11

INFINITE SEQUENCES AND SERIES

11.8
Power Series

In this section, we will learn about:

Power series and testing it

for convergence or divergence.

POWER SERIES

Equation 1

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

where:

- *x* is a variable
- The c_n's are constants called the coefficients of the series.

POWER SERIES

For each fixed x, the series in Equation 1 is a series of constants that we can test for convergence or divergence.

- A power series may converge for some values of x and diverge for other values of x.

POWER SERIES

The sum of the series is a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots$$

whose domain is the set of all *x* for which the series converges.

Notice that *f* resembles a polynomial.

- The only difference is that *f* has infinitely many terms.

For instance, if we take $c_n = 1$ for all n, the power series becomes the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots + x^n + \ldots$$

which converges when $-1 < x < 1$ and diverges when $|x| \geq 1$.

- See Equation 5 in Section 11.2

Equation 2

More generally, a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$$

is called any of the following:

- A power series in (x – a)
- A power series centered at a
- A power series about a

POWER SERIES

Notice that, in writing out the term pertaining to $n = 0$ in Equations 1 and 2, we have adopted the convention that $(x - a)^0 = 1$ even when $x = a$.

Notice also that, when $x = a$, all the terms are 0 for $n \geq 1$.

- So, the power series in Equation 2 always converges when $x = a$.

Example 1

For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

- We use the Ratio Test.
- If we let a_n as usual denote the nth term of the series, then $a_n = n!x^n$.

If $x \neq 0$, we have:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,x^{n+1}}{n!\,x^n}\right|$$

$$= \lim_{n\to\infty}(n+1)|x| = \infty$$

- Notice that:

$$(n+1)! = (n+1)n(n-1)\cdot \ldots \cdot 3 \cdot 2 \cdot 1$$
$$= (n+1)n!$$

By the Ratio Test, the series diverges when $x \neq 0$.

- Thus, the given series converges only when $x = 0$.

For what values of *x* does the series

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$$

converge?

Let $a_n = (x - 3)^n/n$.

Then,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n}\right|$$

$$= \frac{1}{1+\frac{1}{n}}|x-3| \to |x-3| \qquad \text{as } n \to \infty$$

By the Ratio Test, the given series is:

- Absolutely convergent, and therefore convergent, when $|x - 3| < 1$.
- Divergent when $|x - 3| > 1$.

Now,

$$|x-3|<1 \Leftrightarrow -1<x-3<1 \Leftrightarrow 2<x<4$$

- Thus, the series converges when $2 < x < 4$.
- It diverges when $x < 2$ or $x > 4$.

The Ratio Test gives no information when $|x - 3| = 1$.

- So, we must consider $x = 2$ and $x = 4$ separately.

If we put $x = 4$ in the series, it becomes $\Sigma\ 1/n$, the harmonic series, which is divergent.

If we put $x = 2$, the series is $\Sigma\ (-1)^n/n$, which converges by the Alternating Series Test.

- Thus, the given series converges for $2 \le x < 4$.

USE OF POWER SERIES

We will see that the main use of a power series is that it provides a way to represent some of the most important functions that arise in mathematics, physics, and chemistry.

BESSEL FUNCTION

In particular, the sum of the power series in the next example is called a Bessel function, after the German astronomer Friedrich Bessel (1784–1846).

- The function given in Exercise 35 is another example of a Bessel function.

In fact, these functions first arose when Bessel solved Kepler's equation for describing planetary motion.

