11

INFINITE SEQUENCES AND SERIES

11.4
The Comparison Tests

In this section, we will learn:

How to find the value of a series

by comparing it with a known series.

COMPARISON TESTS

In the comparison tests, the idea is to compare a given series with one that is known to be convergent or divergent.

Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$$

- This reminds us of the series $\sum_{n=1}^{\infty} 1/2^n$.
- The latter is a geometric series with a = ½ and r = ½ and is therefore convergent.

COMPARISON TESTS

As the series is similar to a convergent series, we have the feeling that it too must be convergent.

- Indeed, it is.

COMPARISON TESTS

The inequality

$$\frac{1}{2^n + 1} < \frac{1}{2^n}$$

shows that our given series has smaller terms than those of the geometric series.

- Hence, all its partial sums are also smaller than 1 (the sum of the geometric series).

Thus,

- Its partial sums form a bounded increasing sequence, which is convergent.

- It also follows that the sum of the series is less than the sum of the geometric series:

$$\sum_{n=1}^{\infty} \frac{1}{2^n + 1} < 1$$

COMPARISON TESTS

Similar reasoning can be used to prove the following test—which applies only to series whose terms are positive.

COMPARISON TESTS

The first part says that, if we have a series whose terms are smaller than those of a known convergent series, then our series is also convergent.

COMPARISON TESTS

The second part says that, if we start with a series whose terms are larger than those of a known divergent series, then it too is divergent.

THE COMPARISON TEST

Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms.

i. If $\Sigma\, b_n$ is convergent and $a_n \leq b_n$ for all n, then $\Sigma\, a_n$ is also convergent.

ii. If $\Sigma\, b_n$ is divergent and $a_n \geq b_n$ for all n, then $\Sigma\, a_n$ is also divergent.

THE COMPARISON TEST—PROOF **Part i**

Let

$$s_n = \sum_{i=1}^{n} a_i \qquad t_n = \sum_{i=1}^{n} b_i \qquad t = \sum_{n=1}^{\infty} b_n$$

- Since both series have positive terms, the sequences $\{s_n\}$ and $\{t_n\}$ are increasing ($s_{n+1} = s_n + a_{n+1} \geq s_n$).
- Also, $t_n \rightarrow t$; so $t_n \leq t$ for all n.

THE COMPARISON TEST—PROOF **Part i**

Since $a_i \leq b_i$, we have $s_n \leq t_n$.

Hence, $s_n \leq t$ for all n.

- This means that $\{s_n\}$ is increasing and bounded above.
- So, it converges by the Monotonic Sequence Theorem.
- Thus, $\Sigma\, a_n$ converges.

If $\Sigma\, b_n$ is divergent, then $t_n \to \infty$ (since $\{t_n\}$ is increasing).

- However, $a_i \geq b_i$; so $s_n \geq t_n$.
- Thus, $s_n \to \infty$; so $\Sigma\, a_n$ diverges.

SEQUENCE VS. SERIES

It is important to keep in mind the distinction between a sequence and a series.

- A sequence is a list of numbers.
- A series is a sum.

SEQUENCE VS. SERIES

With every series $\Sigma\, a_n$, there are associated two sequences:

1. The sequence $\{a_n\}$ of terms
2. The sequence $\{s_n\}$ of partial sums

COMPARISON TEST

In using the Comparison Test, we must, of course, have some known series $\Sigma\, b_n$ for the purpose of comparison.

COMPARISON TEST

Most of the time, we use one of these:

- A *p*-series
  [$\Sigma\ 1/n^p$ converges if $p > 1$ and diverges if $p \leq 1$]
- A geometric series
  [$\Sigma\ ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1$]

Determine whether the given series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$$

For large n, the dominant term in the denominator is $2n^2$.

- So, we compare the given series with the series $\Sigma\ 5/(2n^2)$.

Observe that

$$\frac{5}{2n^2+4n+3} < \frac{5}{2n^2}$$

since the left side has a bigger denominator.

- In the notation of the Comparison Test, a_n is the left side and b_n is the right side.

We know that

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent because it's a constant times a *p*-series with $p = 2 > 1$.

Therefore,

$$\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$$

is convergent by part i of the Comparison Test.

NOTE 1

Although the condition $a_n \leq b_n$ or $a_n \geq b_n$ in the Comparison Test is given for all n, we need verify only that it holds for $n \geq N$, where N is some fixed integer.

- This is because the convergence of a series is not affected by a finite number of terms.
- This is illustrated in the next example.

Test the given series for convergence or divergence:

$$\sum_{n=1}^{\infty} \frac{\ln n}{n}$$

This series was tested (using the Integral Test) in Example 4 in Section 11.3

- However, it is also possible to test it by comparing it with the harmonic series.

Observe that $\ln n > 1$ for $n \geq 3$.

So,

$$\frac{\ln n}{n} > \frac{1}{n} \qquad n \geq 3$$

- We know that $\Sigma\, 1/n$ is divergent (p-series with $p = 1$).
- Thus, the series is divergent by the Comparison Test.

NOTE 2

The terms of the series being tested must be smaller than those of a convergent series or larger than those of a divergent series.

