11

INFINITE SEQUENCES AND SERIES

In general, it is difficult to find the exact sum of a series.

- We were able to accomplish this for geometric series and the series ∑ $1/[n(n+1)]$.
- This is because, in each of these cases, we can find a simple formula for the *n*th partial sum s_n.
- Nevertheless, usually, it isn't easy to compute $\lim_{n\to\infty} s_n$

So, in the next few sections, we develop several tests that help us determine whether a series is convergent or divergent without explicitly finding its sum.

- In some cases, however, our methods will enable us to find good estimates of the sum.

Our first test involves improper integrals.

11.3

The Integral Test and Estimates of Sums

In this section, we will learn how to:

Find the convergence or divergence of a series and estimate its sum.

INTEGRAL TEST

We begin by investigating the series whose terms are the reciprocals of the squares of the positive integers:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots$$

- There's no simple formula for the sum s_n of the first n terms.

INTEGRAL TEST

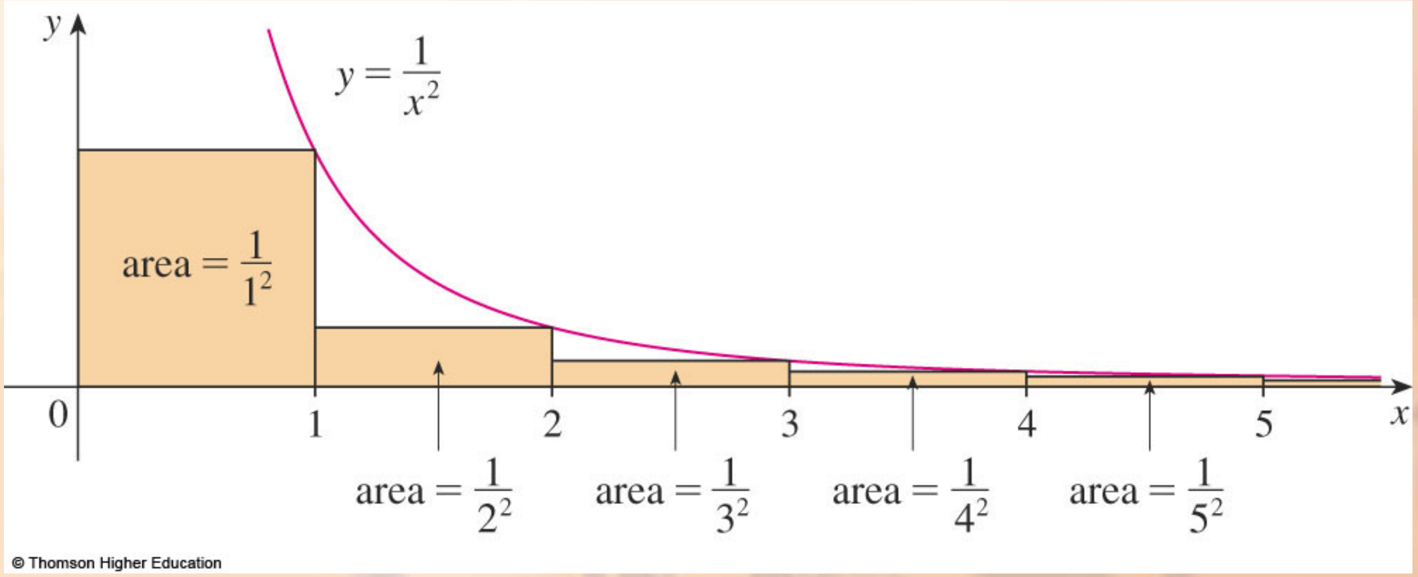
However, the computer-generated values given here suggest that the partial sums are approaching near 1.64 as $n \to \infty$.

- So, it looks as if the series is convergent.
- We can confirm this impression with a geometric argument.

n	$s_n = \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$
5	3.2317
10	5.0210
50	12.7524
100	18.5896
500	43.2834
1000	61.8010
5000	139.9681

INTEGRAL TEST

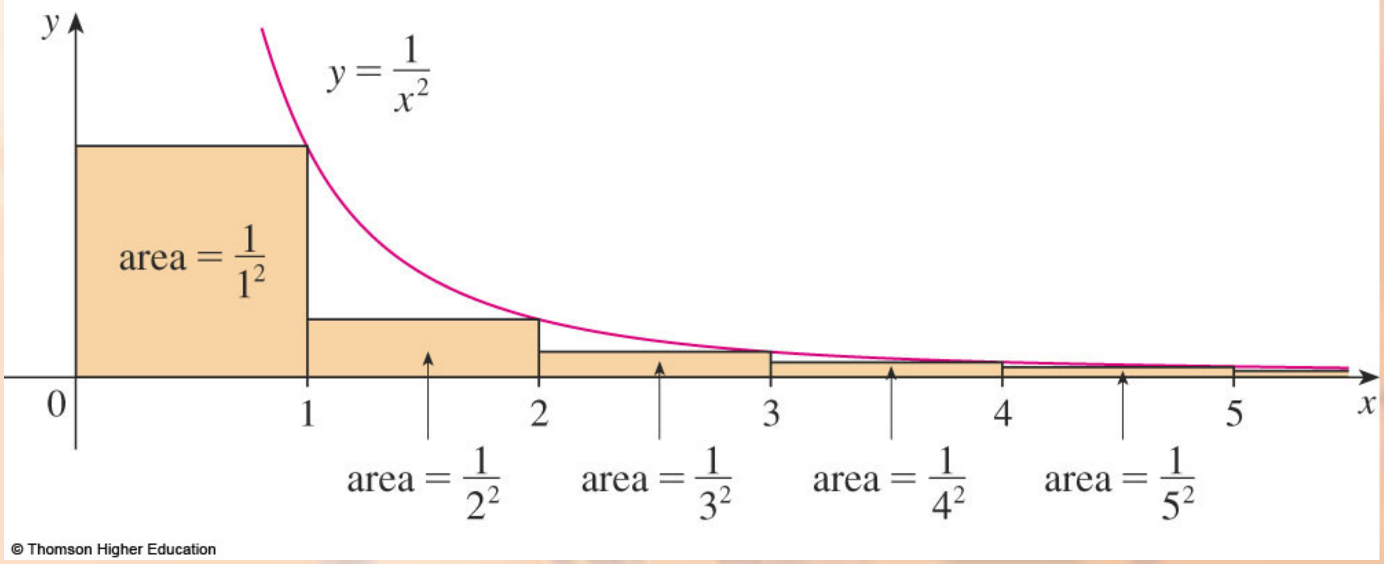
This figure shows the curve $y = 1/x^2$ and rectangles that lie below the curve.



