11

INFINITE SEQUENCES AND SERIES

The convergence tests that we have looked at so far apply only to series with positive terms.

11.5
Alternating Series

In this section, we will learn:

How to deal with series whose terms alternate in sign.

ALTERNATING SERIES

An alternating series is a series whose terms are alternately positive and negative.

- Here are two examples:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$$

$$-\frac{1}{2}+\frac{2}{3}-\frac{3}{4}+\frac{4}{5}-\frac{5}{6}+\frac{6}{7}-\ldots=\sum_{n=1}^{\infty}(=1)^{n}\frac{n}{n+1}$$

From these examples, we see that the *n*th term of an alternating series is of the form

$$a_n = (-1)^{n-1} b_n \quad or \quad a_n = (-1)^n b_n$$

where b_n is a positive number.

- In fact, $b_n = |a_n|$

ALTERNATING SERIES

The following test states that, if the terms of an alternating series decrease toward 0 in absolute value, the series converges.

ALTERNATING SERIES TEST

If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \ldots$$

$$b_n > 0$$

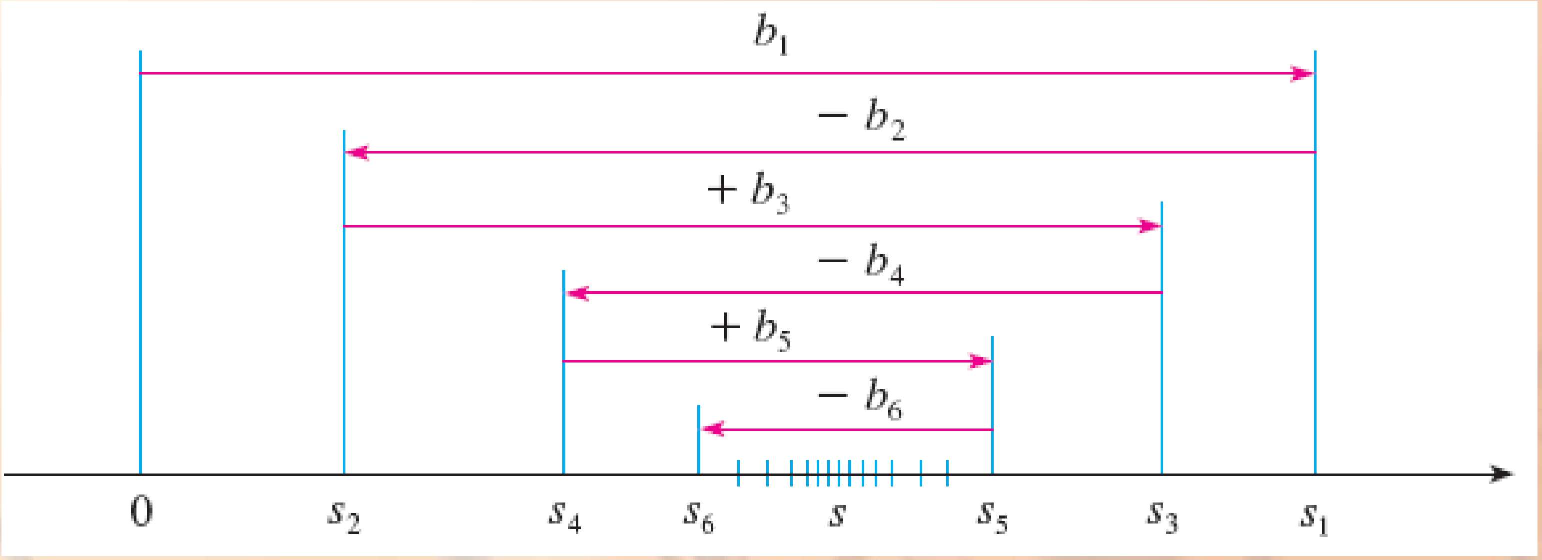
satisfies

i. $b_{n+1} \leq b_n$ for all n

ii. $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

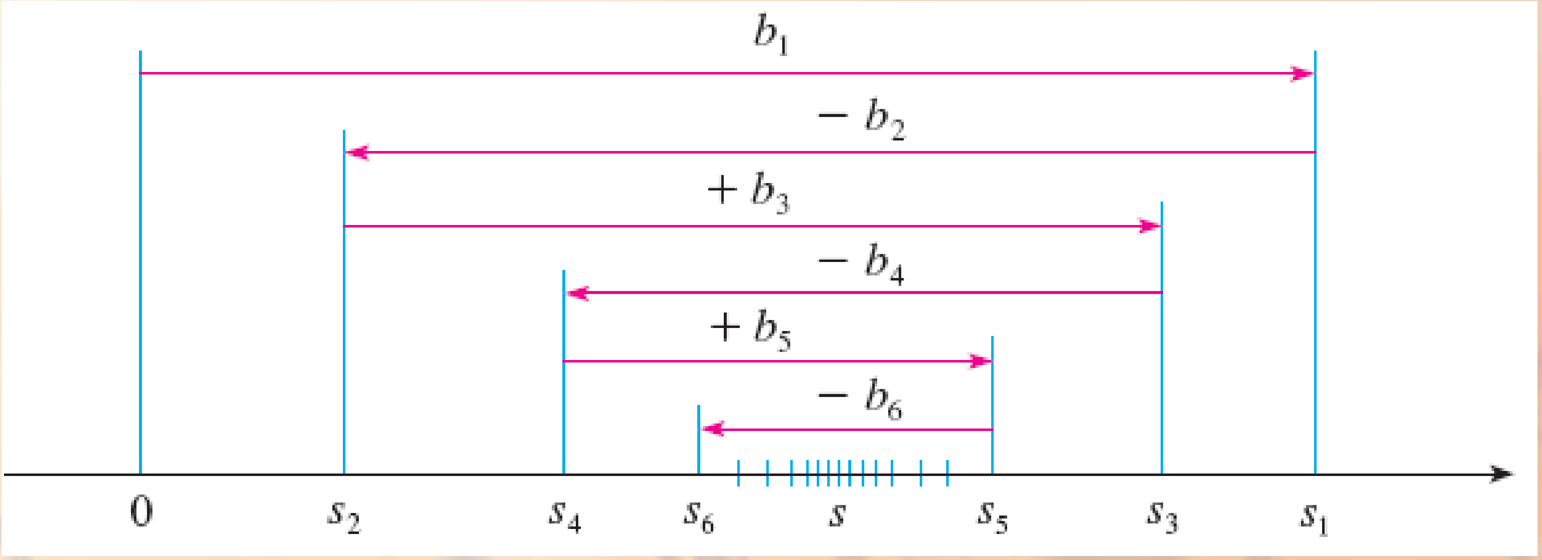
Before giving the proof, let's look at this figure—which gives a picture of the idea behind the proof.



First, we plot $s_1 = b_1$ on a number line.

To find s_2,we subtract b_2.

- So, s_2 is to the left of s_1.