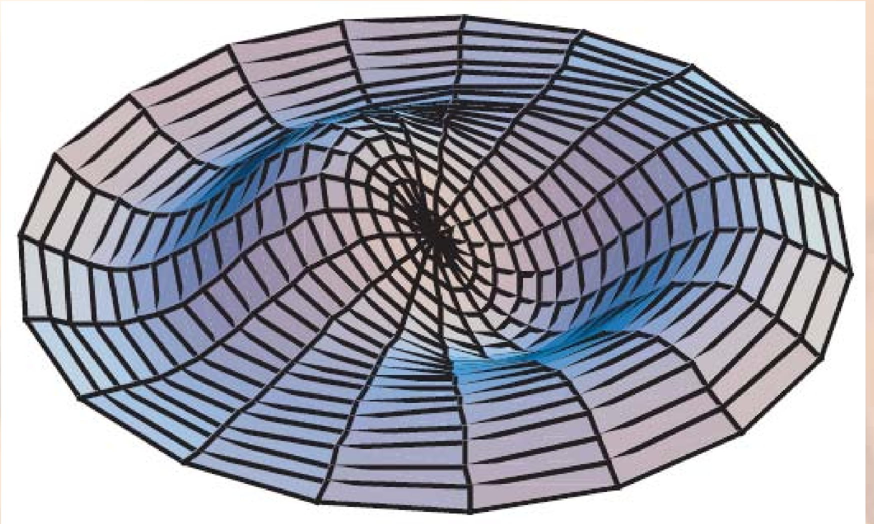
BESSEL FUNCTION

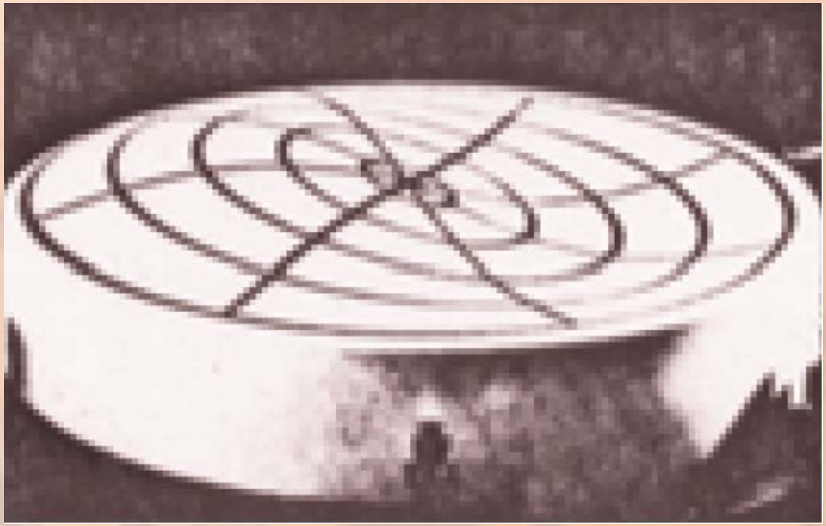
Since then, these functions have been applied in many different physical situations, such as:

- Temperature distribution in a circular plate
- Shape of a vibrating drumhead

BESSEL FUNCTION

Notice how closely the computer-generated model (which involves Bessel functions and cosine functions) matches the photograph of a vibrating rubber membrane.





Find the domain of the Bessel function of order 0 defined by:

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

Let $a_n = \dfrac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$

Then, $\left|\dfrac{a_{n+1}}{a_n}\right| = \left|\dfrac{(-1)^{n+1} x^{2(n+1)}}{2^{2(n+1)}[(n+1)!]^2} \cdot \dfrac{2^{2n}(n!)^2}{(-1)^n x^{2n}}\right|$

$$= \frac{x^{2n+2}}{2^{2n+2}(n+1)^2(n!)^2} \cdot \frac{2^{2n}(n!)^2}{x^{2n}}$$

$$= \frac{x^2}{4(n+1)^2} \to 0 < 1 \qquad \text{for all } x$$

Thus, by the Ratio Test, the given series converges for all values of x.

- In other words, the domain of the Bessel function J_0 is:

$$(-\infty, \infty) = \mathbb{R}$$

Recall that the sum of a series is equal to the limit of the sequence of partial sums.