INTEGRAL TEST

The base of each rectangle is an interval of length 1.

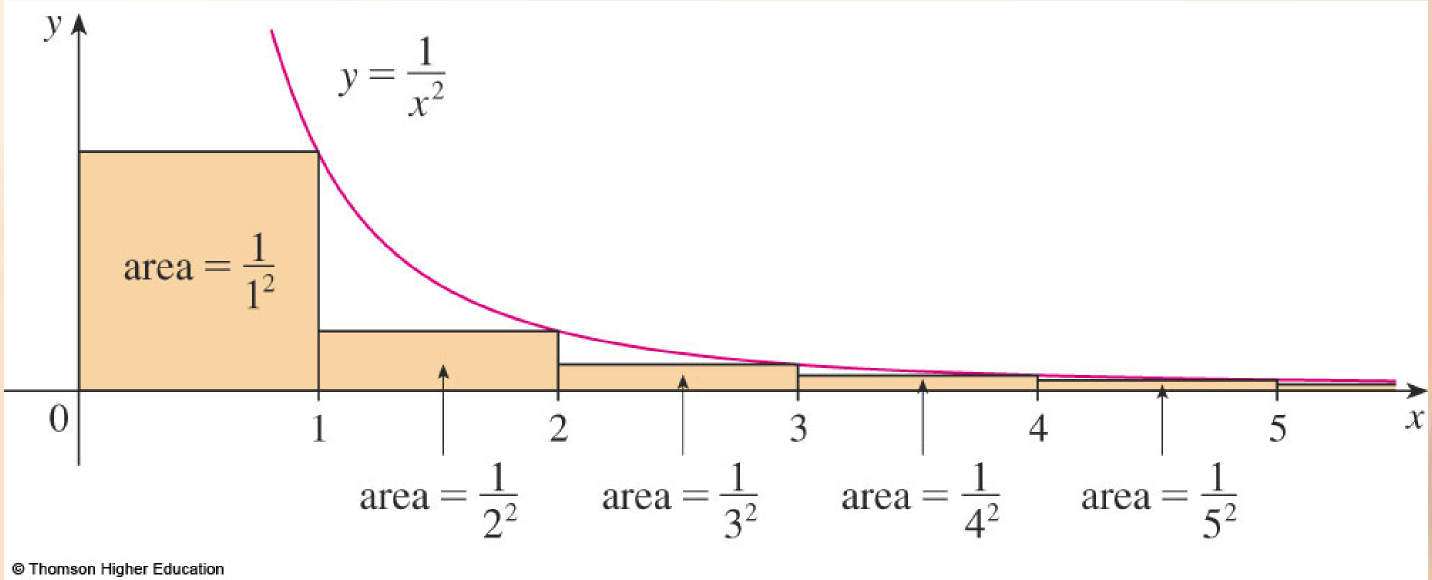
The height is equal to the value of the function $y = 1/x^2$ at the right endpoint of the interval.



INTEGRAL TEST

Thus, the sum of the areas of the rectangles is:

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\dots=\sum_{n=1}^{\infty}\frac{1}{n^2}$$



INTEGRAL TEST

If we exclude the first rectangle, the total area of the remaining rectangles is smaller than the area under the curve $y = 1/x^2$ for $x \geq 1$, which is the value of the integral $\int_1^\infty (1/x^2)\,dx$

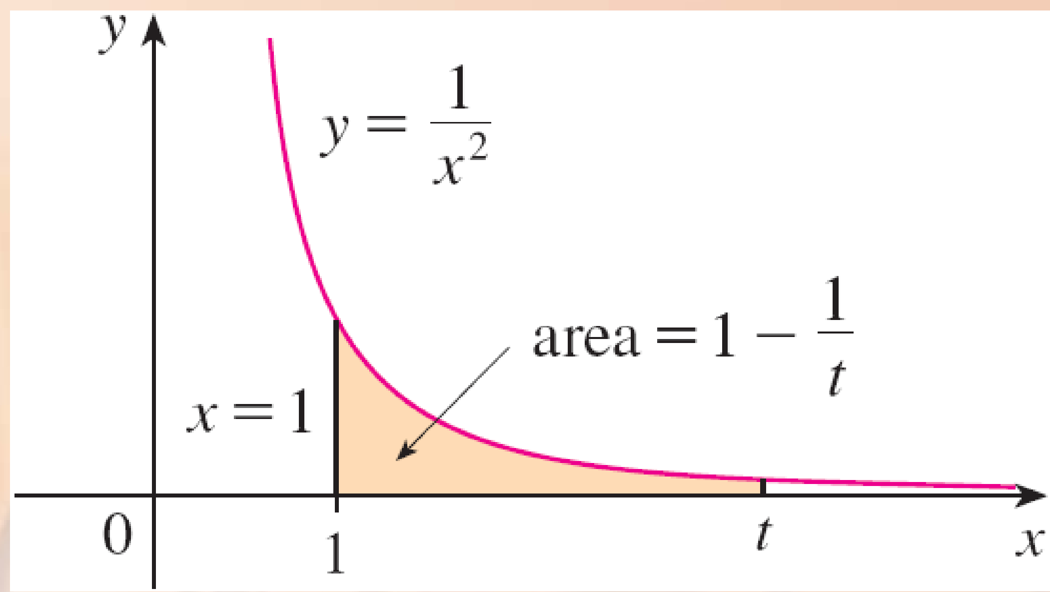
- In Section 7.8, we discovered that this improper integral is convergent and has value 1.