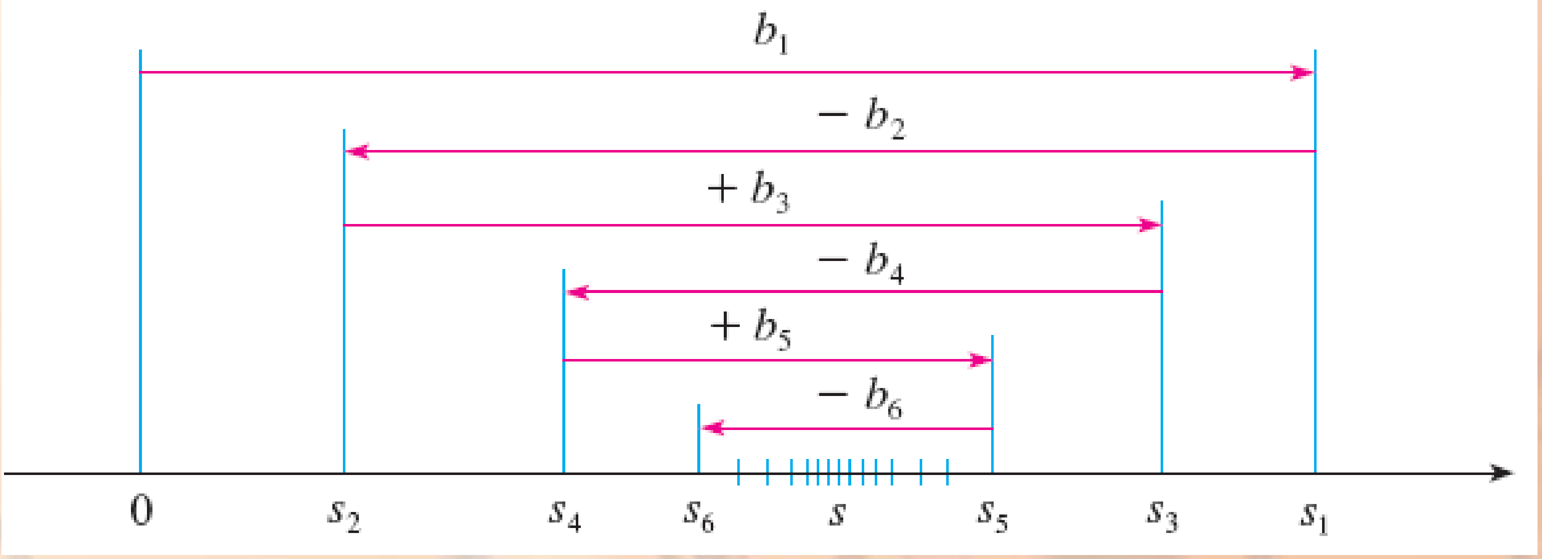


ALTERNATING SERIES

Then, to find s_3, we add b_3.

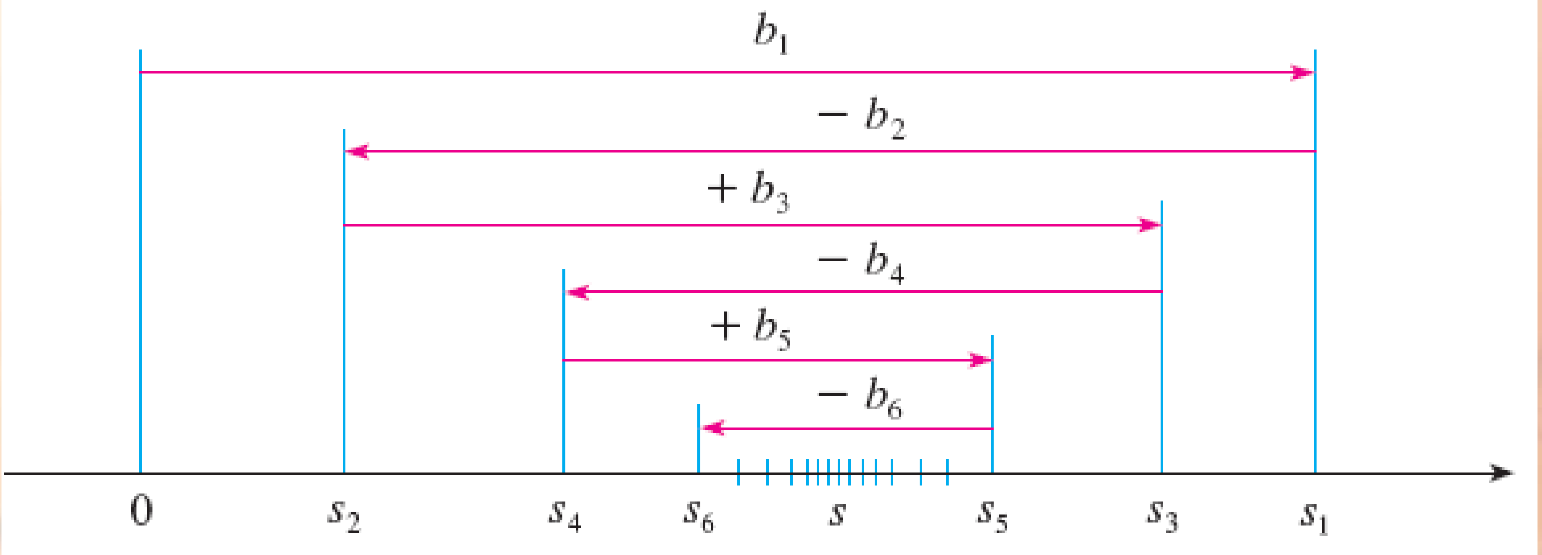
- So, s_3 is to the right of s_2.