So, when we define the Bessel function in Example 3 as the sum of a series, we mean that, for every real number *x*,

$$J_0(x) = \lim_{n \to \infty} s_n(x)$$

where

$$s_n(x) = \sum_{i=0}^{n} \frac{(-1)^i x^{2i}}{2^{2i} (i!)^2}$$

The first few partial sums are:

$$s_0(x) = 1$$

$$s_1(x) = 1 - \frac{x^2}{4}$$

$$s_2(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64}$$

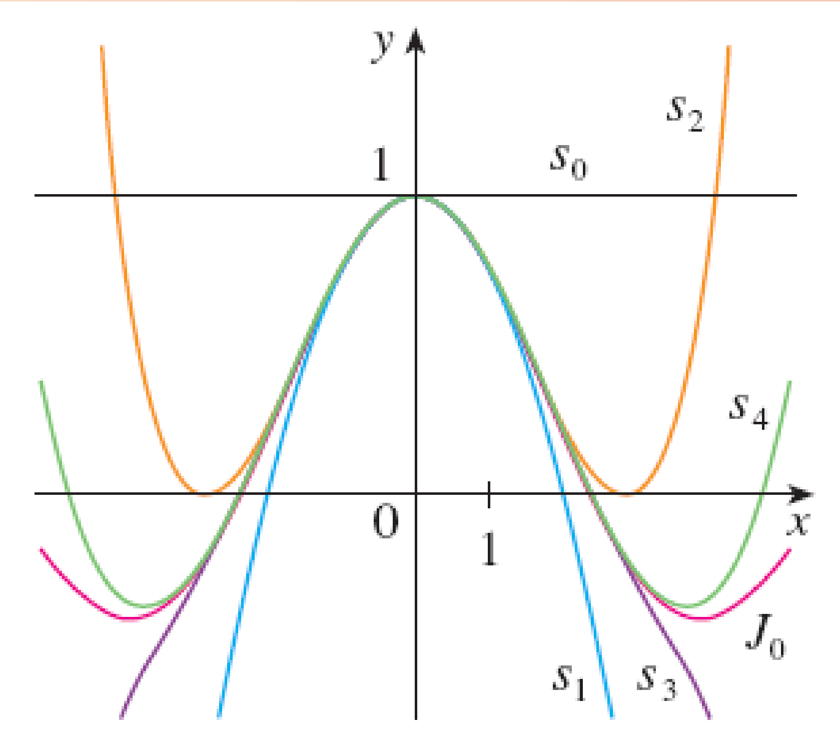
$$s_3(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304}$$

$$s_4(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \frac{x^8}{147,456}$$

BESSEL FUNCTION

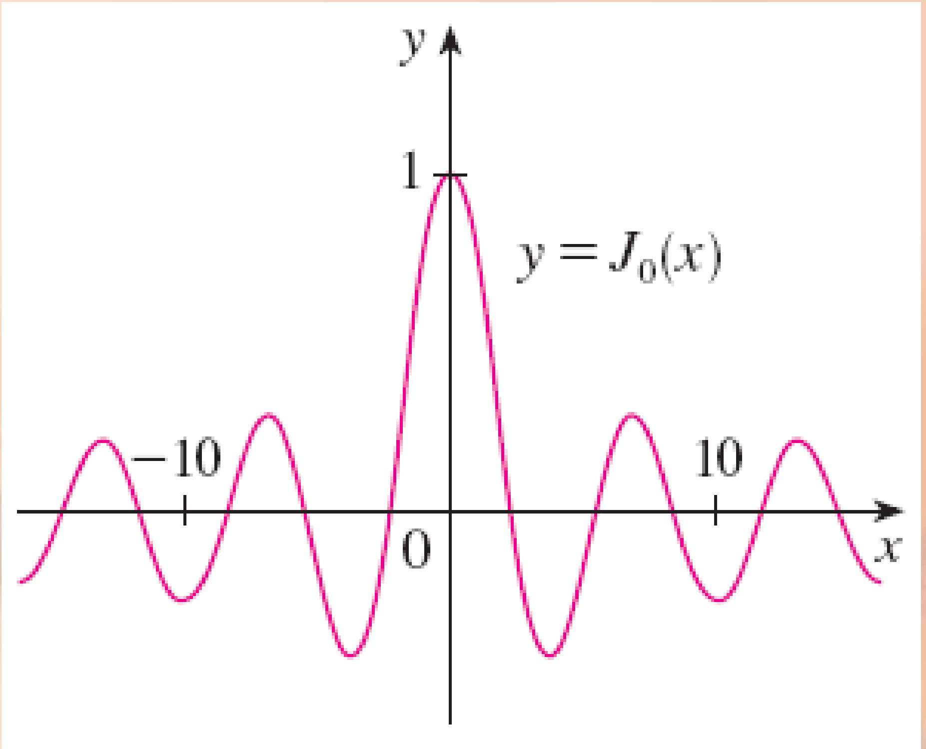
The graphs of these partial sums—which are polynomials—are displayed.

