11

INFINITE SEQUENCES AND SERIES

11.9
Representations of Functions as Power Series

In this section, we will learn:

How to represent certain functions as sums of power series.

FUNCTIONS AS POWER SERIES

We can represent certain types of functions as sums of power series by either:

- Manipulating geometric series
- Differentiating or integrating such a series

You might wonder:

- Why would we ever want to express a known function as a sum of infinitely many terms?

We will see that this strategy is useful for:

- Integrating functions without elementary antiderivatives
- Solving differential equations
- Approximating functions by polynomials

FUNCTIONS AS POWER SERIES

Scientists do this to simplify the expressions they deal with.

Computer scientists do this to represent functions on calculators and computers.

Equation 1

We start with an equation we have seen before:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

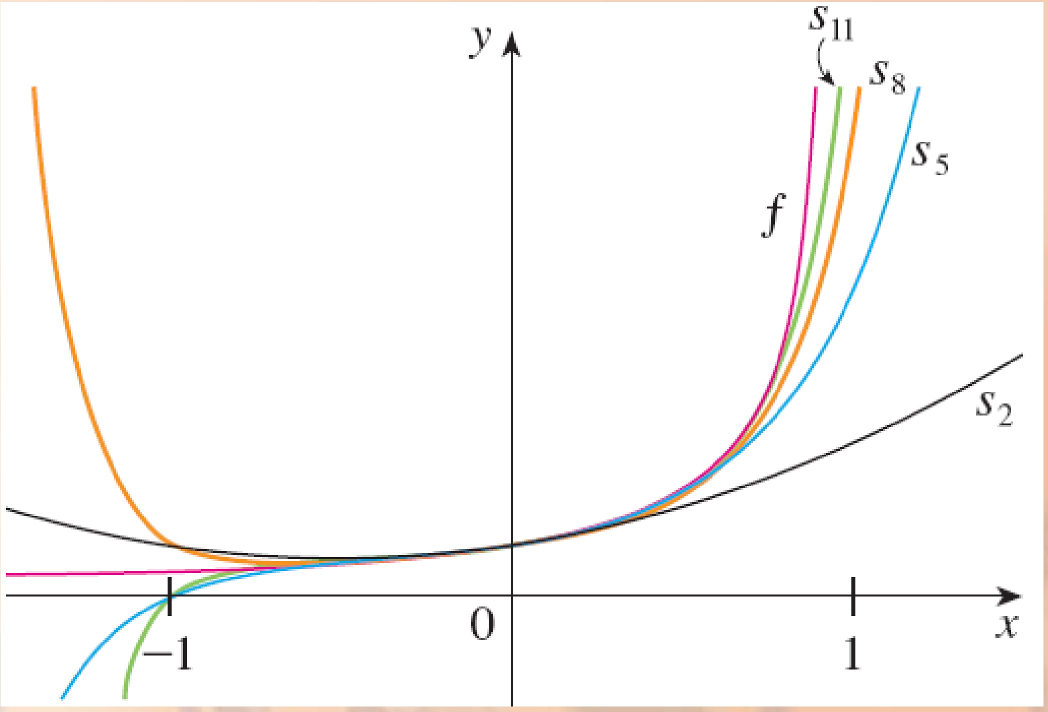
We first saw this equation in Example 5 in Section 11.2

- We obtained it by observing that it is a geometric series with $a = 1$ and $r = x$.

However, here our point of view is different.

- We regard Equation 1 as expressing the function $f(x) = 1/(1 - x)$ as a sum of a power series.

A geometric illustration of Equation 1 is shown.



