

The background of the slide features a warm, orange-toned image of a clock face with Roman numerals and a pair of glasses. The clock is partially visible on the left side, and the glasses are positioned in the center-left. The overall aesthetic is clean and professional.

6

APPLICATIONS OF INTEGRATION

6.4 Work

In this section, we will learn about:
Applying integration to calculate the amount of work done in performing a certain physical task.

WORK

The term 'work' is used in everyday language to mean the total amount of effort required to perform a task.

WORK

In physics, it has a technical meaning that depends on the idea of a 'force.'

- Intuitively, you can think of a force as describing a push or pull on an object.
- Some examples are: the horizontal push of a book across a table or the downward pull of the earth's gravity on a ball.

In general, if an object moves along a straight line with position function $s(t)$, then:

- The force F on the object (in the same direction) is defined by Newton's Second Law of Motion as the product of its mass m and its acceleration.

$$F = m \frac{d^2 s}{dt^2}$$

FORCE

In the SI metric system, the mass is measured in kilograms (kg), the displacement in meters (m), the time in seconds (s), and the force in newtons ($N = \text{kg} \cdot \text{m}/\text{s}^2$).

- Thus, a force of 1 N acting on a mass of 1 kg produces an acceleration of $1 \text{ m}/\text{s}^2$.

FORCE

In the US Customary system, the fundamental unit is chosen to be the unit of force, which is the pound.

In the case of constant acceleration:

- The force F is also constant and the work done is defined to be the product of the force F and the distance that the object moves:

$$W = Fd \quad (\text{work} = \text{force} \times \text{distance})$$

WORK

If F is measured in newtons and d in meters, then the unit for W is a newton-meter called joule (J).

If F is measured in pounds and d in feet, then the unit for W is a foot-pound (ft-lb), which is about 1.36 J.

WORK

Example 1

a. How much work is done in lifting a 1.2-kg book off the floor to put it on a desk that is 0.7 m high?

- Use the fact that the acceleration due to gravity is $g = 9.8 \text{ m/s}^2$.

b. How much work is done in lifting a 20-lb weight 6 ft off the ground?

The force exerted is equal and opposite to that exerted by gravity.

- Therefore, Equation 1 gives:

$$F = mg = (1.2)(9.8) = 11.76 \text{ N}$$

- Thus, Equation 2 gives the work done as:

$$W = Fd = (11.76)(0.7) \approx 8.2 \text{ J}$$

Here, the force is given as $F = 20 \text{ lb}$.

- Therefore, the work done is:

$$W = Fd = 20 \cdot 6 = 120 \text{ ft-lb}$$

- Notice that in (b), unlike (a), we did not have to multiply by g as we were given the 'weight' (which is a force) and not the mass of the object.

WORK

Equation 2 defines work as long as the force is constant.

However, what happens if the force is variable?

WORK

Let's suppose that the object moves along the x -axis in the positive direction, from $x = a$ to $x = b$.

- At each point x between a and b , a force $f(x)$ acts on the object—where f is a continuous function.

WORK

We divide the interval $[a, b]$ into n subintervals with endpoints x_0, x_1, \dots, x_n and equal width Δx .

- We choose a sample point x_i^* in the i th subinterval $[x_{i-1}, x_i]$.
- Then, the force at that point is $f(x_i^*)$.

WORK

If n is large, then Δx is small, and since f is continuous, the values of f don't change very much over the interval $[x_{i-1}, x_i]$.

- In other words, f is almost constant on the interval.
- So, the work W_i that is done in moving the particle from x_{i-1} to x_i is approximately given by Equation 2:
$$W_i \approx f(x_i^*) \Delta x$$

Thus, we can approximate the total work by:

$$W \approx \sum_{i=1}^n f(x_i^*) \Delta x$$

WORK

It seems that this approximation becomes better as we make n larger.

$$W \approx \sum_{i=1}^n f(x_i^*) \Delta x$$

WORK

Definition 4

So, we define the work done in moving the object from a to b as the limit of the quantity as $n \rightarrow \infty$.

- As the right side of Equation 3 is a Riemann sum, we recognize its limit as being a definite integral.
- Thus,

$$W = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x = \int_a^b f(x) dx$$

WORK

Example 2

When a particle is located a distance x feet from the origin, a force of $x^2 + 2x$ pounds acts on it.

