6

APPLICATIONS OF INTEGRATION

6.3

Volumes by Cylindrical Shells

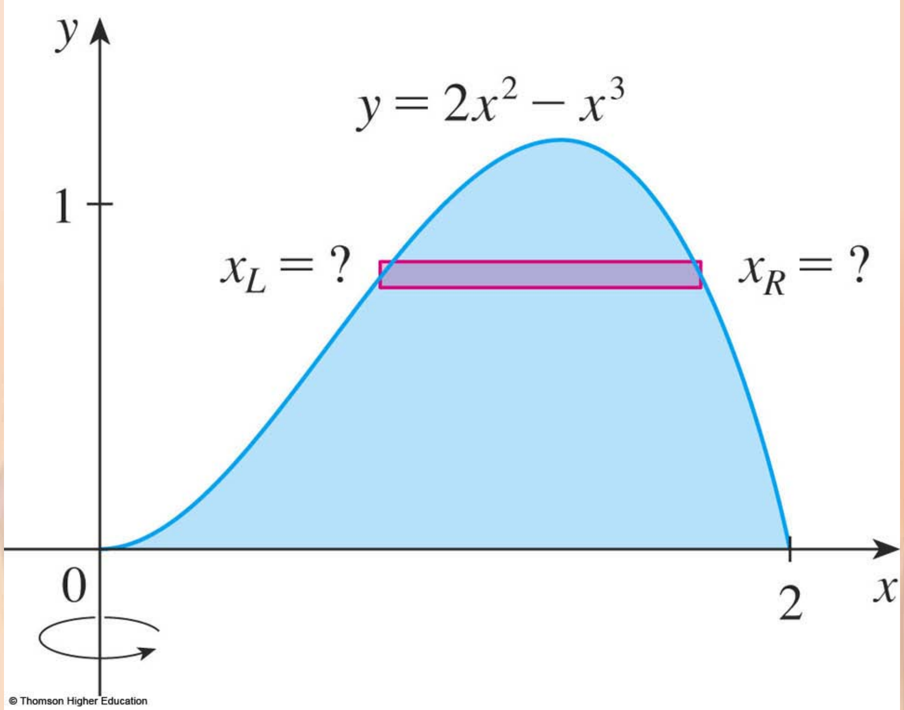
In this section, we will learn:

How to apply the method of cylindrical shells to find out the volume of a solid.

VOLUMES BY CYLINDRICAL SHELLS

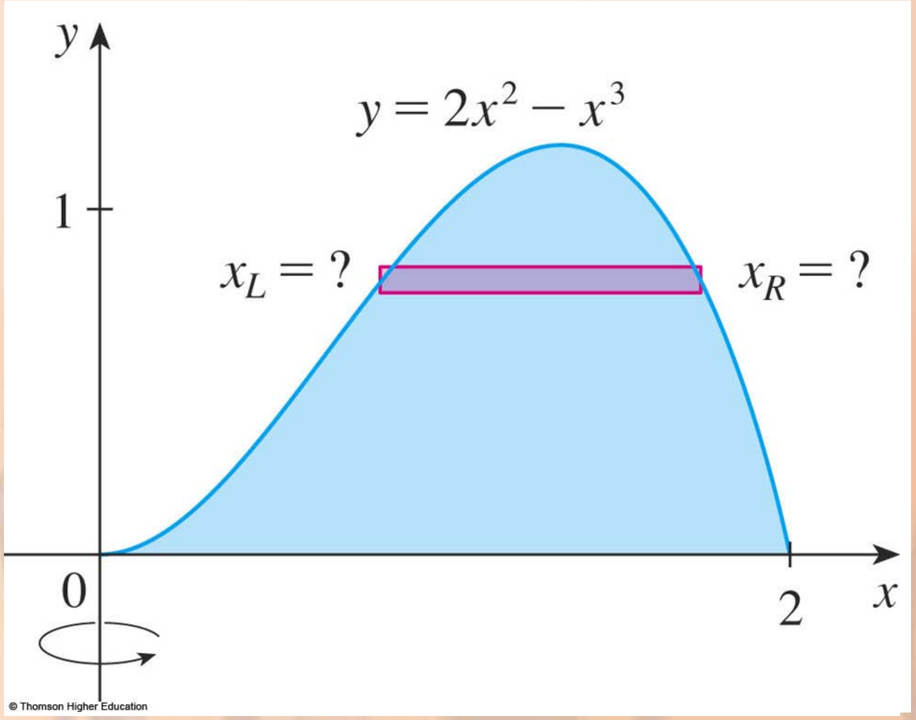
Some volume problems are very difficult to handle by the methods discussed in Section 6.2

Let's consider the problem of finding the volume of the solid obtained by rotating about the *y*-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.



