11

INFINITE SEQUENCES AND SERIES

We now have several ways of testing a series for convergence or divergence.

- The problem is to decide which test to use on which series.

In this respect, testing series is similar to integrating functions.

- Again, there are no hard and fast rules about which test to apply to a given series.
- However, you may find the following advice of some use.

11.7

Strategy for Testing Series

In this section, we will learn about:

The ways of testing a series for convergence or divergence.

STRATEGY FOR TESTING SERIES

It is not wise to apply a list of the tests in a specific order until one finally works.

- That would be a waste of time and effort.

STRATEGY FOR TESTING SERIES

Instead, as with integration, the main strategy is to classify the series according to its form.

STRATEGY 1

If the series is of the form

$$\Sigma\ 1/n^p$$

it is a *p*-series.

- We know this to be convergent if $p > 1$ and divergent if $p \leq 1$.

STRATEGY 2

If the series has the form

$$\Sigma\, ar^{n-1} \text{ or } \Sigma\, ar^{n}$$

it is a geometric series.

- This converges if $|r| < 1$ and diverges if $|r| \geq 1$
- Some preliminary algebraic manipulation may be required to bring the series into this form.

STRATEGY 3

If the series has a form that is similar to a *p*-series or a geometric series, then one of the comparison tests should be considered.

STRATEGY 3

In particular, if a_n is a rational function or an algebraic function of n (involving roots of polynomials), then the series should be compared with a p-series.

- Notice that most of the series in Exercises 11.4 have this form.

STRATEGY 3

The value of *p* should be chosen as in Section 11.4 by keeping only the highest powers of *n* in the numerator and denominator.

The comparison tests apply only to series with positive terms.

- If $\Sigma\ a_n$ has some negative terms, we can apply the Comparison Test to $\Sigma\ |a_n|$ and test for absolute convergence.

STRATEGY 4

If you can see at a glance that

$$\lim_{n \to \infty} a_n \neq 0$$

the Test for Divergence should be used.

STRATEGY 5

If the series is of the form

$$\Sigma\,(-1)^{n-1}b_n \quad \text{or} \quad \Sigma\,(-1)^{n}b_n$$

the Alternating Series Test is an obvious possibility.

STRATEGY 6

Series that involve factorials or other products (including a constant raised to the *n*th power) are often conveniently tested using the Ratio Test.

STRATEGY 6

Bear in mind that $|a_{n+1}/a_n| \to 1$ as $n \to \infty$ for all *p*-series and, therefore, all rational or algebraic functions of *n*.

- Thus, the Ratio Test should not be used for such series.

STRATEGY 7

If a_n is of the form $(b_n)^n$,

the Root Test may be useful.

STRATEGY 8

If $a_n = f(n)$, where $\int_1^\infty f(x)\,dx$ is easily evaluated, the Integral Test is effective.

- This is valid assuming the hypotheses of this test are satisfied.

In the following examples, we don’t work out all the details.

- We simply indicate which tests should be used.

Example 1

$$\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$$

- $a_n \to \frac{1}{2} \neq 0$ as $n \to \infty$.
- So, we should use the Test for Divergence.

Example 2

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$$

- a_n is an algebraic function of n.
- So, we compare the given series with a p-series.

- The comparison series for the Limit Comparison Test is $\Sigma\, b_n$, where:

$$b_n = \frac{\sqrt{n^3}}{3n^3} = \frac{n^{3/2}}{3n^3} = \frac{1}{3n^{3/2}}$$

STRATEGY FOR TESTING

Example 3

$$\sum_{n=1}^{\infty} ne^{-n^2}$$

- The integral $\int_1^\infty xe^{-x^2}\,dx$ is easily evaluated.
- So, we use the Integral Test.
- The Ratio Test also works.

STRATEGY FOR TESTING

Example 4

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+1}$$

- The series is alternating.
- So, we use the Alternating Series Test.

Example 5

$$\sum_{k=1}^{\infty} \frac{2^k}{k!}$$

- The series involves k!
- So, we use the Ratio Test.

Example 6

$$\sum_{n=1}^{\infty} \frac{1}{2+3^n}$$

- The series is closely related to the geometric series $\Sigma\ 1/3^n$.
- So, we use the Comparison Test.