However, since $b_3 < b_2$, s_3 is to the left of s_1.



Continuing in this manner, we see that the partial sums oscillate back and forth.

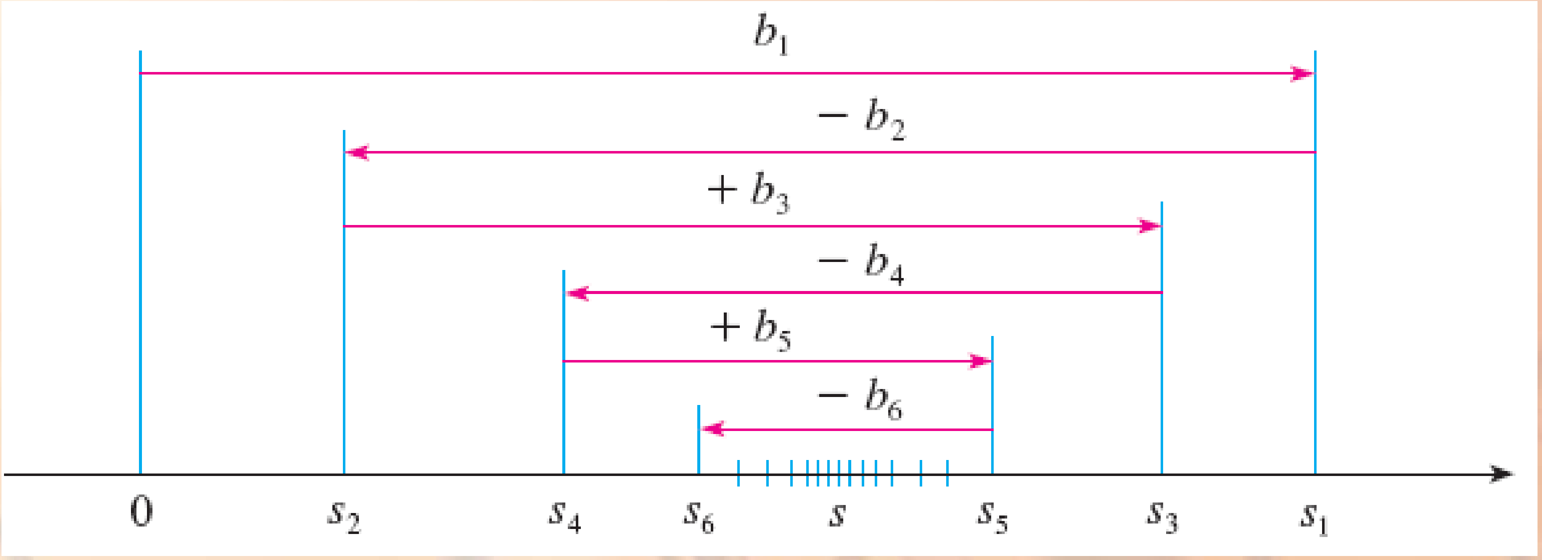
- Since $b_n \to 0$, the successive steps are becoming smaller and smaller.