How much work is done in moving it from $x = 1$ to $x = 3$?

WORK

Example 2

$$W = \int_1^3 (x^2 + 2x) dx = \left. \frac{x^3}{3} + x^2 \right|_1^3 = \frac{50}{3}$$

- The work done is $16\frac{2}{3}$ ft-lb.

WORK

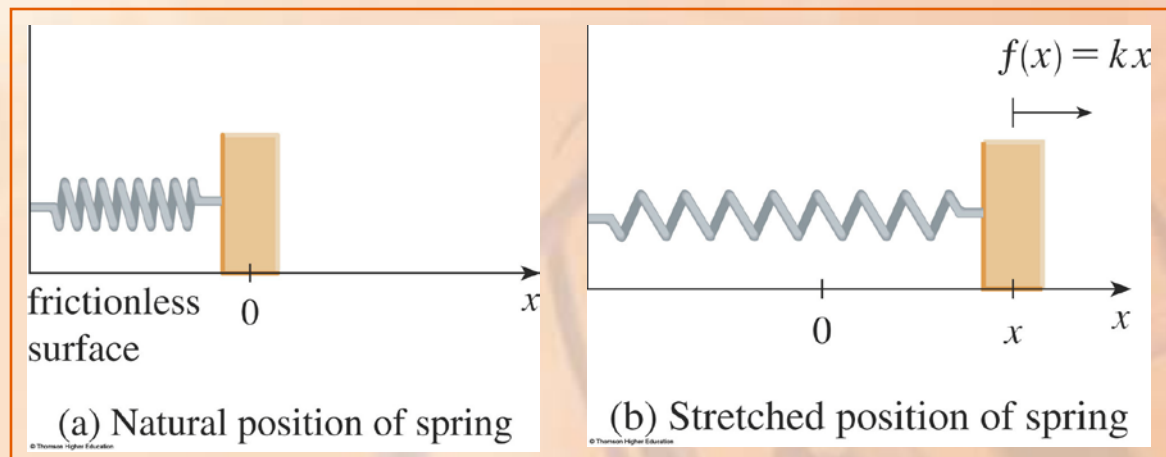
In the next example, we use a law from physics: Hooke's Law.

HOOKE'S LAW

The force required to maintain a spring stretched x units beyond its natural length is proportional to x

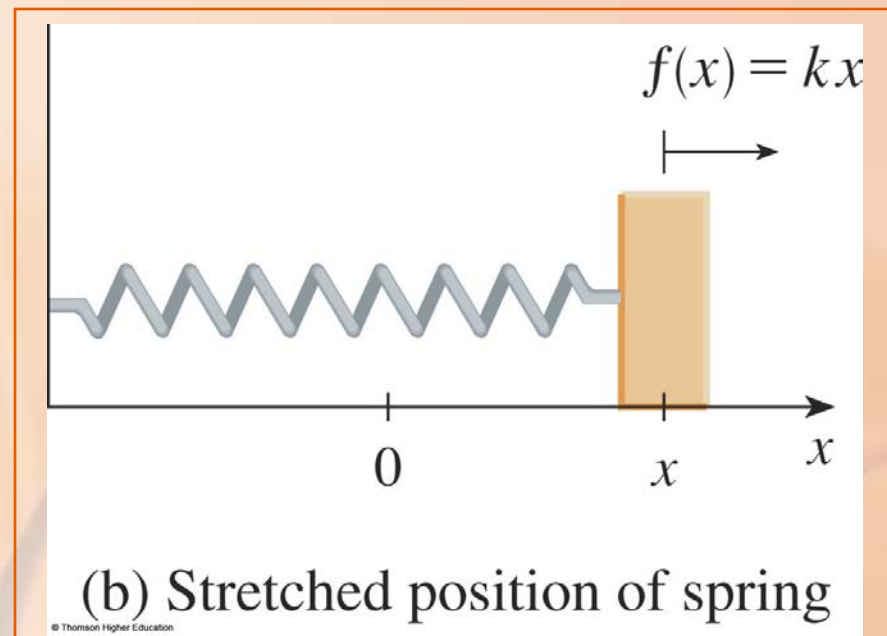
$$f(x) = kx$$

where k is a positive constant (called the spring constant).