Since the sum of a series is the limit of the sequence of partial sums, we have

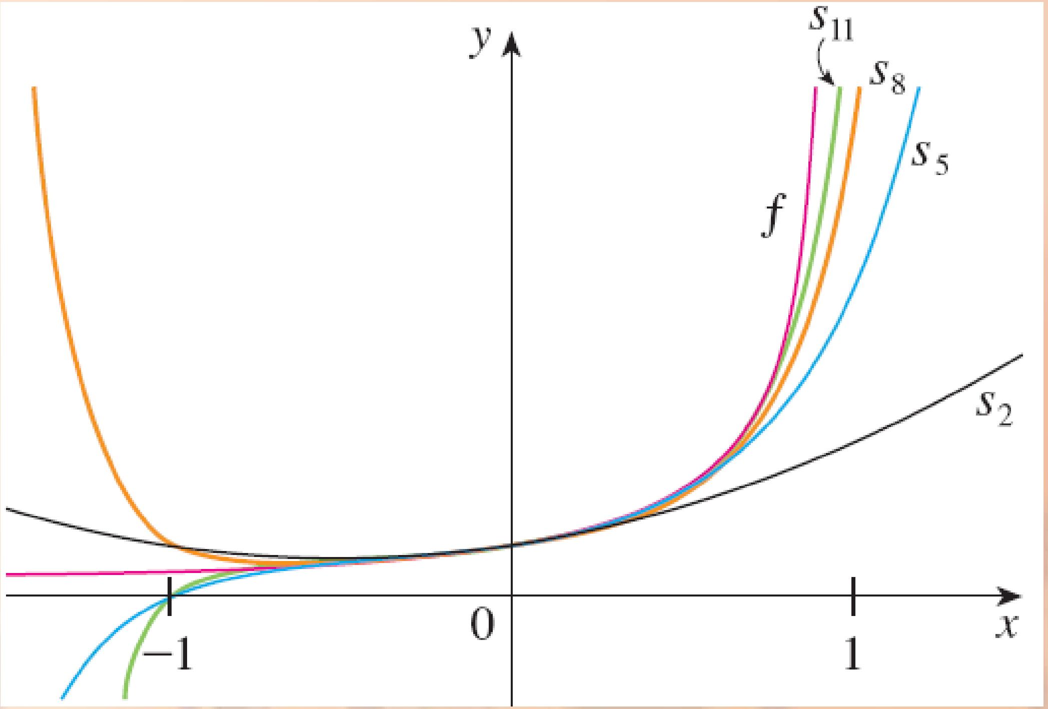
$$\frac{1}{1-x} = \lim_{n\to\infty} s_n(x)$$

where $s_n(x) = 1 + x + x^2 + \ldots + x^n$

is the *n*th partial sum.

FUNCTIONS AS POWER SERIES

Notice that, as n increases, $s_n(x)$ becomes a better approximation to $f(x)$ for $-1 < x < 1$.



Express $1/(1 + x^2)$ as the sum of a power series and find the interval of convergence.

Replacing x by $-x^2$ in Equation 1, we have:

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

$$= \sum_{n=0}^{\infty} (-x^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - \ldots$$

Since this is a geometric series,

it converges when $|-x^2| < 1$, that is,

$x^2 < 1$, or $|x| < 1$.

- Hence, the interval of convergence is $(-1, 1)$.

Of course, we could have determined the radius of convergence by applying the Ratio Test.

- However, that much work is unnecessary here.

Find a power series representation for $1/(x + 2)$.

- We need to put this function in the form of the left side of Equation 1.

- So, we first factor a 2 from the denominator:

$$\frac{1}{2+x} = \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1}{2\left[1-\left(-\frac{x}{2}\right)\right]}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n$$

This series converges when $|-x/2| < 1$, that is, $|x| < 2$.

- So, the interval of convergence is $(-2, 2)$.

Find a power series representation of $x^3/(x + 2)$.

- This function is just x^3 times the function in Example 2.
- So, all we have to do is multiply that series by x^3, as follows.

$$\frac{x^3}{x+2} = x^3 \cdot \frac{1}{x+2} = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n$$

$$= x^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+3}$$

$$= \tfrac{1}{2} x^3 - \tfrac{1}{4} x^4 + \tfrac{1}{8} x^5 - \tfrac{1}{16} x^6 + \ldots$$

- It's legitimate to move x^3 across the sigma sign because it doesn't depend on n.