ALTERNATING SERIES

The even partial sums s_2, s_4, s_6, . . . are increasing.

The odd partial sums s_1, s_3, s_5, . . . are decreasing.



Thus, it seems plausible that both are converging to some number *s*, which is the sum of the series.

- So, we consider the even and odd partial sums separately in the following proof.

ALTERNATING SERIES TEST—PROOF

First, we consider the even partial sums:

$s_2 = b_1 - b_2 \geq 0$ since $b_2 \leq b_1$

$s_4 = s_2 + (b_3 - b_4) \geq s_2$ since $b_4 \leq b_3$

ALTERNATING SERIES TEST—PROOF

In general,

$$s_{2n} = s_{2n-2} + (b_{2n-1} - b_{2n}) \geq s_{2n-2}$$

since $b_{2n} \leq b_{2n-1}$

Thus, $0 \leq s_2 \leq s_4 \leq s_6 \leq \ldots \leq s_{2n} \leq \ldots$

ALTERNATING SERIES TEST—PROOF

However, we can also write:

$$s_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \ldots - (b_{2n-2} - b_{2n-1}) - b_{2n}$$

- Every term in brackets is positive.
- So, $s_{2n} \le b_1$ for all n.

ALTERNATING SERIES TEST—PROOF

Thus, the sequence $\{s_{2n}\}$ of even partial sums is increasing and bounded above.

- Therefore, it is convergent by the Monotonic Sequence Theorem.

ALTERNATING SERIES TEST—PROOF

Let's call its limit *s*, that is,

$$\lim_{n\to\infty} s_{2n} = s$$

- Now, we compute the limit of the odd partial sums:

$$\begin{aligned}
\lim_{n\to\infty} s_{2n+1} &= \lim_{n\to\infty}(s_{2n} + b_{2n+1}) \\
&= \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} b_{2n+1} \\
&= s + 0 \qquad \text{(condition ii)} \\
&= s
\end{aligned}$$

ALTERNATING SERIES TEST—PROOF

As both the even and odd partial sums converge to *s*, we have $\lim_{n \to \infty} s_n = s$

- See Exercise 80(a) in Section 11.1

Thus, the series is convergent.

The alternating harmonic series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$$

satisfies

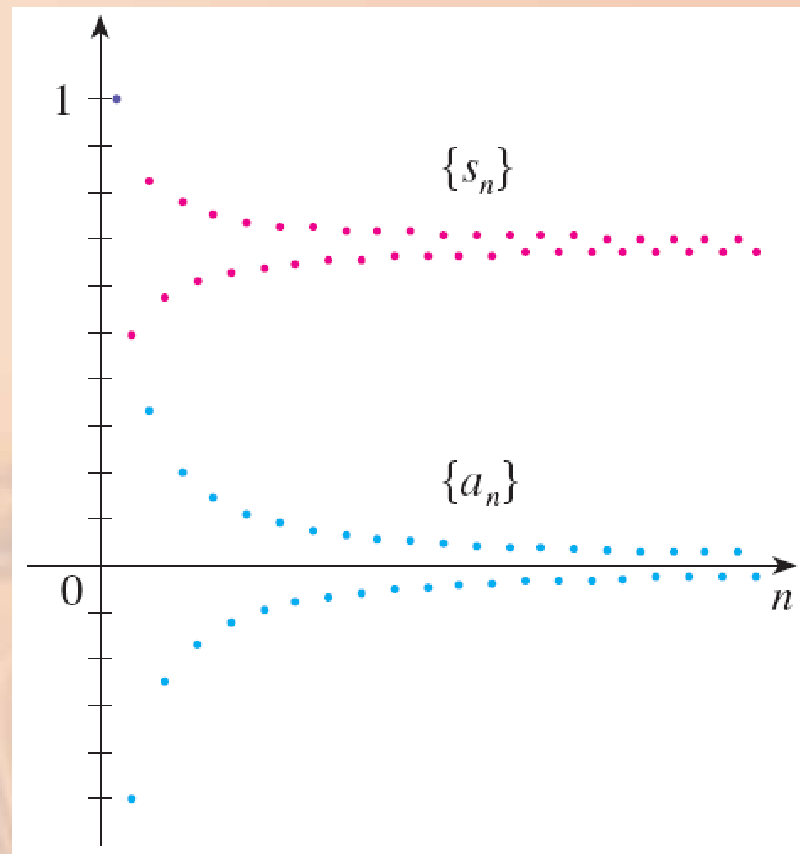
i. $b_{n+1} < b_n$ because $\frac{1}{n+1}<\frac{1}{n}$