HOOKE'S LAW

The law holds provided that x is not too large.



WORK

Example 3

A force of 40 N is required to hold a spring that has been stretched from its natural length of 10 cm to a length of 15 cm.

How much work is done in stretching the spring from 15 cm to 18 cm?

According to Hooke's Law, the force required to hold the spring stretched x meters beyond its natural length is $f(x) = kx$.

- When the spring is stretched from 10 cm to 15 cm, the amount stretched is 5 cm = 0.05 m.
- This means that $f(0.05) = 40$, so $0.05k = 40$.
- Therefore, $k = \frac{40}{0.05} = 800$

WORK

Example 3

Thus, $f(x) = 800x$ and the work done in stretching the spring from 15 cm to 18 cm is:

$$\begin{aligned} W &= \int_{0.05}^{0.08} 800x dx = 800 \left. \frac{x^2}{2} \right|_{0.05}^{0.08} \\ &= 400 \left[(0.08)^2 - (0.05)^2 \right] = 1.56 \text{ J} \end{aligned}$$

A 200-lb cable is 100 ft long and hangs vertically from the top of a tall building. How much work is required to lift the cable to the top of the building?

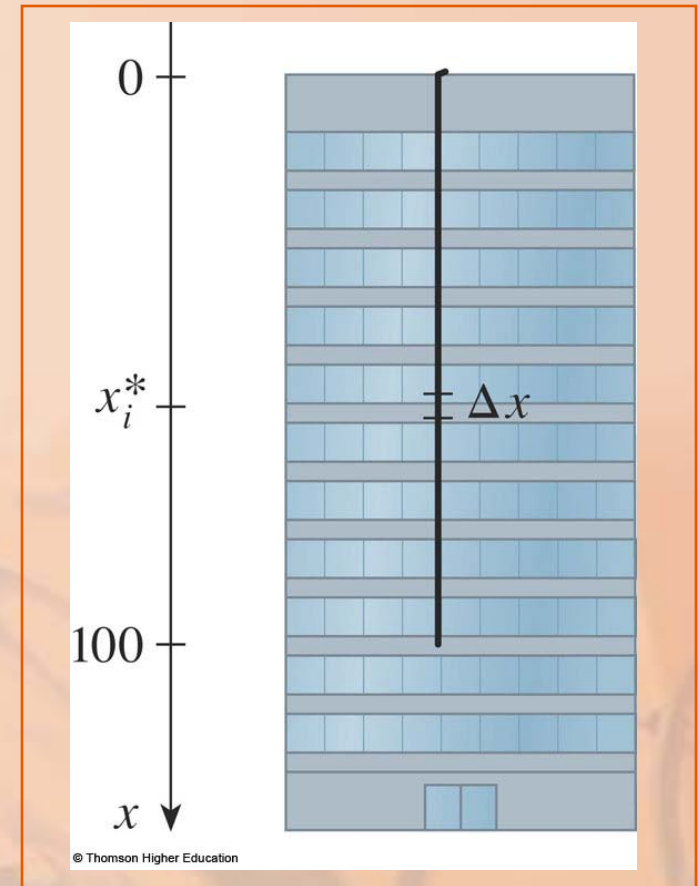
- Here, we don't have a formula for the force function.
- However, we can use an argument similar to the one that led to Definition 4.

WORK

Example 4

Let's place the origin at the top of the building and the x -axis pointing downward.

- We divide the cable into small parts with length Δx .