Another way of writing this series is:

$$\frac{x^3}{x+2} = \sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{2^{n-2}} x^n$$

- As in Example 2, the interval of convergence is (–2, 2).

DIFFERENTIATION & INTEGRATION OF POWER SERIES

The sum of a power series is a function

$$f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n$$

whose domain is the interval of convergence of the series.

- We would like to be able to differentiate and integrate such functions.

TERM–BY–TERM DIFFN. & INTGN.

The following theorem (which we won't prove) says that we can do so by differentiating or integrating each individual term in the series—just as we would for a polynomial.

- This is called term-by-term differentiation and integration.

If the power series $\Sigma\, c_n(x - a)^n$ has radius of convergence $R > 0$, the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

$$= \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a - R, a + R)$.

Also,

i. $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots$

$$= \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

ii. $\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \dots$

$$= C + \sum_{n=0}^{\infty} \frac{c_n(x-a)^{n+1}}{n+1}$$

- The radii of convergence of the power series in Equations i and ii are both *R*.

In part ii, $\int c_0 \, dx = c_0 x + C_1$ is written as $c_0(x - a) + C$, where $C = C_1 + ac_0$.

- So, all the terms of the series have the same form.

NOTE 1

Equations i and ii in Theorem 2 can be rewritten in the form

iii. $$\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n (x-a)^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}\left[c_n (x-a)^n\right]$$

iv. $$\int \left[\sum_{n=0}^{\infty} c_n (x-a)^n\right] dx = \sum_{n=0}^{\infty} \int c_n (x-a)^n \, dx$$

NOTE 1

For finite sums, we know that:

- The derivative of a sum is the sum of the derivatives.
- The integral of a sum is the sum of the integrals.

NOTE 1

Equations iii and iv assert that the same is true for infinite sums—provided we are dealing with power series.

- For other types of series of functions, the situation is not as simple (see Exercise 36).

NOTE 2

Theorem 2 says that the radius of convergence remains the same when a power series is differentiated or integrated.

- However, this does not mean that the interval of convergence remains the same.

NOTE 2

It may happen that the original series converges at an endpoint, whereas the differentiated series diverges there.

- See Exercise 37.

NOTE 3

The idea of differentiating a power series term by term is the basis for a powerful method for solving differential equations.

- We will discuss this method in Chapter 17.

In Example 3 in Section 11.8, we saw that the Bessel function

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

is defined for all *x*.

Thus, by Theorem 2, J_0 is differentiable for all *x* and its derivative is found by term-by-term differentiation as follows:

$$J_0'(x) = \sum_{n=0}^{\infty} \frac{d}{dx} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2n\, x^{2n-1}}{2^{2n}(n!)^2}$$

Express $1/(1 - x)^2$ as a power series by differentiating Equation 1.

What is the radius of convergence?

Differentiating each side of the equation

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$$

we get:

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots = \sum_{n=1}^{\infty} nx^{n-1}$$

If we wish, we can replace *n* by *n* + 1 and write the answer as:

$$\frac{1}{(1-x)^2}=\sum_{n=0}^{\infty}(n+1)x^n$$

- By Theorem 2, the radius of convergence of the differentiated series is the same as that of the original series, namely, $R = 1$.

Find a power series representation for $\ln(1 - x)$ and its radius of convergence.

- We notice that, except for a factor of –1, the derivative of this function is $1/(1 - x)$.

- So, we integrate both sides of Equation 1:

$$-\ln(1-x)=\int\frac{1}{1-x}\,dx$$

$$=\int(1+x+x^2+\ldots)\,dx$$

$$=x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots+C$$

$$=\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}+C=\sum_{n=1}^{\infty}\frac{x^n}{n}+C \qquad |x|<1$$

To determine the value of C, we put $x = 0$ in this equation and obtain $-\ln(1 - 0) = C$.

Thus, $C = 0$ and

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots = -\sum_{n=1}^{\infty} \frac{x^n}{n} \quad |x| < 1$$

- The radius of convergence is the same as for the original series: $R = 1$.