- If the terms are larger than the terms of a convergent series or smaller than those of a divergent series, the Comparison Test doesn't apply.

NOTE 2

For instance, consider $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

- The inequality $\frac{1}{2^n - 1} > \frac{1}{2^n}$ is useless

  as far as the Comparison Test is concerned.

- This is because $\Sigma\, b_n = \Sigma\, (½)^n$ is convergent and $a_n > b_n$.

NOTE 2

Nonetheless, we have the feeling that $\Sigma 1/(2^n - 1)$ ought to be convergent because it is very similar to the convergent geometric series $\Sigma (½)^n$.

- In such cases, the following test can be used.

LIMIT COMPARISON TEST

Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms.

If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, either both series converge or both diverge.

LIMIT COMPARISON TEST—PROOF

Let m and M be positive numbers such that $m < c < M$.

- Since a_n/b_n is close to c for large n, there is an integer N such that

$$m < \frac{a_n}{b_n} < M \qquad \text{when } n > N$$

and so $$mb_n < a_n < Mb_n \qquad \text{when } n > N$$

LIMIT COMPARISON TEST—PROOF

If $\Sigma\ b_n$ converges, so does $\Sigma\ Mb_n$.

- Thus, $\Sigma\ a_n$ converges by part i of the Comparison Test.

LIMIT COMPARISON TEST—PROOF

If $\Sigma\, b_n$ diverges, so does $\Sigma\, mb_n$.

- Thus, $\Sigma\, a_n$ diverges by part ii of the Comparison Test.

Test the given series for convergence or divergence:

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$$

We use the Limit Comparison Test with:

$$a_n = \frac{1}{2^n - 1} \qquad b_n = \frac{1}{2^n}$$

We obtain:

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{1/\left(2^n-1\right)}{1/2^n}$$

$$=\lim_{n\to\infty}\frac{2^n}{2^n-1}$$

$$=\lim_{n\to\infty}\frac{1}{1-1/2^n}=1>0$$

This limit exists and $\Sigma\ 1/2^n$ is a convergent geometric series.

- Thus, the given series converges by the Limit Comparison Test.

Determine whether the given series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$$

The dominant part of the numerator is $2n^2$.

The dominant part of the denominator is $n^{5/2}$.

- This suggests taking:

$$a_n = \frac{2n^2 + 3n}{\sqrt{5 + n^5}} \qquad b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}}$$

We obtain:

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{2n^2+3n}{\sqrt{5+n^5}}\cdot\frac{n^{1/2}}{2}$$

$$=\lim_{n\to\infty}\frac{2n^{5/2}+3n^{3/2}}{2\sqrt{5+n^5}}$$

$$=\lim_{n\to\infty}\frac{2+\frac{3}{n}}{2\sqrt{\frac{5}{n^5}+1}}=\frac{2+0}{2\sqrt{0+1}}=1$$

$\Sigma\, b_n = 2\, \Sigma\, 1/n^{1/2}$ is divergent

(p-series with $p = ½ < 1$).

- Thus, the given series diverges by the Limit Comparison Test.

COMPARISON TESTS

Notice that, in testing many series, we find a suitable comparison series $\Sigma\, b_n$ by keeping only the highest powers in the numerator and denominator.

ESTIMATING SUMS

We have used the Comparison Test to show that a series Σ a_n converges by comparison with a series Σ b_n.

- It follows that we may be able to estimate the sum Σ a_n by comparing remainders.

ESTIMATING SUMS

As in Section 11.3, we consider the remainder

$$R_n = s - s_n = a_{n+1} + a_{n+2} + \dots$$

- For the comparison series $\Sigma\, b_n$, we consider the corresponding remainder

$$T_n = t - t_n = b_{n+1} + b_{n+2} + \dots$$

As $a_n \leq b_n$ for all n, we have $R_n \leq T_n$.

- If $\Sigma\, b_n$ is a *p*-series, we can estimate its remainder T_n as in Section 11.3
- If $\Sigma\, b_n$ is a geometric series, then T_n is the sum of a geometric series and we can sum it exactly (Exercises 35 and 36).
- In either case, we know that R_n is smaller than T_n

Use the sum of the first 100 terms to approximate the sum of the series $\Sigma\ 1/(n^3+1)$.

Estimate the error involved in this approximation.

Since

$$\frac{1}{n^3+1} < \frac{1}{n^3}$$

the given series is convergent by the Comparison Test.

The remainder T_n for the comparison series Σ $1/n^3$ was estimated in Example 5 in Section 11.3 (using the Remainder Estimate for the Integral Test).

- We found that:

$$T_n \le \int_n^\infty \frac{1}{x^3}\,dx = \frac{1}{2n^2}$$

Therefore, the remainder for the given series satisfies:

$$R_n \le T_n \le 1/2n^2$$

With n = 100, we have:

$$R_{100} \leq \frac{1}{2(100)^2} = 0.00005$$

- With a programmable calculator or a computer, we find that

$$\sum_{n=1}^{\infty} \frac{1}{n^3+1} \approx \sum_{n=1}^{100} \frac{1}{n^3+1} \approx 0.6864538$$

with error less than 0.00005