If we slice perpendicular to the *y*-axis, we get a washer.

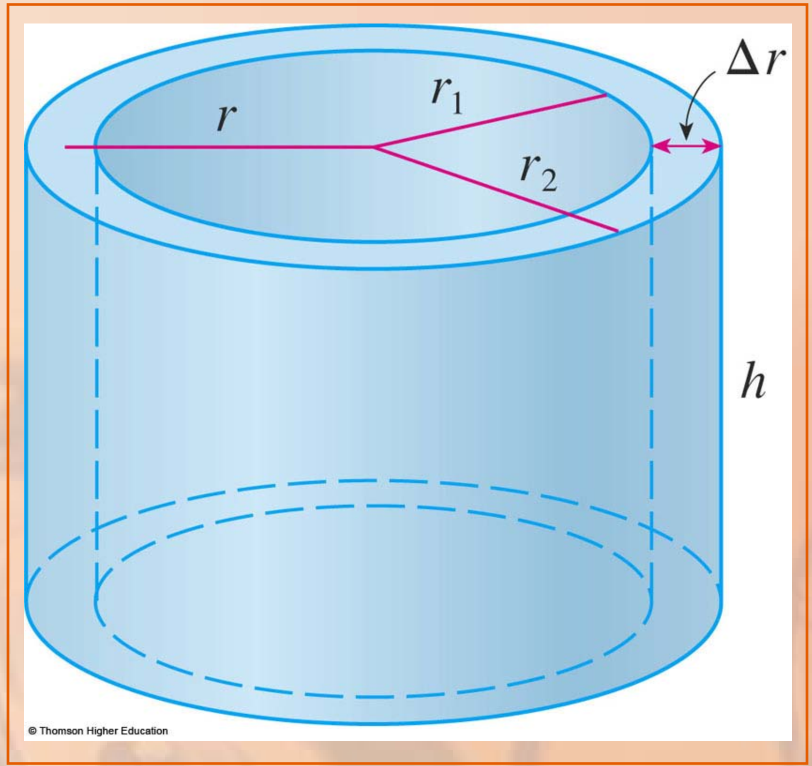
- However, to compute the inner radius and the outer radius of the washer, we would have to solve the cubic equation $y = 2x^2 - x^3$ for x in terms of y.
- That's not easy.



Fortunately, there is a method—the method of cylindrical shells—that is easier to use in such a case.

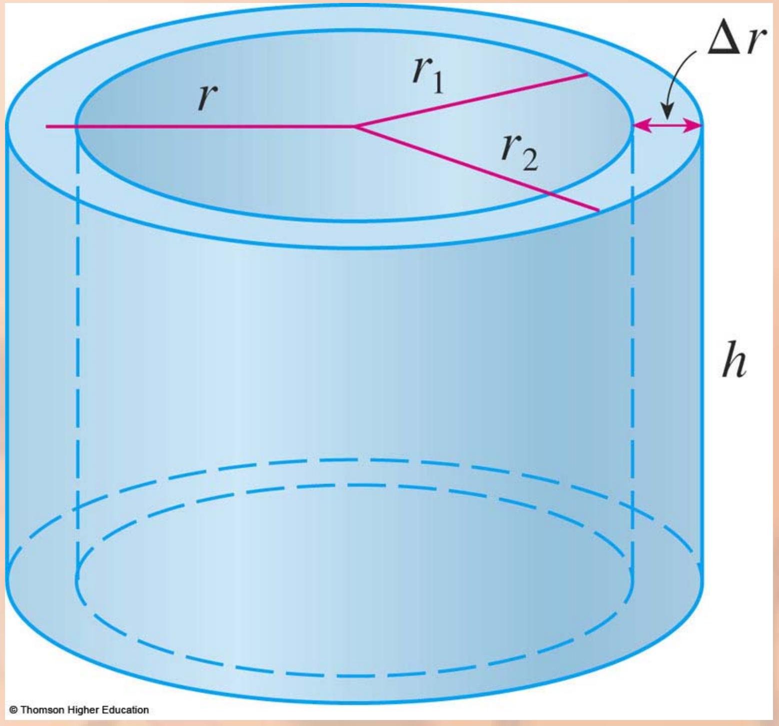
CYLINDRICAL SHELLS METHOD

The figure shows a cylindrical shell with inner radius r_1, outer radius r_2, and height h.



CYLINDRICAL SHELLS METHOD

Its volume V is calculated by subtracting the volume V_1 of the inner cylinder from the volume of the outer cylinder V_2 .