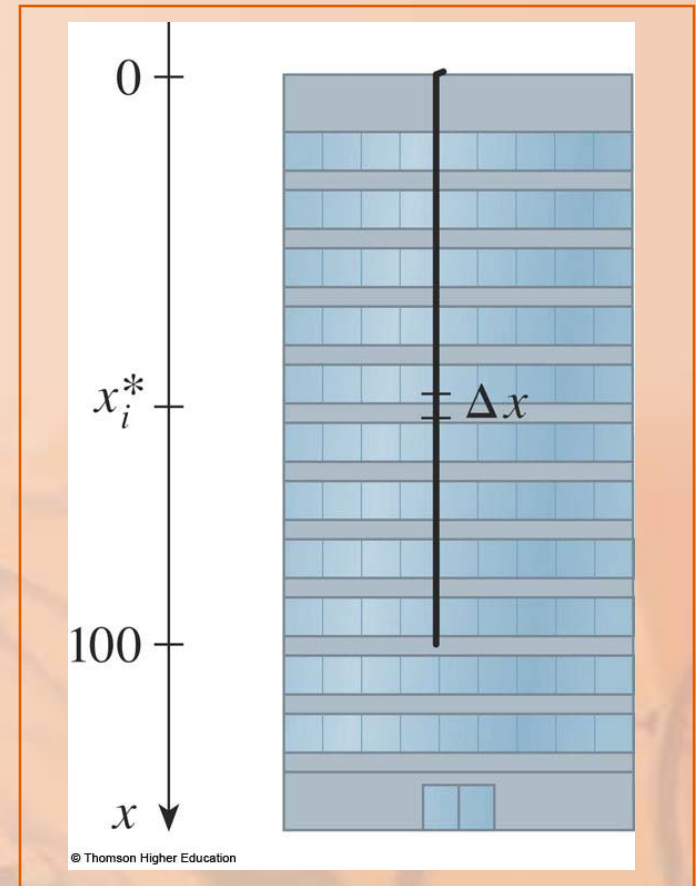


WORK

Example 4

If x_i^* is a point in the i th such interval, then all points in the interval are lifted by roughly the same amount, namely x_i^* .

- The cable weighs 2 lb/foot.
- So, the weight of the i th part is $2\Delta x$.



Thus, the work done on the i th part, in foot-pounds, is:

$$\underbrace{(2\Delta x)}_{\text{force}} \underbrace{x_i}_{\text{distance}} = 2x_i \Delta x$$

WORK

Example 4

We get the total work done by adding all these approximations and letting the number of parts become large (so $\Delta x \rightarrow 0$):

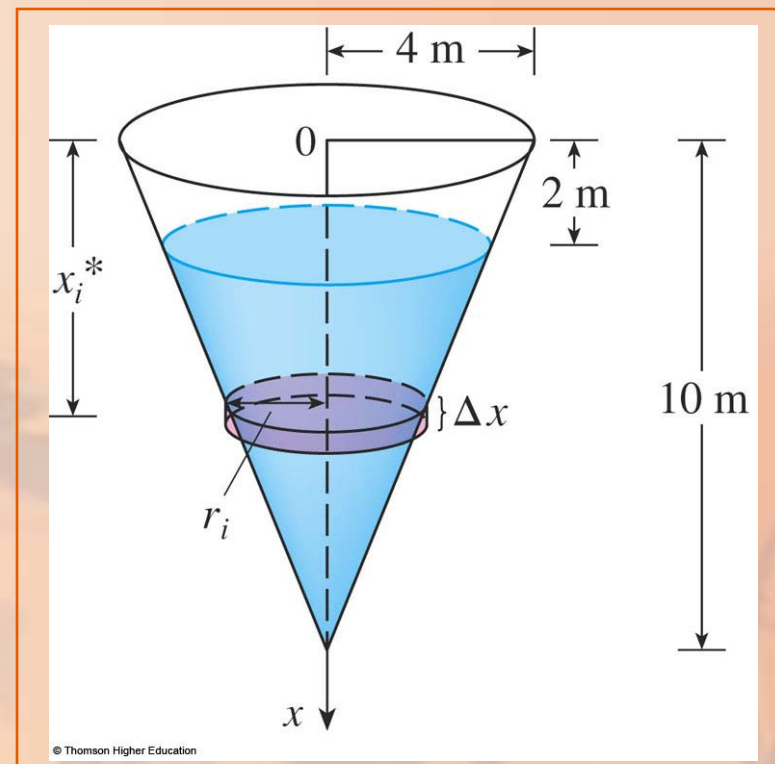
$$\begin{aligned} W &= \lim_{n \rightarrow \infty} \sum_{i=1}^n 2x_i * \Delta x \\ &= \int_0^{100} 2x dx = x^2 \Big|_0^{100} \\ &= 10,000 \text{ ft-lb} \end{aligned}$$

WORK

Example 5

A tank has the shape of an inverted circular cone with height 10 m and base radius 4 m. It is filled with water to a height of 8 m.