- They are all approximations to the function J_0.
- However, the approximations become better when more terms are included.



BESSEL FUNCTION

This figure shows a more complete graph of the Bessel function.



In the series we have seen so far, the set of values of *x* for which the series is convergent has always turned out to be an interval:

- A finite interval for the geometric series and the series in Example 2
- The infinite interval $(-\infty, \infty)$ in Example 3
- A collapsed interval $[0, 0] = \{0\}$ in Example 1

The following theorem, proved in Appendix F, states that this is true in general.

Theorem 3

For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$

there are only three possibilities:

I. The series converges only when $x = a$.

II. The series converges for all x.

III. There is a positive number R such that the series converges if $|x – a| < R$ and diverges if $|x – a| > R$.

RADIUS OF CONVERGENCE

The number R in case iii is called the radius of convergence of the power series.

- By convention, the radius of convergence is $R = 0$ in case i and $R = \infty$ in case ii.

INTERVAL OF CONVERGENCE

The interval of convergence of a power series is the interval that consists of all values of *x* for which the series converges.

In case i, the interval consists of just a single point *a*.

In case ii, the interval is $(-\infty, \infty)$.

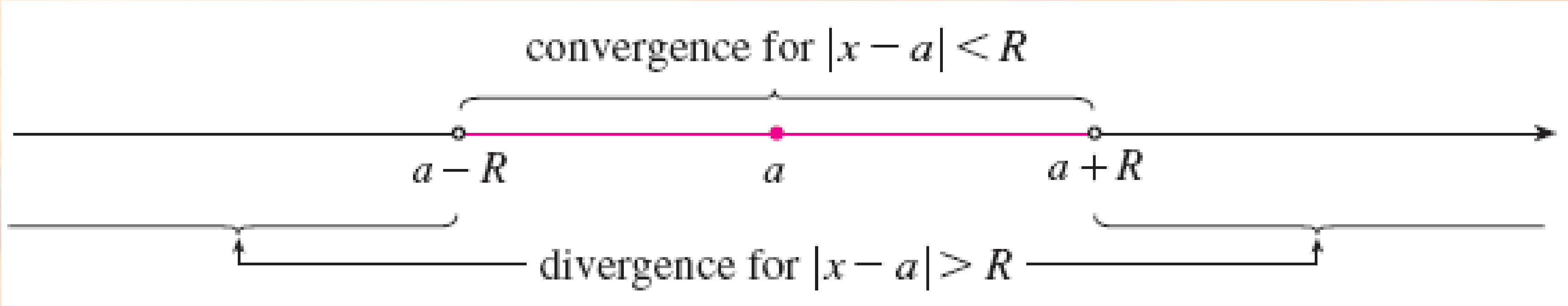
In case iii, note that the inequality $|x - a| < R$ can be rewritten as $a - R < x < a + R$.