CYLINDRICAL SHELLS METHOD

Thus, we have:

$$\begin{aligned} V &= V_2 - V_1 \\ &= \pi r_2^{\,2} h - \pi r_1^{\,2} h \\ &= \pi (r_2^{\,2} - r_1^{\,2}) h \\ &= \pi (r_2 + r_1)(r_2 - r_1) h \\ &= 2\pi \frac{r_2 + r_1}{2} h (r_2 - r_1) \end{aligned}$$

Formula 1

Let $\Delta r = r_2 - r_1$ (thickness of the shell) and $r = \frac{1}{2}(r_2 + r_1)$ (average radius of the shell).

Then, this formula for the volume of a cylindrical shell becomes:

$$V = 2\pi r h \Delta r$$

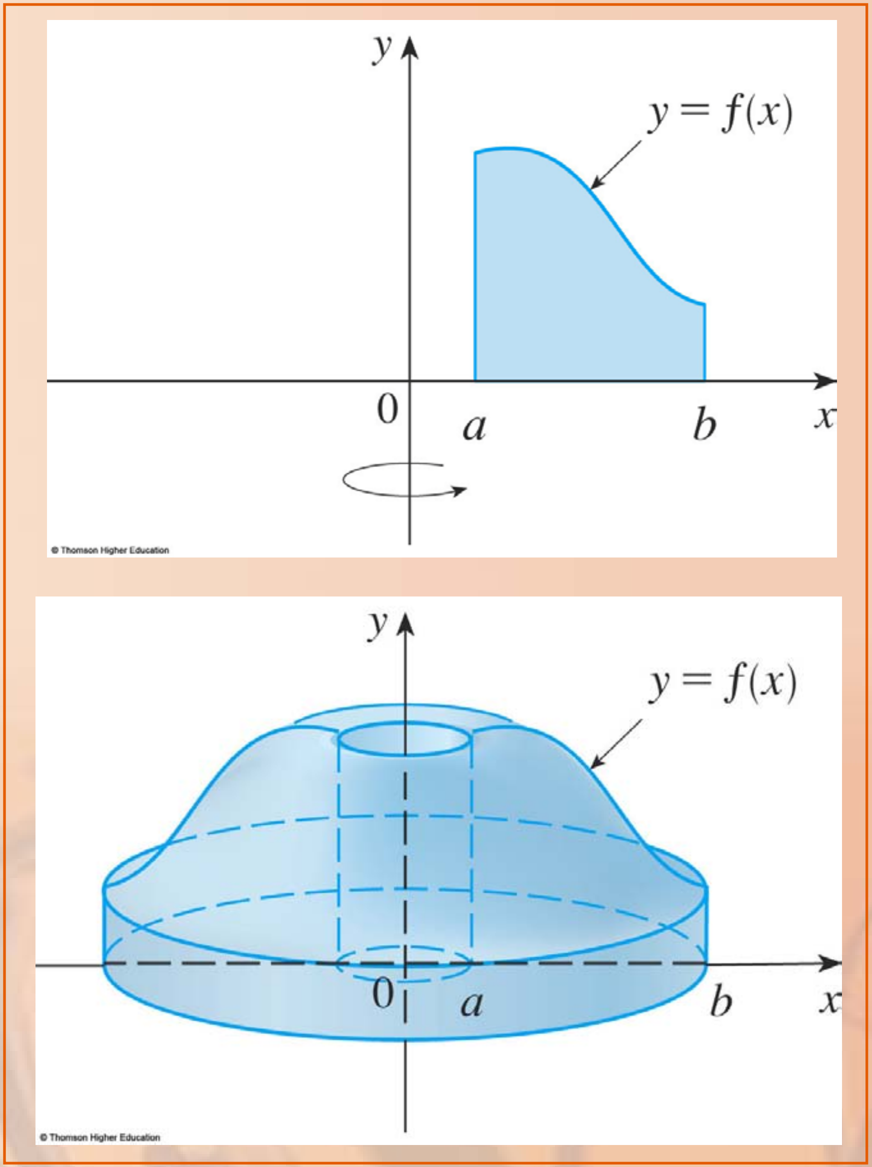
$$V = 2\pi rh\Delta r$$

The equation can be remembered as:

V = [circumference] [height] [thickness]