- Find the work required to empty the tank by pumping all the water to the top of the tank.
- The density of water is $1,000 \text{ kg/m}^3$.

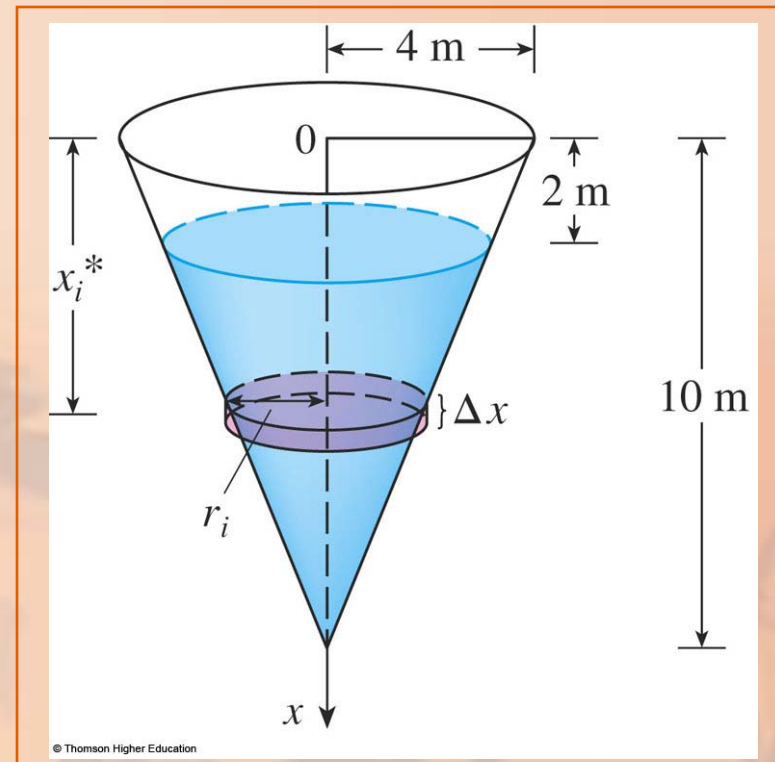


WORK

Example 5

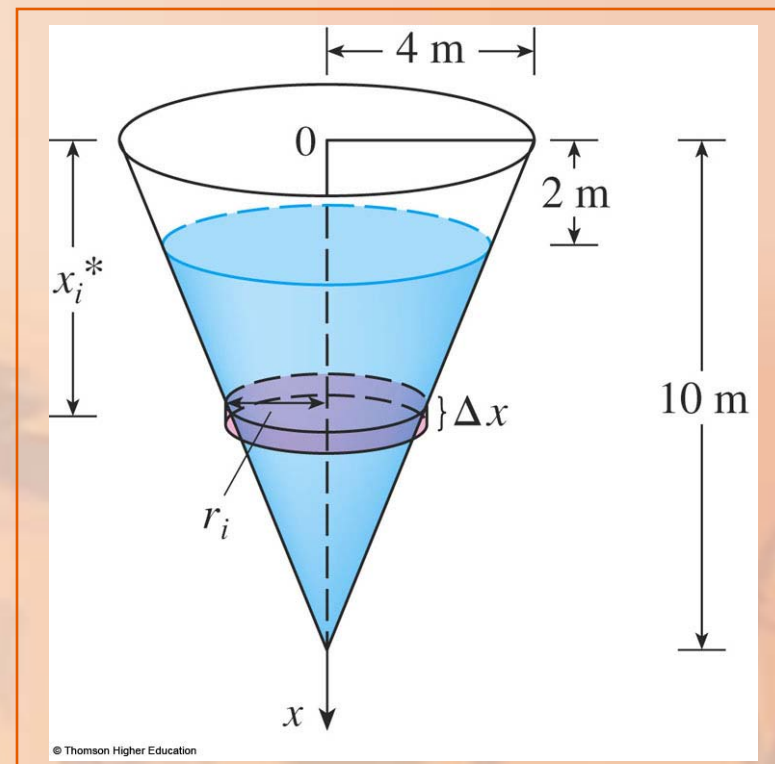
Let's measure depths from the top of the tank by introducing a vertical coordinate line.