INTEGRAL TEST

So, the image shows that all the partial sums are less than

$$\frac{1}{1^2}+\int_1^{\infty}\frac{1}{x^2}\,dx=2$$

- Therefore, the partial sums are bounded.



INTEGRAL TEST

We also know that the partial sums are increasing (as all the terms are positive).

Thus, the partial sums converge (by the Monotonic Sequence Theorem).

- So, the series is convergent.

INTEGRAL TEST

The sum of the series (the limit of the partial sums) is also less than 2:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots < 2$$

INTEGRAL TEST

The exact sum of this series was found by the mathematician Leonhard Euler (1707–1783) to be $\pi^2/6$.

- However, the proof of this fact is quite difficult.
- See Problem 6 in the Problems Plus, following Chapter 15.

Now, let's look at this series:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \ldots$$

INTEGRAL TEST

The table of values of s_n suggests that the partial sums aren't approaching a finite number.

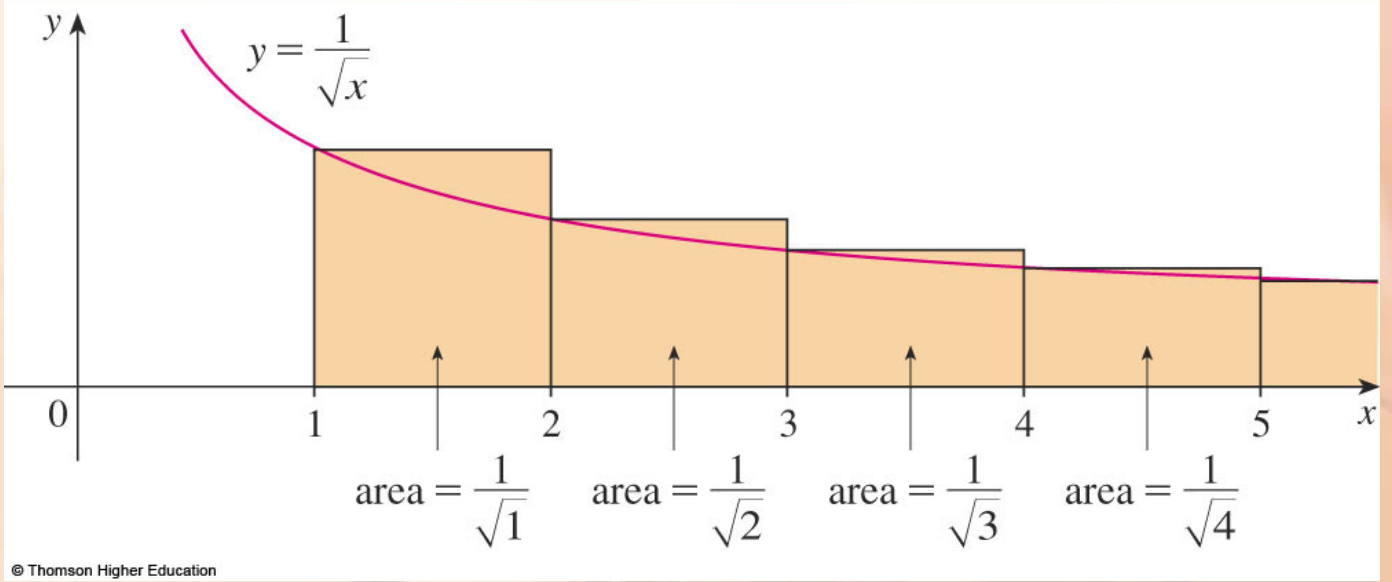
- So, we suspect that the given series may be divergent.
- Again, we use a picture for confirmation.

n	$s_n = \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$
5	3.2317
10	5.0210
50	12.7524
100	18.5896
500	43.2834
1000	61.8010
5000	139.9681

INTEGRAL TEST

The figure shows the curve $y = 1/\sqrt{x}$.

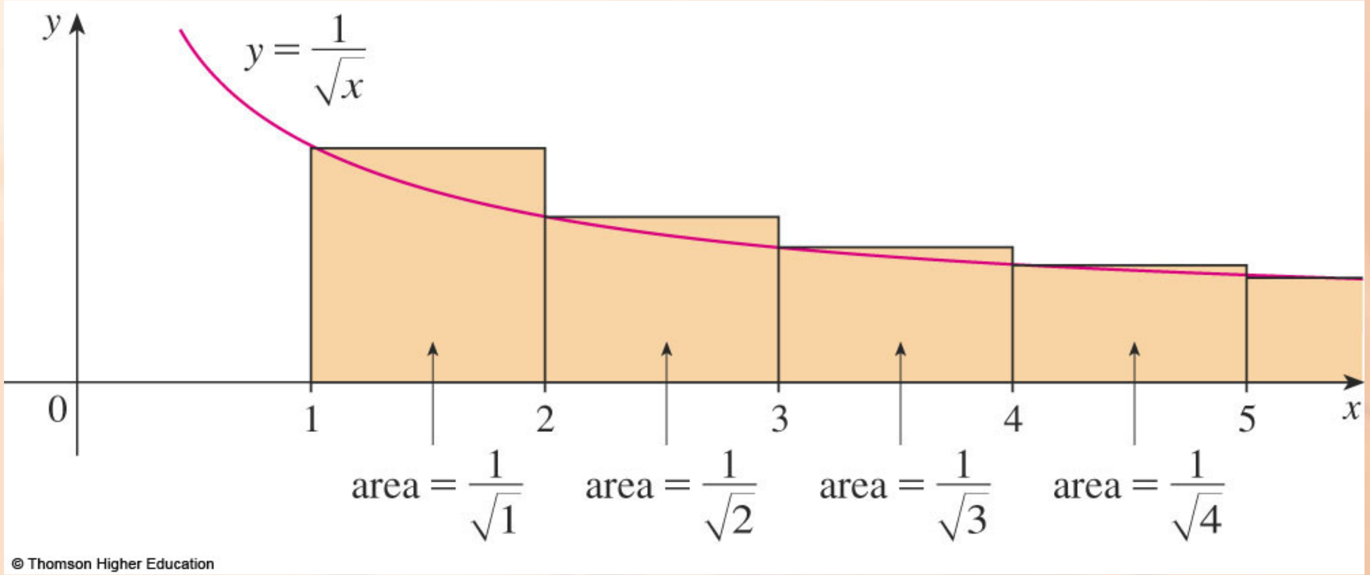
However, this time, we use rectangles whose tops lie above the curve.