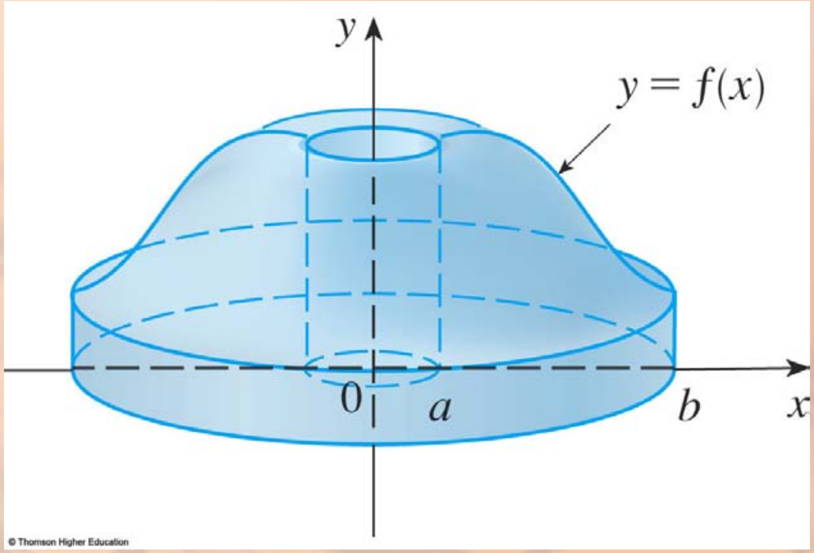
CYLINDRICAL SHELLS METHOD

Now, let *S* be the solid obtained by rotating about the *y*-axis the region bounded by $y = f(x)$ [where $f(x) \geq 0$], $y = 0$, $x = a$ and $x = b$, where $b > a \geq 0$.



CYLINDRICAL SHELLS METHOD

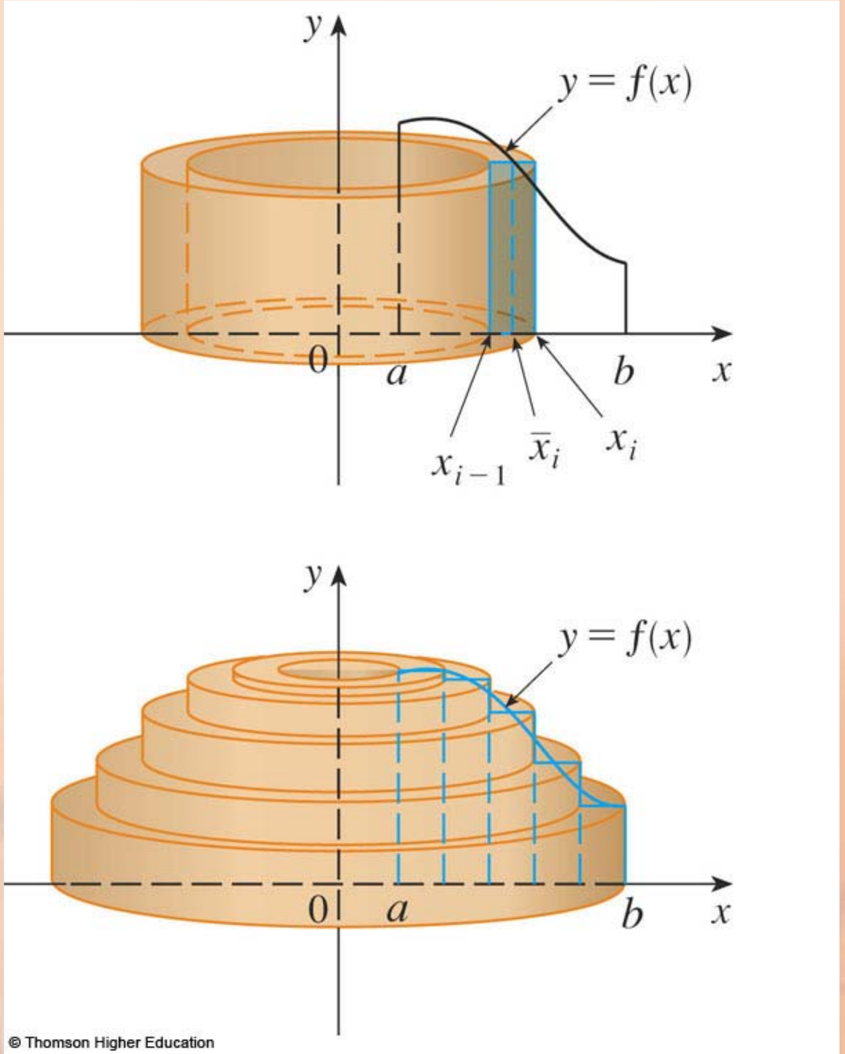
Divide the interval [*a*, *b*] into *n* subintervals $[x_{i-1}, x_i]$ of equal width $\overline{x}_i$ and let be Δx the midpoint of the *i* th subinterval.



CYLINDRICAL SHELLS METHOD

The rectangle with base $[x_{i-1}, x_i]$ and height $f(\bar{x}_i)$ is rotated about the y-axis.

- The result is a cylindrical shell with average radius $\bar{x}_i$, height $f(\bar{x}_i)$, and thickness Δx.