- The water extends from a depth of 2 m to a depth of 10 m.
- So, we divide the interval $[2, 10]$ into n subintervals with endpoints x_0, x_1, \dots, x_n and choose x_i^* in the i th subinterval.



This divides the water into n layers.

- The i th layer is approximated by a circular cylinder with radius r_i and height Δx .



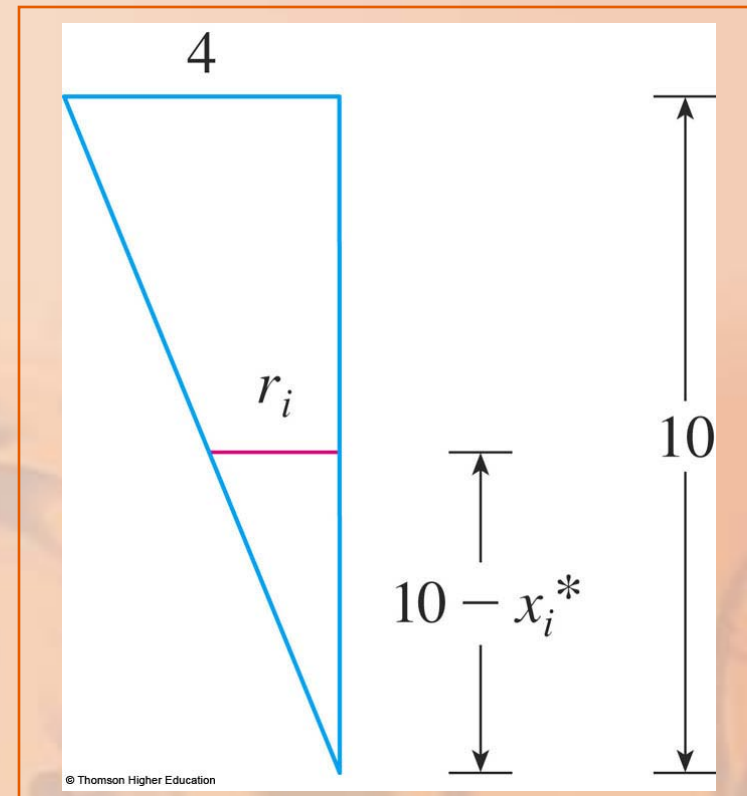
WORK

Example 5

We can compute r_i from similar triangles:

$$\frac{r_i}{10 - x_i^*} = \frac{4}{10}$$

$$r_i = \frac{2}{5}(10 - x_i^*)$$



Thus, an approximation to the volume of the i th layer of water is:

$$V_i \approx \pi r_i^2 \Delta x = \frac{4\pi}{25} (10 - x_i^*)^2 \Delta x$$

- So, its mass is: $m_i = \text{density} \times \text{volume}$

$$\approx 1000 \cdot \frac{4\pi}{25} (10 - x_i^*)^2 \Delta x$$

$$= 160\pi (10 - x_i^*)^2 \Delta x$$

The force required to raise this layer must overcome the force of gravity and so:

$$F_i = m_i g$$

$$\approx (9.8)160\pi(10 - x_i^*)^2 \Delta x$$

$$\approx 1570\pi(10 - x_i^*)^2 \Delta x$$

Each particle in the layer must travel a distance of approximately x_i^* .

- The work W_i done to raise this layer to the top is approximately the product of the force F_i and the distance x_i^* :

$$W_i \approx F_i x_i^*$$

$$\approx 1570\pi x_i^* (10 - x_i^*)^2 \Delta x$$

WORK

Example 5

To find the total work done in emptying the entire tank, we add the contributions of each of the layers and then take the limit as $n \rightarrow \infty$.

WORK

Example 5

$$\begin{aligned}W &= \lim_{n \rightarrow \infty} \sum_{i=1}^n 1570\pi x_i^* (10 - x_i^*)^2 \Delta x \\&= \int_2^{10} 1570\pi x_i^* (10 - x)^2 dx \\&= 1570\pi \int_2^{10} (100x - 20x + x^2) dx \\&= 1570\pi \left[50x^2 - \frac{20x^3}{3} + \frac{x^4}{4} \right]_2^{10} \\&= 1570\pi \left(\frac{2048}{3} \right) \approx 3.4 \times 10^6 \text{ J}\end{aligned}$$