INTEGRAL TEST

The base of each rectangle is an interval of length 1.

The height is equal to the value of the function $y = 1/\sqrt{x}$ at the left endpoint of the interval.

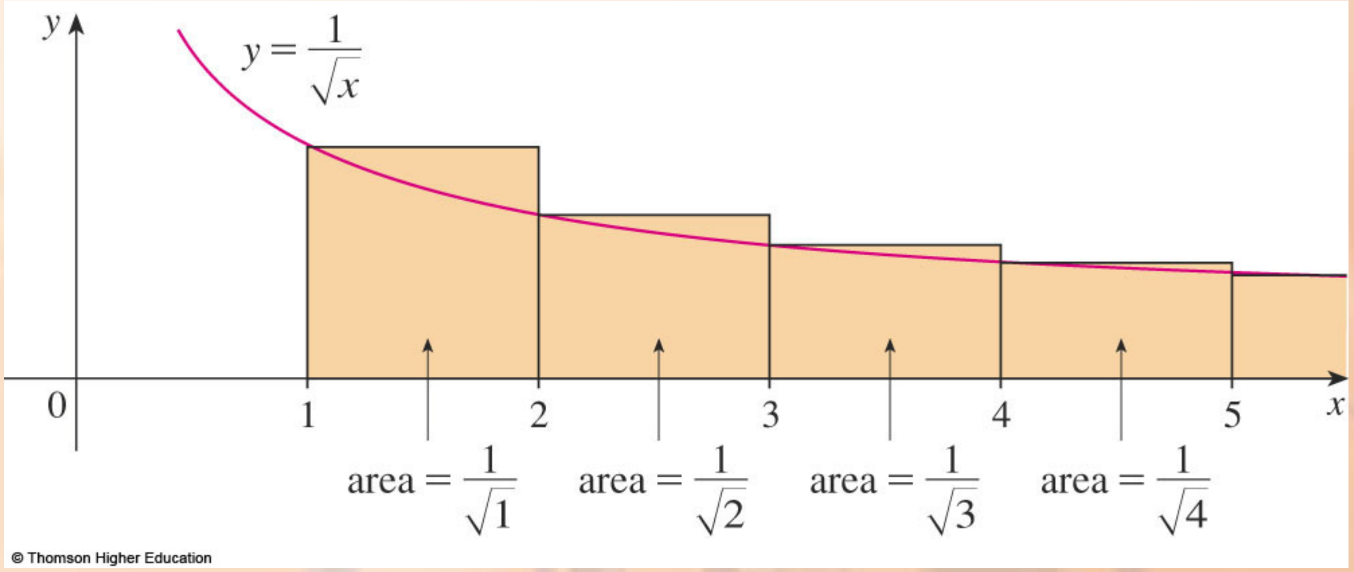


INTEGRAL TEST

So, the sum of the areas

of all the rectangles is:

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{5}}+\ldots=\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$$



INTEGRAL TEST

This total area is greater than the area under the curve $y = 1/\sqrt{x}$ for $x \geq 1$, which is equal to the integral $\int_1^\infty (1/\sqrt{x})\,dx$

- However, we know from Section 7.8 that this improper integral is divergent.
- In other words, the area under the curve is infinite.

INTEGRAL TEST

Thus, the sum of the series must be infinite.

- That is, the series is divergent.

INTEGRAL TEST

The same sort of geometric reasoning that we used for these two series can be used to prove the following test.

- The proof is given at the end of the section.

THE INTEGRAL TEST

Suppose *f* is a continuous, positive, decreasing function on [1, ∞) and let $a_n = f(n)$.

Then, the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent.

THE INTEGRAL TEST

In other words,

i. If $\int_1^\infty f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

ii. If $\int_1^\infty f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

NOTE

When we use the Integral Test, it is not necessary to start the series or the integral at $n = 1$.

- For instance, in testing the series $\sum_{n=4}^{\infty} \frac{1}{(n-3)^2}$ we use $\int_4^{\infty} \frac{1}{(x-3)^2}\, dx$

NOTE

Also, it is not necessary that *f* be always decreasing.

- What is important is that *f* be ultimately decreasing, that is, decreasing for *x* larger than some number *N*.
- Then, $\sum_{n=N}^{\infty} a_n$ is convergent.
- So, $\sum_{n=1}^{\infty} a_n$ is convergent by Note 4 of Section 11.2

Test the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence or divergence.

- The function $f(x) = 1/(x^2 + 1)$ is continuous, positive, and decreasing on $[1, \infty)$.

- So, we use the Integral Test:

$$\int_1^\infty \frac{1}{x^2+1}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2+1}\,dx$$

$$= \lim_{t\to\infty} \tan^{-1} x\Big]_1^t$$

$$= \lim_{t\to\infty}\left(\tan^{-1} t - \frac{\pi}{4}\right)$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

- So, $\int_1^\infty 1/(x^2+1)\,dx$ is a convergent integral.

- Thus, by the Integral Test,
  the series $\sum 1/(n^2 + 1)$ is convergent.

For what values of p is the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$ convergent?

If p < 0, then $\lim_{n\to\infty}(1/n^p)=\infty$

If p = 0, then $\lim_{n\to\infty}(1/n^p)=1$

- In either case, $\lim_{n\to\infty}(1/n^p)\neq 0$
- So, the given series diverges by the Test for Divergence (Section 11.2).