Thus, by Formula 1, its volume is calculated as follows:

$$V_i = (2\pi \bar{x}_i)[f(\bar{x}_i)]\Delta x$$

So, an approximation to the volume *V* of *S* is given by the sum of the volumes of these shells:

$$V \approx \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} 2\pi \bar{x}_i f(\bar{x}_i) \Delta x$$

CYLINDRICAL SHELLS METHOD

The approximation appears to become better as $n \to \infty$.

However, from the definition of an integral, we know that:

$$\lim_{n\to\infty} \sum_{i=1}^{n} 2\pi \overline{x}_i f(\overline{x}_i) \Delta x = \int_a^b 2\pi x f(x) dx$$

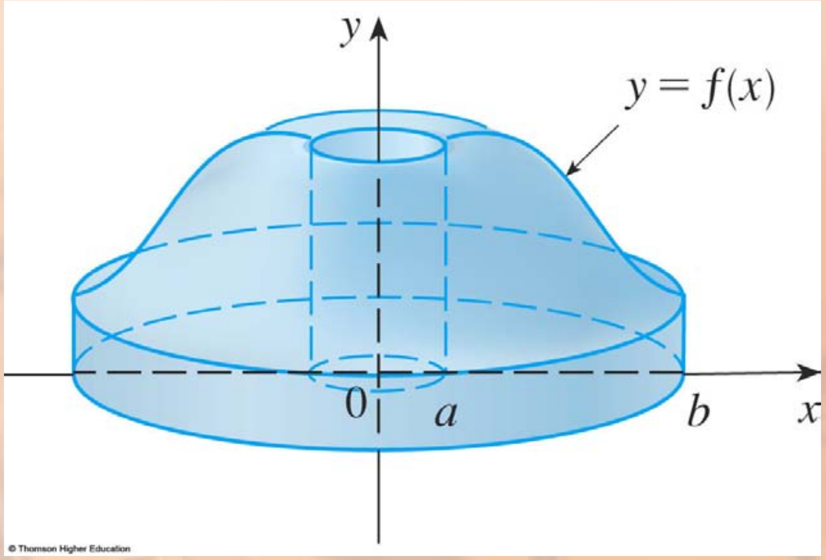
Formula 2

Thus, the following appears plausible.

- The volume of the solid obtained by rotating about the *y*-axis the region under the curve $y = f(x)$ from a to b, is:

$$V = \int_a^b 2\pi x f(x)\,dx$$

where $0 \leq a < b$



CYLINDRICAL SHELLS METHOD

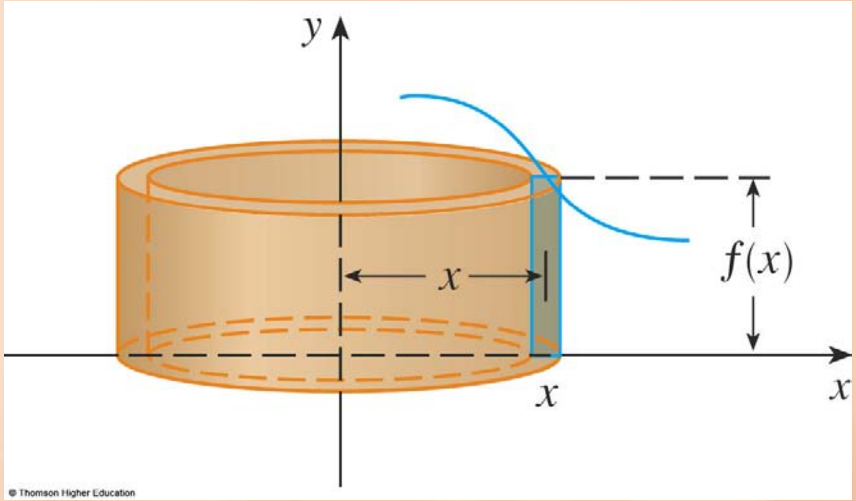
The argument using cylindrical shells makes Formula 2 seem reasonable, but later we will be able to prove it.

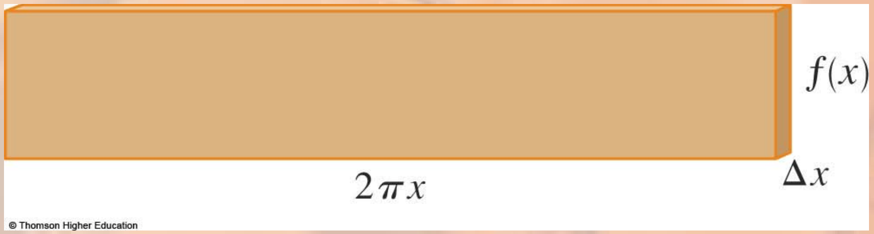
CYLINDRICAL SHELLS METHOD

Here's the best way to remember the formula.