Notice what happens if we put $x = ½$ in the result of Example 6.

- Since ln ½ = -ln 2, we see that:

$$\ln 2 = \frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n2^n}$$

Find a power series representation for $f(x) = \tan^{-1} x$.

- We observe that $f'(x) = 1/(1 + x^2)$.

- Thus, we find the required series by integrating the power series for $1/(1 + x^2)$ found in Example 1.

$$\tan^{-1} x = \int \frac{1}{1+x^2}\,dx$$

$$= \int (1 - x^2 + x^4 - x^6 + \ldots)\,dx$$

$$= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

To find C, we put $x = 0$ and obtain $C = \tan^{-1} 0$.

Hence,

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

- Since the radius of convergence of the series for $1/(1 + x^2)$ is 1, the radius of convergence of this series for $\tan^{-1} x$ is also 1.

GREGORY'S SERIES

The power series for $\tan^{-1}x$ obtained in Example 7 is called Gregory's series.

- It is named after the Scottish mathematician James Gregory (1638–1675), who had anticipated some of Newton's discoveries.

We have shown that Gregory's series is valid when $-1 < x < 1$.

- However, it turns out that it is also valid when $x = \pm 1$.
- This isn't easy to prove, though.

GREGORY'S SERIES

Notice that, when $x = 1$, the series becomes:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

- This beautiful result is known as the Leibniz formula for π.

a. Evaluate $\int [1/(1 + x^7)]\, dx$ as a power series.

b. Use part (a) to approximate $\int_0^{0.5} [1/(1+x^7)]\, dx$ correct to within 10^{-7}.

The first step is to express the integrand, $1/(1 + x^7)$, as the sum of a power series.

- As in Example 1, we start with Equation 1 and replace x by $–x^7$:

$$\frac{1}{1+x^7}=\frac{1}{1-(-x^7)}=\sum_{n=0}^{\infty}(-x^7)^n$$

$$=\sum_{n=0}^{\infty}(-1)^n x^{7n}$$

$$=1-x^7+x^{14}-\dots$$

Now, we integrate term by term:

$$\int \frac{1}{1+x^2}\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^{7n}\,dx$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{7n+1}}{7n+1}$$

$$= C + x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \ldots$$

- This series converges for $|-x^7| < 1$, that is, for $|x| < 1$.

 Example 8 b

In applying the Fundamental Theorem of Calculus (FTC), it doesn’t matter which antiderivative we use.

So, let's use the antiderivative from part (a) with $C = 0$:

$$\int_0^{0.5} \frac{1}{1+x^7}\,dx = \left[x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \ldots \right]_0^{1/2}$$

$$= \frac{1}{2} - \frac{1}{8 \cdot 2^8} + \frac{1}{15 \cdot 2^{15}} - \frac{1}{22 \cdot 2^{22}}$$

$$+ \ldots + \frac{(-1)^n}{(7n+1)2^{7n+1}} + \ldots$$

This infinite series is the exact value of the definite integral.

However, since it is an alternating series, we can approximate the sum using the Alternating Series Estimation Theorem.

If we stop adding after the term with n = 3, the error is smaller than the term with n = 4:

$$\frac{1}{29 \cdot 2^{29}} \approx 6.4 \times 10^{-11}$$

So, we have:

$$\int_0^{0.5} \frac{1}{1+x^7}\,dx$$

$$\approx \frac{1}{2} - \frac{1}{8\cdot 2^8} + \frac{1}{15\cdot 2^{15}} - \frac{1}{22\cdot 2^{22}}$$

$$\approx 0.49951374$$

This example demonstrates one way in which power series representations are useful.

- Integrating $1/(1 + x^7)$ by hand is incredibly difficult.
- Different computer algebra systems (CAS) return different forms of the answer, but they are all extremely complicated.

TERM–BY–TERM DIFFN. & INTGN.

The infinite series answer that we obtain in Example 8 a is actually much easier to deal with than the finite answer provided by a CAS.