When x is an endpoint of the interval, that is, $x = a \pm R$, anything can happen:

- The series might converge at one or both endpoints.
- It might diverge at both endpoints.

Thus, in case iii, there are four possibilities for the interval of convergence:

- $(a - R, a + R)$
- $(a - R, a + R]$
- $[a - R, a + R)$
- $[a - R, a + R]$



Here, we summarize the radius and interval of convergence for each of the examples already considered in this section.

	Series	Radius of convergence	Interval of convergence
Geometric series	$\sum_{n=0}^{\infty} x^n$	$R = 1$	$(-1, 1)$
Example 1	$\sum_{n=0}^{\infty} n!\, x^n$	$R = 0$	$\{0\}$
Example 2	$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	$R = 1$	$[2, 4)$
Example 3	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$	$R = \infty$	$(-\infty, \infty)$

In general, the Ratio Test (or sometimes the Root Test) should be used to determine the radius of convergence R.

- The Ratio and Root Tests always fail when x is an endpoint of the interval of convergence.
- So, the endpoints must be checked with some other test.

Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

Example 4

Let $a_n = (-3)^n x^n / \sqrt{n+1}$

Then,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n}\right|$$

$$= \left|-3x\sqrt{\frac{n+1}{n+2}}\right|$$

$$= 3\sqrt{\frac{1+(1/n)}{1+(2/n)}}\,|x| \to 3|x| \qquad \text{as } n \to \infty$$

By the Ratio Test, the series converges if $3|x| < 1$ and diverges if $3|x| > 1$.

- Thus, it converges if $|x| < \frac{1}{3}$ and diverges if $|x| > \frac{1}{3}$.
- This means that the radius of convergence is $R = \frac{1}{3}$.

We know the series converges in the interval $(-\frac{1}{3}, \frac{1}{3})$.

Now, however, we must test for convergence at the endpoints of this interval.

If x = -⅓, the series becomes:

$$\sum_{n=0}^{\infty}\frac{(-3)^n(-\frac{1}{3})^n}{\sqrt{n+1}}=\sum_{n=0}^{\infty}\frac{1}{\sqrt{n+1}}$$

$$=\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\dots$$

- This diverges.
- Use the Integral Test or simply observe that it is a p-series with $p = \frac{1}{2} < 1$.

If $x = \frac{1}{3}$, the series is:

$$\sum_{n=0}^{\infty} \frac{(-3)^n \left(\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

- This converges by the Alternating Series Test.

Therefore, the given series converges when $-\frac{1}{3} < x \le \frac{1}{3}$.

- Thus, the interval of convergence is $(-\frac{1}{3}, \frac{1}{3}]$.

Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$$

If $a_n = n(x + 2)^n/3^{n+1}$,

then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n(x+2)^n}\right|$$

$$= \left(1+\frac{1}{n}\right)\frac{|x+2|}{3} \to \frac{|x+2|}{3} \qquad \text{as } n \to \infty$$

Using the Ratio Test, we see that the series converges if $|x + 2|/3 < 1$ and it diverges if $|x + 2|/3 > 1$.

- So, it converges if $|x + 2| < 3$ and diverges if $|x + 2| > 3$.
- Thus, the radius of convergence is $R = 3$.

The inequality $|x + 2| < 3$ can be written as $-5 < x < 1$.

- So, we test the series at the endpoints –5 and 1.

When $x = –5$, the series is:

$$\sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \frac{1}{3}\sum_{n=0}^{\infty} (-1)^n n$$

- This diverges by the Test for Divergence.
- $(–1)^n n$ doesn’t converge to 0.

When $x = 1$, the series is:

$$\sum_{n=0}^{\infty} \frac{n(3)^n}{3^{n+1}} = \frac{1}{3}\sum_{n=0}^{\infty} n$$

- This also diverges by the Test for Divergence.

Thus, the series converges only when $-5 < x < 1$.

- So, the interval of convergence is $(-5, 1)$.