- Think of a typical shell, cut and flattened, with radius *x*, circumference $2\pi x$, height $f(x)$, and thickness Δx or dx:

$$\int_a^b \underbrace{(2\pi x)}_{\text{circumference}} \underbrace{[f(x)]}_{\text{height}} \underbrace{dx}_{\text{thickness}}$$





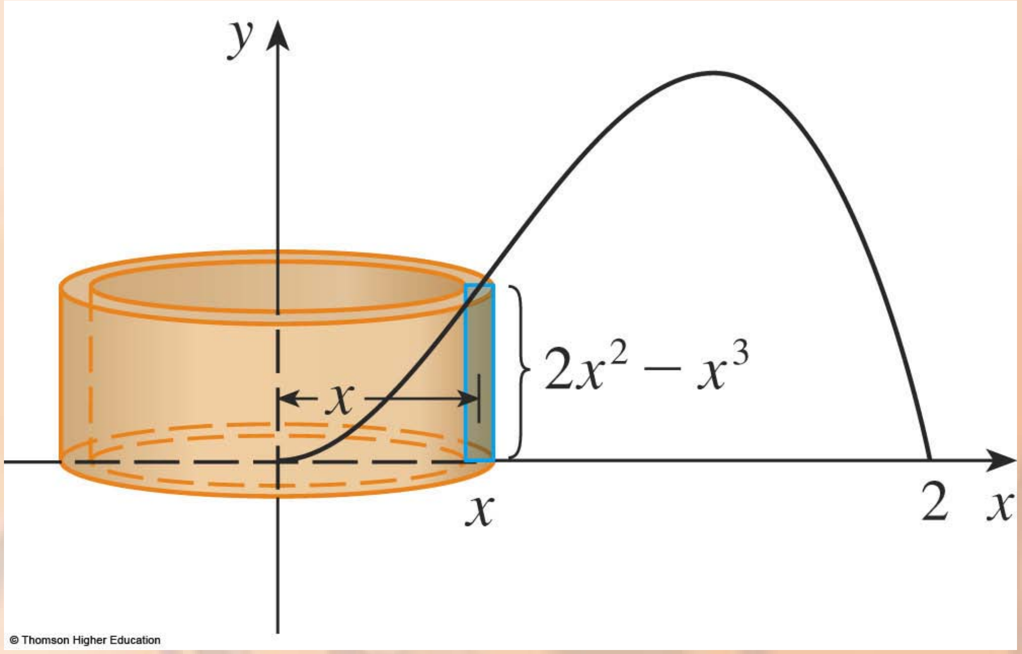
This type of reasoning will be helpful in other situations—such as when we rotate about lines other than the *y*-axis.

Example 1

Find the volume of the solid obtained by rotating about the *y*-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

 Example 1

We see that a typical shell has radius x, circumference $2\pi x$, and height $f(x) = 2x^2 - x^3$.



So, by the shell method, the volume is:

$$V = \int_0^2 (2\pi x)\left(2x^2 - x^3\right)dx$$

$$= \int_0^2 (2\pi x)(2x^3 - x^4)dx$$

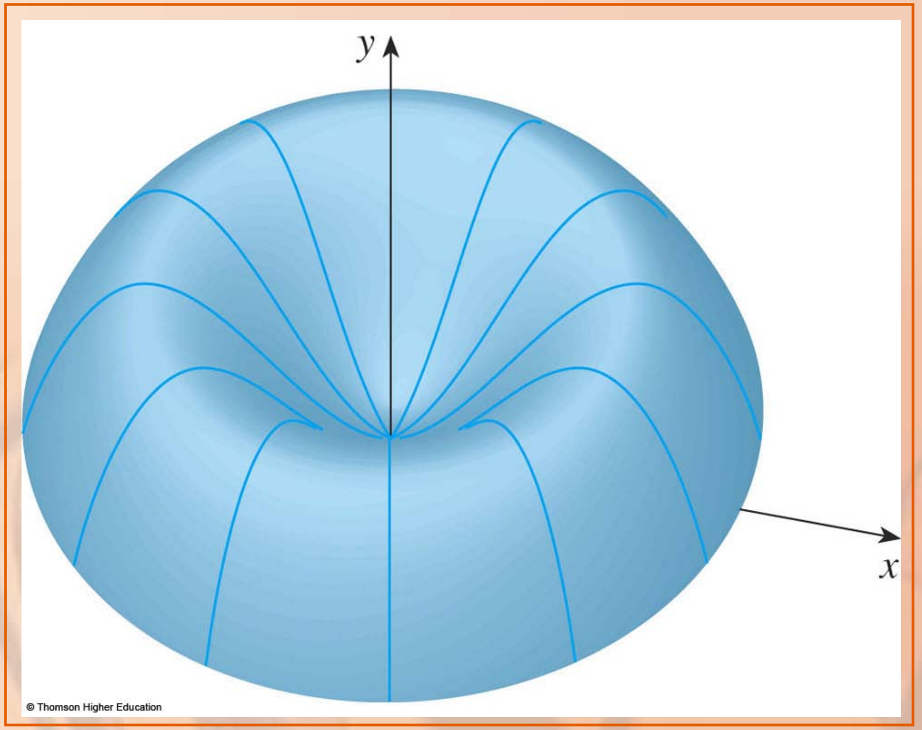
$$= 2\pi\left[\tfrac{1}{2}x^4 - \tfrac{1}{5}x^5\right]_0^2$$

