-1, 1]$ R $[0, \pi]$

$\cos(\cos^{-1} x) = x$ for $-1 \leq x \leq 1$

$\cos^{-1}(\cos x) = x$ for $0 \leq x \leq \pi$





arctan(x) $\tan^{-1} x$: D : $\mathbb{R}$ R $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$\tan(\tan^{-1} x) = x, \quad x \in \mathbb{R}$

$\tan^{-1}(\tan x) = x, \quad -\pi/2 < x < \pi/2$