ii. $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n}=0$

- It is convergent by the Alternating Series Test.

ALTERNATING SERIES

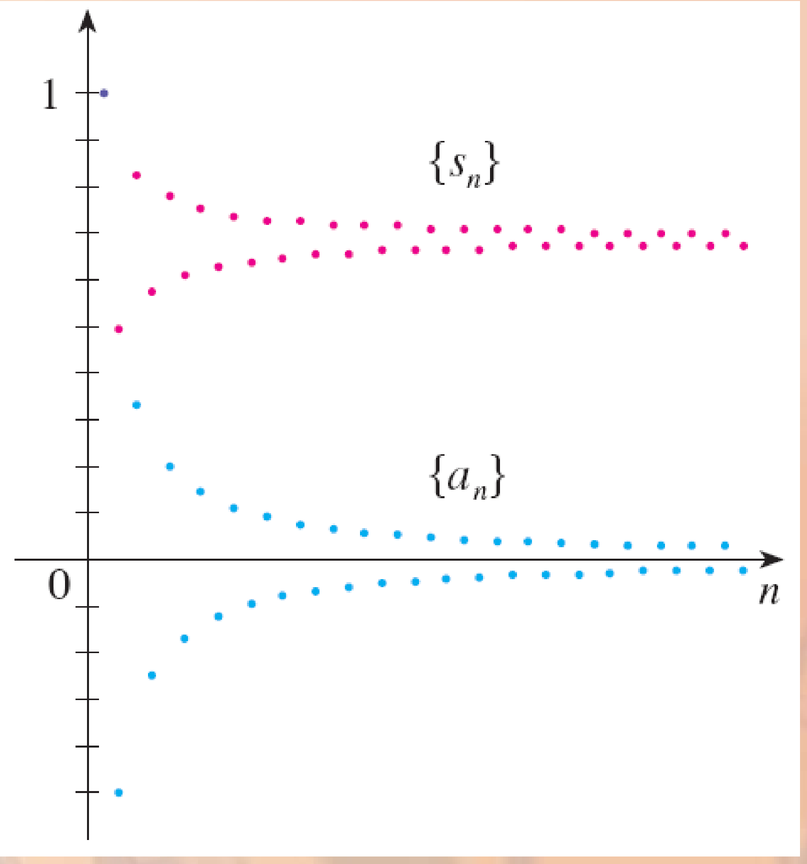
The figure illustrates Example 1 by showing the graphs of the terms $a^n = (–1)^{n – 1}/n$ and the partial sums s_n.



ALTERNATING SERIES

Notice how the values of s_n zigzag across the limiting value, which appears to be about 0.7

- In fact, it can be proved that the exact sum of the series is $\ln 2 \approx 0.693$ (Exercise 36).



The series $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ is alternating.

However,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3n}{4n-1} = \lim_{n\to\infty} \frac{3}{4-\frac{1}{n}} = \frac{3}{4}$$

- So, condition ii is not satisfied.

