11

INFINITE SEQUENCES AND SERIES

11.6
Absolute Convergence and the Ratio and Root tests

In this section, we will learn about:

Absolute convergence of a series and tests to determine it.

ABSOLUTE CONVERGENCE

Given any series $\Sigma\, a_n$, we can consider the corresponding series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \ldots$$

whose terms are the absolute values of the terms of the original series.

Definition 1

A series $\Sigma\, a_n$ is called absolutely convergent if the series of absolute values $\Sigma\, |a_n|$ is convergent.

ABSOLUTE CONVERGENCE

Notice that, if $\Sigma\ a_n$ is a series with positive terms, then $|a_n| = a_n$.

- So, in this case, absolute convergence is the same as convergence.

The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$

is absolutely convergent because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$$

is a convergent *p*-series ($p = 2$).

We know that the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

is convergent.

- See Example 1 in Section 11.5.

However, it is not absolutely convergent because the corresponding series of absolute values is:

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n-1}}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$$

- This is the harmonic series (*p*-series with *p* = 1) and is, therefore, divergent.

CONDITIONAL CONVERGENCE **Definition 2**

A series $\Sigma\, a_n$ is called conditionally convergent if it is convergent but not absolutely convergent.

ABSOLUTE CONVERGENCE

Example 2 shows that the alternating harmonic series is conditionally convergent.

- Thus, it is possible for a series to be convergent but not absolutely convergent.
- However, the next theorem shows that absolute convergence implies convergence.

Theorem 3

If a series $\Sigma\, a_n$ is absolutely convergent, then it is convergent.

Observe that the inequality

$$0 \le a_n + |a_n| \le 2|a_n|$$

is true because $|a_n|$ is either a_n or $-a_n$.

Theorem 3—Proof

If $\Sigma\, a_n$ is absolutely convergent, then $\Sigma\, |a_n|$ is convergent.

So, $\Sigma\, 2|a_n|$ is convergent.

- Thus, by the Comparison Test, $\Sigma\, (a_n + |a_n|)$ is convergent.

Then,

$$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$$

is the difference of two convergent series and is, therefore, convergent.

Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \dots$$

is convergent or divergent.

ABSOLUTE CONVERGENCE **Example 3**

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \ldots$$

The series has both positive and negative terms, but it is not alternating.

- The first term is positive.
- The next three are negative.
- The following three are positive—the signs change irregularly.

We can apply the Comparison Test to the series of absolute values:

$$\sum_{n=1}^{\infty}\left|\frac{\cos n}{n^2}\right| = \sum_{n=1}^{\infty}\frac{|\cos n|}{n^2}$$

Since $|\cos n| \leq 1$ for all n, we have:

$$\frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$$

- We know that $\Sigma\, 1/n^2$ is convergent (p-series with $p = 2$).
- Hence, $\Sigma\, (\cos n)/n^2$ is convergent by the Comparison Test.

Thus, the given series Σ (cos n)/n^2 is absolutely convergent and, therefore, convergent by Theorem 3.

ABSOLUTE CONVERGENCE

The following test is very useful in determining whether a given series is absolutely convergent.

THE RATIO TEST

Case i

If $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$$

then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

If

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1 \qquad \text{or} \qquad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$$

then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

THE RATIO TEST

Case iii

If

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$$

the Ratio Test is inconclusive.

- That is, no conclusion can be drawn about the convergence or divergence of $\Sigma\, a_n$.

The idea is to compare the given series with a convergent geometric series.

- Since $L < 1$, we can choose a number r such that $L < r < 1$.

Since
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \qquad \text{and} \qquad L < r$$

the ratio $|a_{n+1}/a_n|$ will eventually be less than *r*.

- That is, there exists an integer *N* such that:

$$\left|\frac{a_{n+1}}{a_n}\right| < r \qquad \text{whenever } n \geq N$$

Equivalently,

$$|a_{n+1}| < |a_n|r \qquad \text{whenever } n \geq N$$

Putting n successively equal to N, $N + 1$, $N + 2$, . . . in Equation 4, we obtain:

$$|a_{N+1}| < |a_N|r$$

$$|a_{N+2}| < |a_{N+1}|r < |a_N|r^2$$

$$|a_{N+3}| < |a_{N+2}| < |a_N|r^3$$

i-Proof (Inequality 5)

In general,

$$|a_{N+k}| < |a_N| r^k \qquad \text{for all } k \geq 1$$

Now, the series

$$\sum_{k=1}^{\infty} |a_N| r^k = |a_N| r + |a_N| r^2 + |a_N| r^3 + \ldots$$

is convergent because it is a geometric series with $0 < r < 1$.

Thus, the inequality 5, together with the Comparison Test, shows that the series

$$\sum_{n=N+1}^{\infty} |a_n| = \sum_{k=1}^{\infty} |a_{N+k}| = |a_{N+1}| + |a_{N+2}| + |a_{N+3}| + \ldots$$

is also convergent.

Case i—Proof

It follows that the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Recall that a finite number of terms doesn't affect convergence.

- Therefore, $\Sigma\, a_n$ is absolutely convergent.

If $|a_{n+1}/a_n| \to L > 1$ or $|a_{n+1}/a_n| \to \infty$

then the ratio $|a_{n+1}/a_n|$ will eventually be greater than 1.

- That is, there exists an integer N such that:

$$\left|\frac{a_{n+1}}{a_n}\right| > 1 \quad \text{whenever } n \geq N$$

This means that $|a_{n+1}| > |a_n|$ whenever $n \geq N$, and so

$$\lim_{n\to\infty} a_n \neq 0$$

- Therefore, Σa_n diverges by the Test for Divergence.