If $p > 0$, then the function $f(x) = 1/x^p$ is clearly continuous, positive, and decreasing on $[1, \infty)$.

In Section 7.8 (Definition 2), we found that $\int_1^\infty \frac{1}{x^p}\,dx$:

- Converges if $p > 1$
- Diverges if $p \leq 1$

It follows from the Integral Test that the series $\sum 1/n^p$ converges if $p > 1$ and diverges if $0 < p \leq 1$.

- For $p = 1$, this series is the harmonic series discussed in Example 7 in Section 11.2

INTEGRAL TEST

To use the Integral Test, we need to be able to evaluate $\int_1^\infty f(x)\,dx$

Therefore, we have to be able to find an antiderivative of *f*.

- Frequently, this is difficult or impossible.
- So, we need other tests for convergence too.

p-SERIES

The series in Example 2 is called the *p*-series.

- It is important in the rest of this chapter.
- So, we summarize the results of Example 2 for future reference—as follows.

The *p*-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is convergent if $p > 1$

and divergent if $p \leq 1$

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \ldots$$

is convergent because it is a *p*-series with $p = 3 > 1$

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/3}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \dots$$

is divergent because it is a *p*-series

with $p = \frac{1}{3} < 1$.

NOTE

We should not infer from the Integral Test that the sum of the series is equal to the value of the integral.

- In fact, $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ whereas $\int_1^{\infty} \frac{1}{x^2}\,dx = 1$
- Thus, in general, $\sum_{n=1}^{\infty} a_n \neq \int_1^{\infty} f(x)\,dx$

Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

- The function $f(x) = (\ln x)/x$ is positive and continuous for $x > 1$ because the logarithm function is continuous.
- However, it is not obvious that f is decreasing.

So, we compute its derivative:

$$f'(x) = \frac{(1/x)x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

- Thus, $f'(x) < 0$ when $\ln x > 1$, that is, $x > e$.
- It follows that f is decreasing when $x > e$.

So, we can apply the Integral Test:

$$\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\left.\frac{(\ln x)^2}{2}\right]_1^t$$

$$= \lim_{t\to\infty}\frac{(\ln t)^2}{2} = \infty$$

- Since this improper integral is divergent, the series $\Sigma\,(\ln n)/n$ is also divergent by the Integral Test.

ESTIMATING THE SUM OF A SERIES

Suppose we have been able to use the Integral Test to show that a series $\sum a_n$ is convergent.

Now, we want to find an approximation to the sum *s* of the series.s

ESTIMATING THE SUM OF A SERIES

Of course, any partial sum s_n is an approximation to s because $\lim_{n \to \infty} = s_n$

- How good is such an approximation?

ESTIMATING THE SUM OF A SERIES

To find out, we need to estimate the size of the remainder

$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \dots$$

- The remainder R_n is the error made when s_n, the sum of the first n terms, is used as an approximation to the total sum.

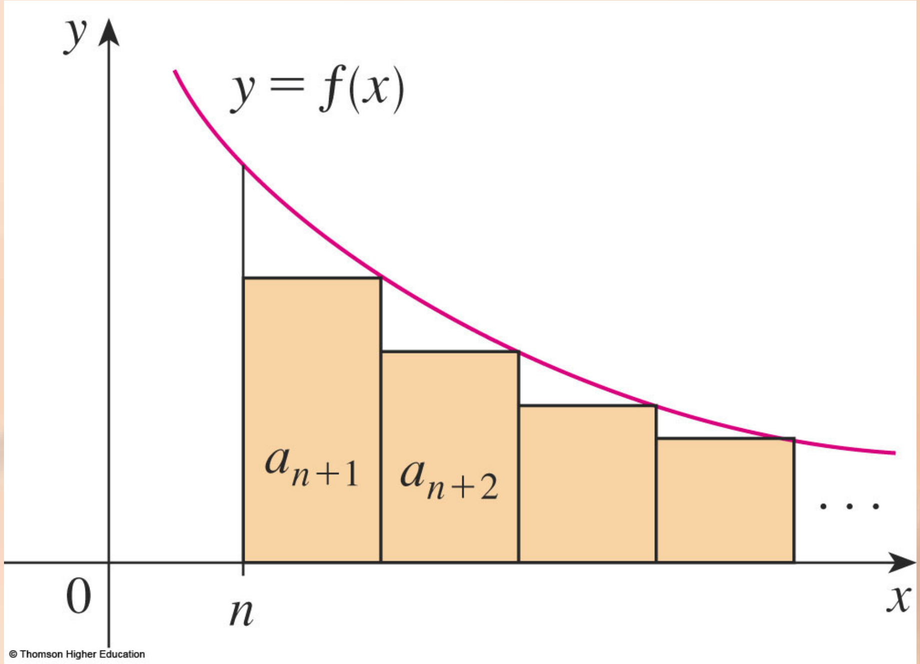
ESTIMATING THE SUM OF A SERIES

We use the same notation and ideas as in the Integral Test, assuming that f is decreasing on $[n, \infty)$.

ESTIMATING THE SUM OF A SERIES

Comparing the areas of the rectangles with the area under *y* = *f*(*x*) for *x* > *n* in the figure, we see that:

$$R_n = a_{n+1} + a_{n+2} + \ldots \leq \int_n^{\infty} f(x)\,dx$$

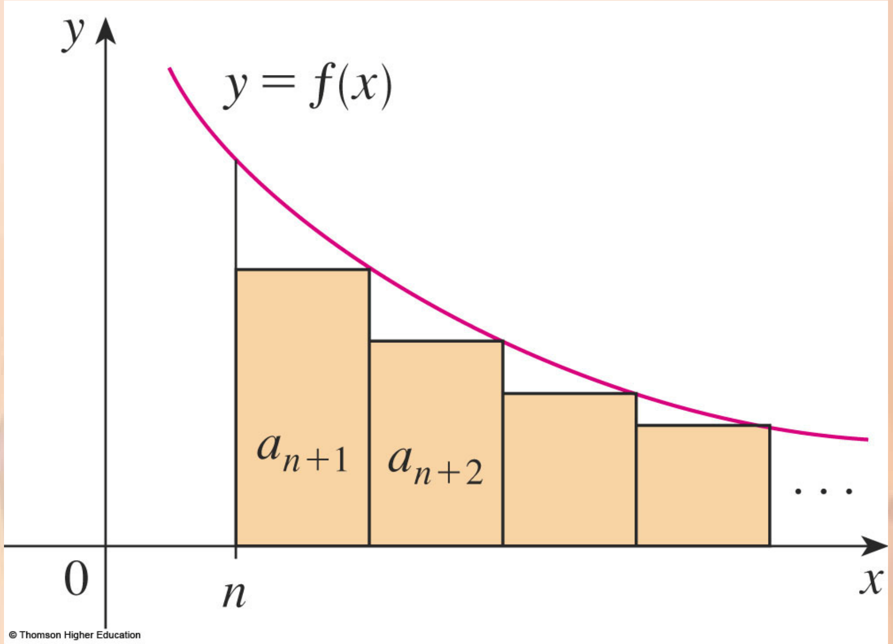


ESTIMATING THE SUM OF A SERIES

Similarly, from this figure,

we see that:

$$R_n = a_{n+1} + a_{n+2} + \ldots \geq \int_{n+1}^{\infty} f(x)\,dx$$



ESTIMATING THE SUM OF A SERIES

Thus, we have proved

the following error estimate.

REMAINDER ESTIMATE (INT. TEST) Estimate 2

Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent.

If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$$

a. Approximate the sum of the series $\sum 1/n^3$ by using the sum of the first 10 terms. Estimate the error involved.

b. How many terms are required to ensure the sum is accurate to within 0.0005?

In both parts, we need to know $\int_n^\infty f(x)\,dx$

- With $f(x) = 1/x^3$, which satisfies the conditions of the Integral Test, we have:

$$\int_n^\infty \frac{1}{x^3}\,dx = \lim_{t\to\infty}\left[-\frac{1}{2x^2}\right]_n^t$$

$$= \lim_{t\to\infty}\left(-\frac{1}{2t^2}+\frac{1}{2n^2}\right) = \frac{1}{2n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx s_{10} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{10^3} \approx 1.1975$$

- As per the remainder estimate 2, we have:

$$R_{10} \leq \int_{10}^{\infty} \frac{1}{x^3}\, dx = \frac{1}{2(10)^2} = \frac{1}{200}$$

- So, the size of the error is at most 0.005

Accuracy to within 0.0005 means that we have to find a value of n such that $R_n \leq 0.0005$

- Since $R_n \leq \int_n^\infty \frac{1}{x^3}\,dx = \frac{1}{2n^2}$

  we want $\frac{1}{2n^2} < 0.0005$

Solving this inequality, we get:

$$n^2 > \frac{1}{0.001} = 1000 \quad \text{or} \quad n > \sqrt{1000} \approx 31.6$$

- We need 32 terms to ensure accuracy to within 0.0005

If we add s_n to each side of the inequalities in Estimate 2, we get

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \le s \le s_n + \int_{n}^{\infty} f(x)\,dx$$

because $s_n + R_n = s$

The inequalities in Estimate 3 give a lower bound and an upper bound for *s*.

- They provide a more accurate approximation to the sum of the series than the partial sum s_n does.

Use Estimate 3 with n = 10 to estimate the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

The inequalities in Estimate 3 become:

$$s_{10} + \int_{11}^{\infty} \frac{1}{x^3}\,dx \le s \le s_{10} + \int_{10}^{\infty} \frac{1}{x^3}\,dx$$

From Example 5, we know that:

$$\int_n^\infty \frac{1}{x^3}\,dx = \frac{1}{2n^2}$$

Thus,

$$s_{10} + \frac{1}{2(11)^2} \le s \le s_{10} + \frac{1}{2(10)^2}$$

Using $s_{10} \approx 1.197532$, we get:

$$1.201664 \le s \le 1.202532$$

- If we approximate *s* by the midpoint of this interval, then the error is at most half the length of the interval.
- Thus, $$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.2021 \quad \text{with error} < 0.0005$$