Instead, we look at the limit of the *n*th term of the series:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n 3n}{4n-1}$$

This limit does not exist.

- So, the series diverges by the Test for Divergence.

Test the series $$\sum_{n=1}^{\infty}(-1)^{n+1}=\frac{n^2}{n^3+1}$$

for convergence or divergence.

- The given series is alternating.
- So, we try to verify conditions i and ii of the Alternating Series Test.

Example 3

Unlike the situation in Example 1, it is not obvious that the sequence given by $b_n = n^2/(n^3 + 1)$ is decreasing.

However, if we consider the related function $f(x) = x^2/(x^3 + 1)$, we find that:

$$f'(x) = \frac{x(2 - x^3)}{(x^3 + 1)^2}$$

Since we are considering only positive x, we see that $f'(x) < 0$ if $2 - x^3 < 0$, that is, $x > \sqrt[3]{2}$.

Thus, f is decreasing on the interval $(\sqrt[3]{2}, \infty)$.

This means that $f(n + 1) < f(n)$ and, therefore, $b_{n+1} < b_n$ when $n \geq 2$.

- The inequality $b_2 < b_1$ can be verified directly.
- However, all that really matters is that the sequence $\{b_n\}$ is eventually decreasing.

Condition ii is readily verified:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^3+1} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n^3}} = 0$$

- Thus, the given series is convergent by the Alternating Series Test.

ESTIMATING SUMS

A partial sum s_n of any convergent series can be used as an approximation to the total sum s.

- However, this is not of much use unless we can estimate the accuracy of the approximation.
- The error involved in using $s \approx s_n$ is the remainder $R_n = s - s_n$.

ESTIMATING SUMS

The next theorem says that, for series that satisfy the conditions of the Alternating Series Test, the size of the error is smaller than b_{n+1}.

- This is the absolute value of the first neglected term.

ALTERNATING SERIES ESTIMATION THEOREM

If $s = \Sigma\, (-1)^{n-1} b_n$ is the sum of an alternating series that satisfies

i. $0 \le b_{n+1} \le b_n$

ii. $\lim_{n \to \infty} b_n = 0$

then $|R_n| = |s - s_n| \le b_{n+1}$

ALTERNATING SERIES ESTIMATION THM.—PROOF

From the proof of the Alternating Series Test, we know that s lies between any two consecutive partial sums s_n and s_{n+1}.