$$= 2\pi\left(8 - \tfrac{32}{5}\right) = \tfrac{16}{5}\pi$$

Example 1

It can be verified that the shell method gives the same answer as slicing.

- The figure shows a computer-generated picture of the solid whose volume we computed in the example.



NOTE

Comparing the solution of Example 1 with the remarks at the beginning of the section, we see that the cylindrical shells method is much easier than the washer method for the problem.

- We did not have to find the coordinates of the local maximum.
- We did not have to solve the equation of the curve for x in terms of y.

NOTE

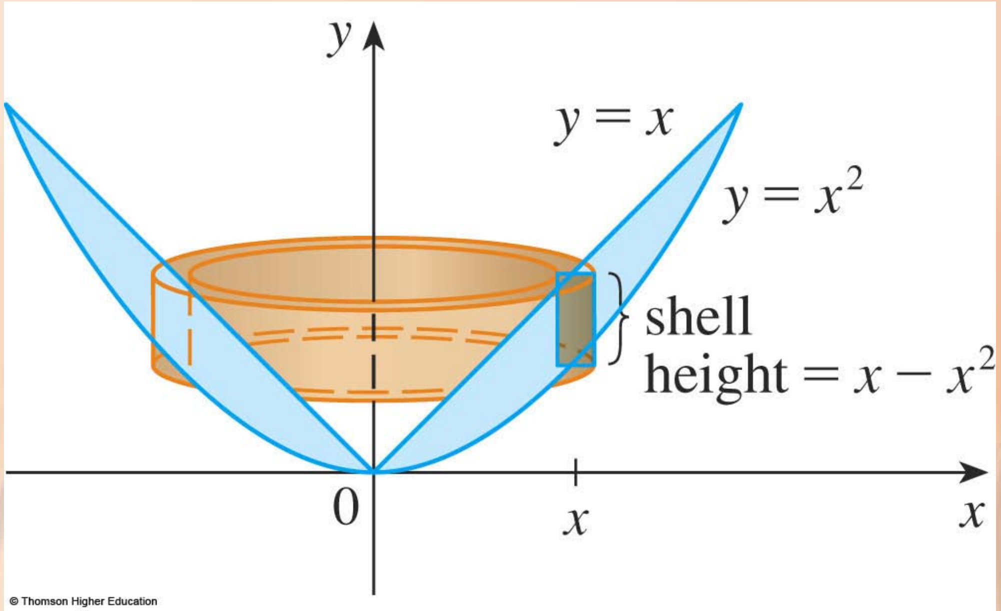
However, in other examples,
the methods learned in Section 6.2
may be easier.

Example 2

Find the volume of the solid obtained by rotating about the *y*-axis the region between $y = x$ and $y = x^2$.

The region and a typical shell are shown here.

- We see that the shell has radius *x*, circumference $2\pi x$, and height $x - x^2$.



Thus, the volume of the solid is:

$$V = \int_0^1 (2\pi x)\left(x - x^2\right)dx$$

$$= 2\pi \int_0^1 \left(x^2 - x^3\right)dx$$

$$= 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{\pi}{6}$$

CYLINDRICAL SHELLS METHOD

As the following example shows, the shell method works just as well if we rotate about the *x*-axis.

- We simply have to draw a diagram to identify the radius and height of a shell.