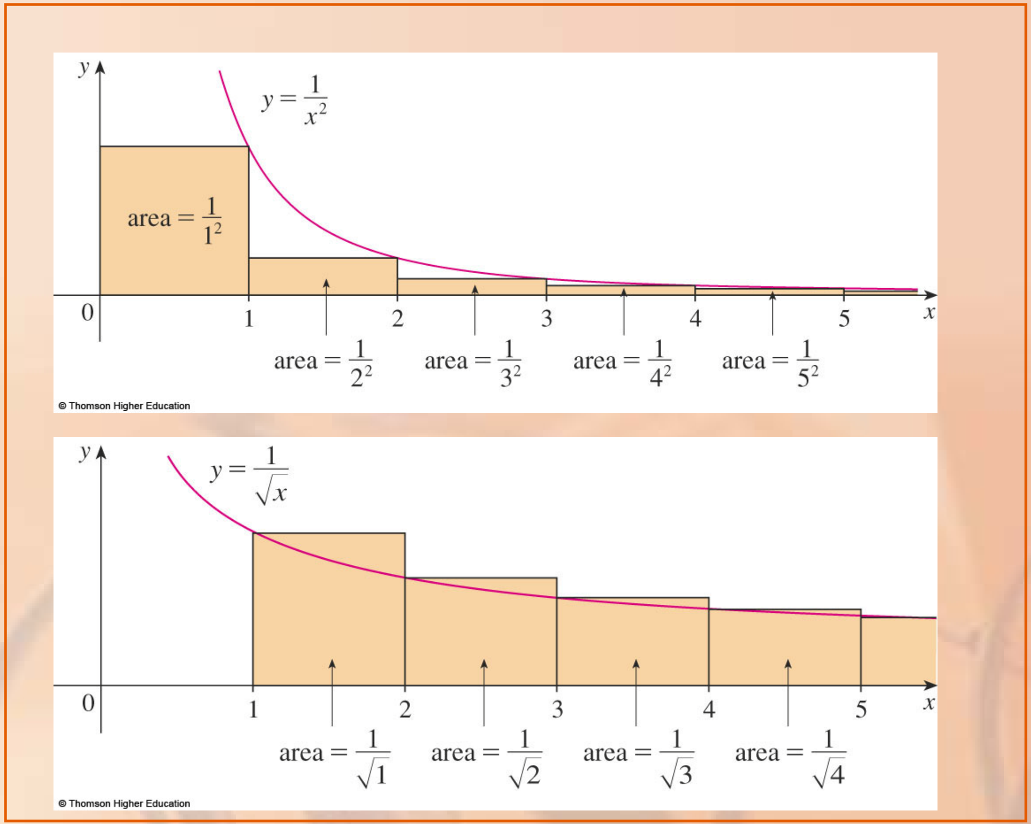
REMAINDER ESTIMATE

If we compare Example 6 with Example 5, we see that the improved estimate 3 can be much better than the estimate $s \approx s_n$.

- To make the error smaller than 0.0005, we had to use 32 terms in Example 5, but only 10 terms in Example 6.

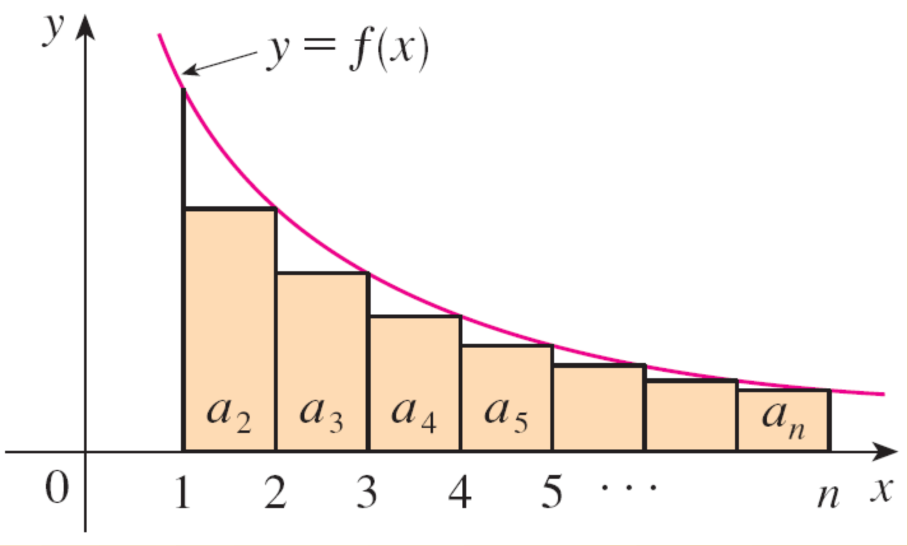
PROOF OF THE INTEGRAL TEST

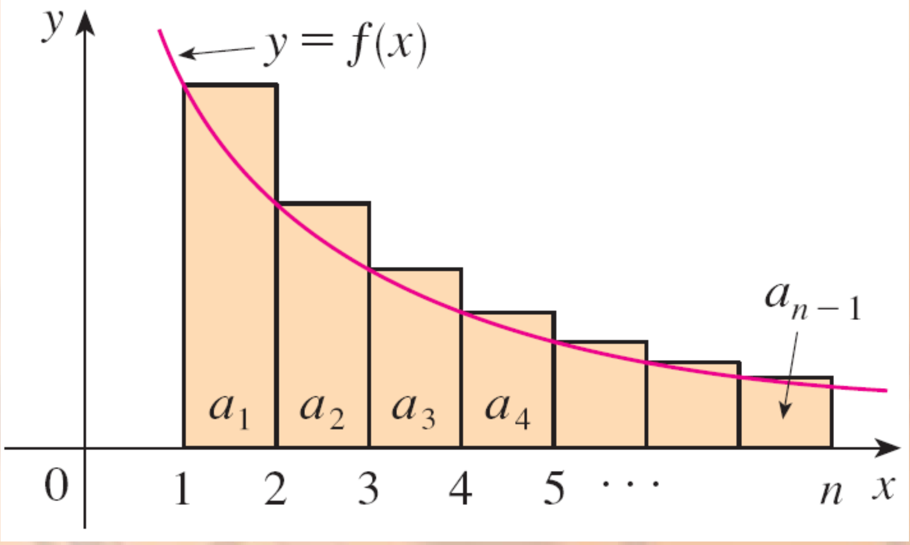
We have already seen the basic idea behind the proof of the Integral Test for the series $\sum 1/n^2$ and $\sum 1/\sqrt{n}$.



PROOF OF THE INTEGRAL TEST

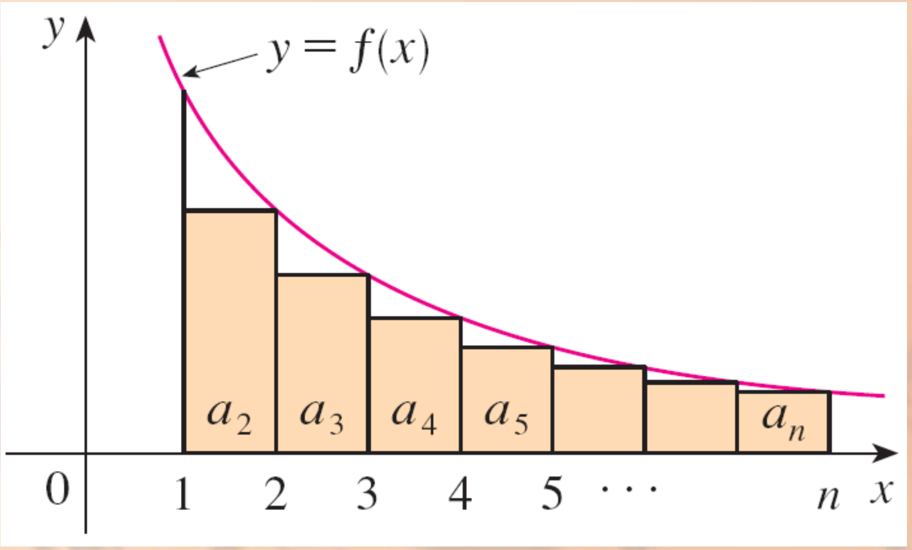
For the general series $\sum a_n$, consider these figures.