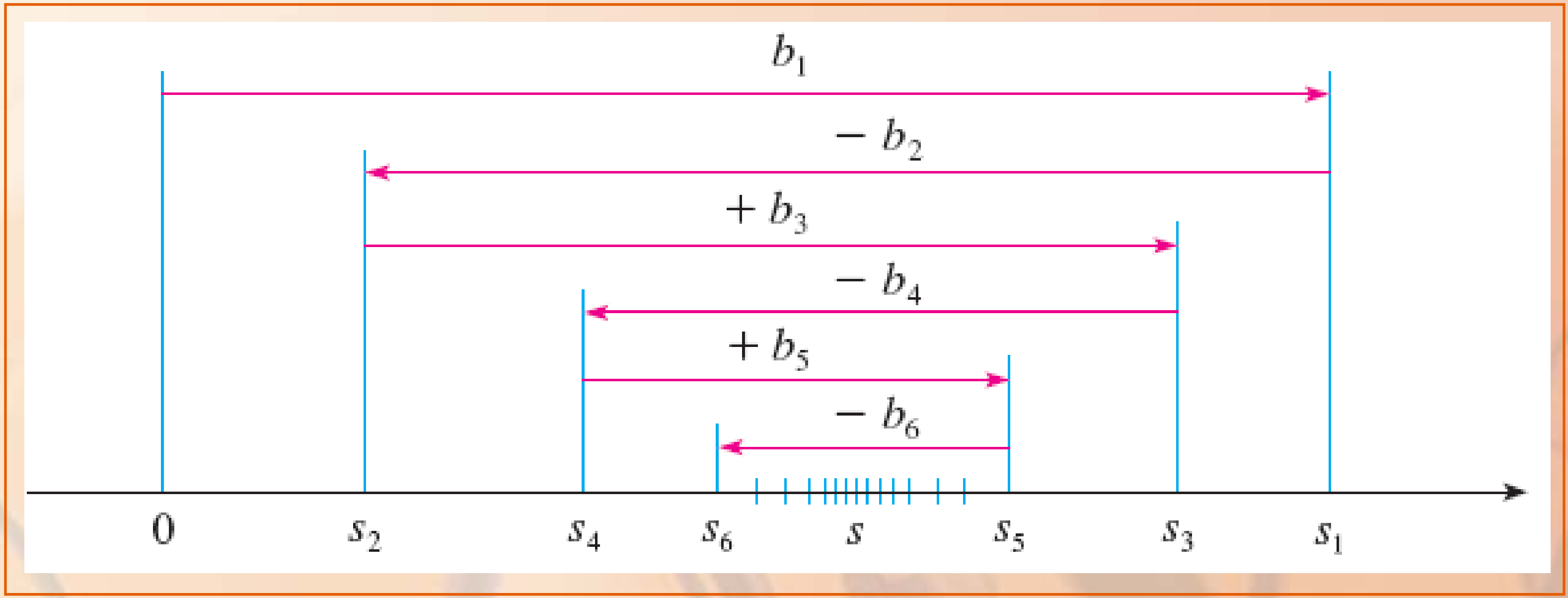
It follows that:

$$|s - s_n| \leq |s_{n+1} - s_n| = b_{n+1}$$

ALTERNATING SERIES ESTIMATION THEOREM

You can see geometrically why the theorem is true by looking at this figure.

- Notice that $s - s_4 < b_5$, $|s - s_5| < b_6$, and so on.
- Notice also that s lies between any two consecutive partial sums.



Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

correct to three decimal places.

- By definition, 0! = 1.

First, we observe that the series is convergent by the Alternating Series Test because:

i. $$\frac{1}{(n+1)!} = \frac{1}{n!(n+1)} < \frac{1}{n!}$$

ii. $$0 < \frac{1}{n!} < \frac{1}{n} \to 0 \quad \text{so} \; \frac{1}{n!} \to 0 \quad \text{as} \;\; n \to \infty$$

To get a feel for how many terms we need to use in our approximation, let's write out the first few terms of the series:

$$s = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \dots$$

$$= 1 - 1 + \tfrac{1}{2} - \tfrac{1}{6} + \tfrac{1}{24} - \tfrac{1}{120} + \tfrac{1}{720} - \tfrac{1}{5040} + \dots$$

$$s = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \ldots$$

$$= 1 - 1 + \tfrac{1}{2} - \tfrac{1}{6} + \tfrac{1}{24} - \tfrac{1}{120} + \tfrac{1}{720} - \tfrac{1}{5040} + \ldots$$

- Notice that $b_7 = \tfrac{1}{5040} < \tfrac{1}{5000} = 0.0002$

  and $s_6 = 1 - 1 + \tfrac{1}{2} - \tfrac{1}{6} + \tfrac{1}{24} - \tfrac{1}{120} + \tfrac{1}{720}$

  ≈ 0.368056

By the Alternating Series Estimation Theorem, we know that:

$$| s - s_6 | \leq b_7 < 0.0002$$

- This error of less than 0.0002 does not affect the third decimal place.
- So, we have $s \approx 0.368$ correct to three decimal places.

NOTE

The rule that the error (in using s_n to approximate s) is smaller than the first neglected term is, in general, valid only for alternating series that satisfy the conditions of the Alternating Series Estimation Theorem.

- The rule does not apply to other types of series.