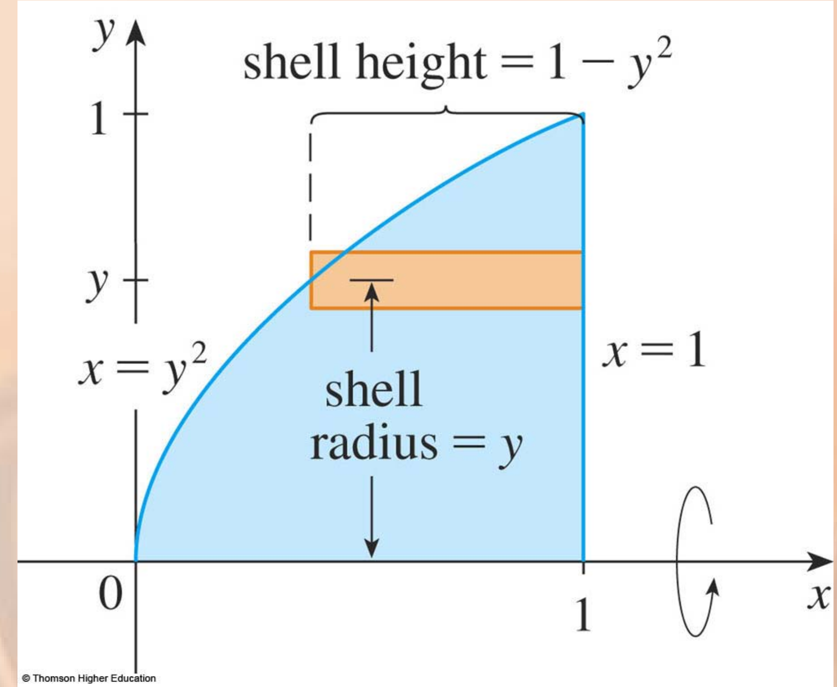
Example 3

Use cylindrical shells to find the volume of the solid obtained by rotating about the *x*-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

- This problem was solved using disks in Example 2 in Section 6.2

To use shells, we relabel the curve $y = \sqrt{x}$ as $x = y^2$.

- For rotation about the *x*-axis, we see that a typical shell has radius *y*, circumference $2\pi y$, and height $1 - y^2$.



So, the volume is:

$$V = \int_0^1 (2\pi y)(1 - y^2)\,dy$$

$$= 2\pi \int_0^1 (y - y^3)\,dy$$

$$= 2\pi \left[\frac{y^2}{2} - \frac{y^4}{4} \right]_0^1 = \frac{\pi}{2}$$

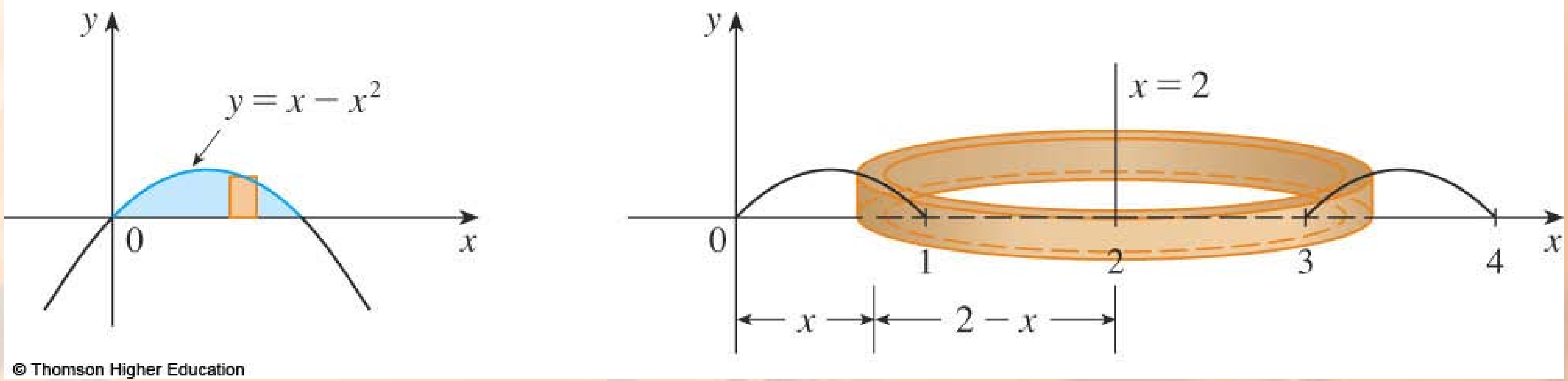
- In this problem, the disk method was simpler.

Example 4

Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.

CYLINDRICAL SHELLS METHOD   Example 4

The figures show the region and a cylindrical shell formed by rotation about the line $x = 2$, which has radius $2 - x$, circumference $2\pi(2 - x)$, and height $x - x^2$.



So, the volume of the solid is:

$$V = \int_1^0 2\pi(2-x)\left(x-x^2\right)dx$$

$$= 2\pi\int_1^0 \left(x^3-3x^2+2x\right)dx$$

$$= 2\pi\left[\frac{x^4}{4}-x^3+x^2\right]_0^1 = \frac{\pi}{2}$$