Part iii of the Ratio Test says that,

if

$$\lim_{n\to\infty} \left| a_{n+1} / a_n \right| = 1$$

the test gives no information.

For instance, for the convergent series Σ 1/n^2, we have:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\dfrac{1}{(n+1)^2}}{\dfrac{1}{n^2}} = \frac{n^2}{(n+1)^2}$$

$$= \frac{1}{\left(1+\dfrac{1}{n}\right)^2} \to 1 \qquad \text{as } n \to \infty$$

For the divergent series Σ 1/*n*,

we have:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1}$$

$$= \frac{1}{1+\frac{1}{n}} \to 1 \qquad \text{as } n \to \infty$$

NOTE

Therefore, if $\lim_{n\to\infty} \left| a_{n+1} / a_n \right| = 1,$

the series Σ a_n might converge or it might diverge.

- In this case, the Ratio Test fails.
- We must use some other test.

Test the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$

for absolute convergence.

- We use the Ratio Test with $a_n = (-1)^n n^3 / 3^n$, as follows.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\dfrac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\dfrac{(-1)^n n^3}{3^n}}\right| = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}$$

$$= \frac{1}{3}\left(\frac{n+1}{n}\right)^3$$

$$= \frac{1}{3}\left(1+\frac{1}{n}\right)^3 \to \frac{1}{3} < 1$$

Thus, by the Ratio Test, the given series is absolutely convergent and, therefore, convergent.

Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{n^n}{n!}$$

- Since the terms $a_n = n^n/n!$ are positive, we don't need the absolute value signs.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \frac{(n+1)(n+1)^n}{(n+1)n!} \cdot \frac{n!}{n^n}$$

$$= \left(\frac{n+1}{n}\right)^n$$

$$= \left(1+\frac{1}{n}\right)^n \to e \quad \text{as } n \to \infty$$

- See Equation 6 in Section 3.6
- Since $e > 1$, the series is divergent by the Ratio Test.

NOTE

Although the Ratio Test works in Example 5, an easier method is to use the Test for Divergence.

- Since $a_n = \dfrac{n^n}{n!} = \dfrac{n \cdot n \cdot n \cdot \cdots \cdot n}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n} \geq n$

  it follows that a_n does not approach 0 as $n \to \infty$.

- Thus, the series is divergent by the Test for Divergence.

ABSOLUTE CONVERGENCE

The following test is convenient to apply when *n*th powers occur.

- Its proof is similar to the proof of the Ratio Test and is left as Exercise 37.

If

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$$

then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

If

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1 \quad \text{or} \quad \lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$$

then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

If

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$$

the Root Test is inconclusive.

ROOT TEST

If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then part iii of the Root Test says that the test gives no information.

- The series $\Sigma\, a_n$ could converge or diverge.

ROOT TEST VS. RATIO TEST

If $L = 1$ in the Ratio Test, don’t try the Root Test—because L will again be 1.

If $L = 1$ in the Root Test, don’t try the Ratio Test—because it will fail too.

Test the convergence of the series

$$\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+2}\right)^n$$

$$a_n=\left(\frac{2n+3}{3n+2}\right)^n$$

$$\sqrt[n]{|a_n|}=\frac{2n+3}{3n+2}=\frac{2+\frac{3}{n}}{3+\frac{2}{n}}\to\frac{2}{3}<1$$

- Thus, the series converges by the Root Test.

REARRANGEMENTS

The question of whether a given convergent series is absolutely convergent or conditionally convergent has a bearing on the question of whether infinite sums behave like finite sums.

REARRANGEMENTS

If we rearrange the order of the terms in a finite sum, then of course the value of the sum remains unchanged.

- However, this is not always the case for an infinite series.

REARRANGEMENT

By a rearrangement of an infinite series $\Sigma\, a_n$, we mean a series obtained by simply changing the order of the terms.

- For instance, a rearrangement of $\Sigma\, a_n$ could start as follows:

$$a_1 + a_2 + a_5 + a_3 + a_4 + a_{15} + a_6 + a_7 + a_{20} + \ldots$$

REARRANGEMENTS

It turns out that, if $\Sigma\, a_n$ is an absolutely convergent series with sum s, then any rearrangement of $\Sigma\, a_n$ has the same sum s.

REARRANGEMENTS

However, any conditionally convergent series can be rearranged to give a different sum.

Equation 6

To illustrate that fact, let's consider the alternating harmonic series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\ldots=\ln 2$$

- See Exercise 36 in Section 11.5

If we multiply this series by ½,

we get:

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \ldots = \frac{1}{2}\ln 2$$

Inserting zeros between the terms of this series, we have:

$$0+\frac{1}{2}+0-\frac{1}{4}+0+\frac{1}{6}+0-\frac{1}{8}+\ldots=\frac{1}{2}\ln 2$$

Now, we add the series in Equations 6 and 7 using Theorem 8 in Section 11.2:

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\ldots=\frac{3}{2}\ln 2$$

REARRANGEMENTS

Notice that the series in Equation 8 contains the same terms as in Equation 6, but rearranged so that one negative term occurs after each pair of positive terms.

However, the sums of these series are different.

- In fact, Riemann proved that, if Σa_n is a conditionally convergent series and r is any real number whatsoever, then there is a rearrangement of Σa_n that has a sum equal to r.

- A proof of this fact is outlined in Exercise 40.