PROOF OF THE INTEGRAL TEST

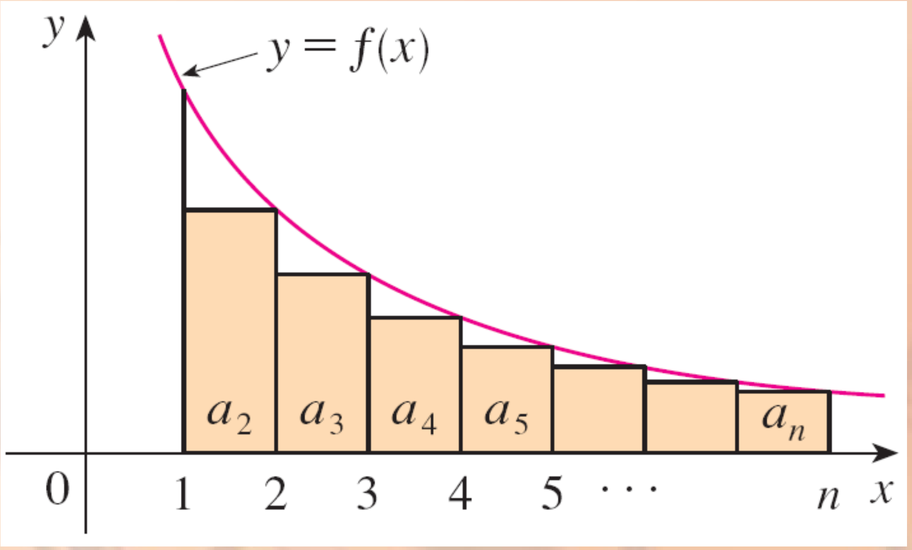
The area of the first shaded rectangle in this figure is the value of *f* at the right endpoint of [1, 2], that is, $f(2) = a_2$.



So, comparing the areas of the rectangles with the area under $y = f(x)$ from 1 to n, we see that:

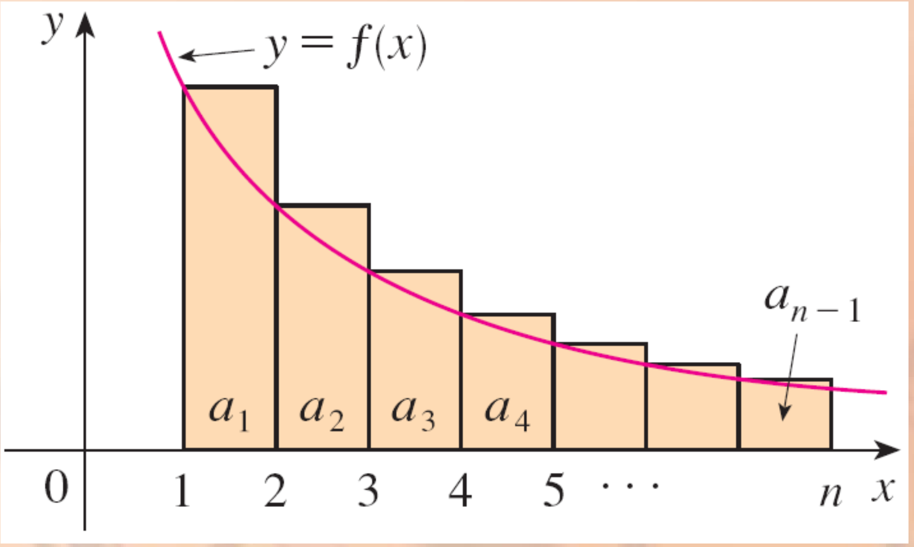
$$a_2 + a_3 + \cdots + a_n \leq \int_1^n f(x)\,dx$$

- Notice that this inequality depends on the fact that *f* is decreasing.



Likewise, the figure shows that:

$$\int_1^n f(x)\,dx \le a_1 + a_2 + \cdots + a_{n-1}$$



PROOF OF THE INTEGRAL TEST Case i

If $\int_1^\infty f(x)\,dx$ is convergent, then Estimate 4 gives

$$\sum_{i=2}^{n} a_i \le \int_1^n f(x)\,dx \le \int_1^\infty f(x)\,dx$$

since $f(x) \ge 0$.

PROOF OF THE INTEGRAL TEST   Case i

Therefore,

$$s_n = a_1 + \sum_{i=2}^{n} a_i \le a_1 + \int_1^\infty f(x)\,dx$$

$$= M,\ say$$

- Since $s_n \le M$ for all n, the sequence $\{s_n\}$ is bounded above.

Also,

$$s_{n+1} = s_n + a_{n+1} \geq s_n$$

since $a_{n+1} = f(n + 1) \geq 0$.

- Thus, $\{s_n\}$ is an increasing bounded sequence.

Thus, it is convergent by the Monotonic Sequence Theorem (Section 11.1).

- This means that $\sum a_n$ is convergent.

PROOF OF THE INTEGRAL TEST Case ii

If $\int_1^{\infty} f(x)\,dx$ is divergent,

then $\int_1^{n} f(x)\,dx \to \infty$

as $n \to \infty$ because $f(x) \geq 0$.

However, Estimate 5 gives:

$$\int_1^n f(x)\,dx \le \sum_{i=1}^{n-1} a_i = s_{n-1}$$

- Hence, $s_{n-1} \to \infty$.
- This implies that $s_n \to \infty$, and so $\sum a_n$ diverges.