11

INFINITE SEQUENCES AND SERIES

11.2
Series

In this section, we will learn about:

Various types of series.

If we try to add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$ we get an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

This is called an infinite series (or just a series).

- It is denoted, for short, by the symbol

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad \sum a_n$$

However, does it make sense to talk about the sum of infinitely many terms?

It would be impossible to find a finite sum for the series

$$1 + 2 + 3 + 4 + 5 + \cdots + n + \cdots$$

- If we start adding the terms, we get the cumulative sums 1, 3, 6, 10, 15, 21, . . .
- After the nth term, we get $n(n + 1)/2$, which becomes very large as n increases.

However, if we start to add the terms of the series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\cdots+\frac{1}{2^n}+\cdots$$

we get:

$$\frac{1}{2},\frac{3}{4},\frac{7}{8},\frac{15}{16},\frac{31}{32},\frac{63}{64},\cdots,1-1/2^n,\cdots$$

The table shows that, as we add more and more terms, these partial sums become closer and closer to 1.

- In fact, by adding sufficiently many terms of the series, we can make the partial sums as close as we like to 1.

n	Sum of first n terms
1	0.50000000
2	0.75000000
3	0.87500000
4	0.93750000
5	0.96875000
6	0.98437500
7	0.99218750
10	0.99902344
15	0.99996948
20	0.99999905
25	0.99999997

So, it seems reasonable to say that the sum of this infinite series is 1 and to write:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots = 1$$

We use a similar idea to determine whether or not a general series (Series 1) has a sum.

INFINITE SERIES

We consider the partial sums

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$s_3 = a_1 + a_2 + a_3 + a_4$$

- In general, $s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$

These partial sums form a new sequence $\{s_n\}$, which may or may not have a limit.

SUM OF INFINITE SERIES

If $\lim_{n\to\infty} s_n = s$ exists (as a finite number), then, as in the preceding example, we call it the sum of the infinite series $\Sigma\, a_n$.

Definition 2

Given a series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

let s_n denote its *n*th partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

Definition 2

If the sequence $\{s_n\}$ is convergent and $\lim_{n \to \infty} s_n = s$ exists as a real number, then the series $\Sigma\, a_n$ is called convergent and we write:

$$a_1 + a_2 + \cdots + a_n + \cdots = s \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = s$$

- The number s is called the sum of the series.
- Otherwise, the series is called divergent.

SUM OF INFINITE SERIES

Thus, the sum of a series is the limit of the sequence of partial sums.

- So, when we write $\sum_{n=1}^{\infty} a_n = s$,

  we mean that, by adding sufficiently many terms of the series, we can get as close as we like to the number *s*.

Notice that:

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

SUM OF INFINITE SERIES VS. IMPROPER INTEGRALS

Compare with the improper integral

$$\int_1^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_1^{t} f(x)\,dx$$

- To find this integral, we integrate from 1 to *t* and then let $t \to \infty$.
- For a series, we sum from 1 to *n* and then let $n \to \infty$.

Example 1

An important example of an infinite series is the geometric series

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots$$

$$= \sum_{n=1}^{\infty} ar^{n-1} \quad a \neq 0$$

Each term is obtained from the preceding one by multiplying it by the common ratio r.

- We have already considered the special case where $a = ½$ and $r = ½$ earlier in the section.

If $r = 1$, then

$$s_n = a + a + \cdots + a = na \to \pm\infty$$

- Since $\lim_{n\to\infty} s_n$ doesn't exist, the geometric series diverges in this case.

If $r \neq 1$, we have

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

and

$$rs_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n$$

Subtracting these equations,

we get:

$$s_n - rs_n = a - ar^n$$

$$s_n = \frac{a(1-r^n)}{1-r}$$

If $-1 < r < 1$, we know from Result 9 in Section 11.1 that $r^n \to 0$ as $n \to \infty$.

So,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{a}{1-r}\lim_{n\to\infty} r^n = \frac{a}{1-r}$$

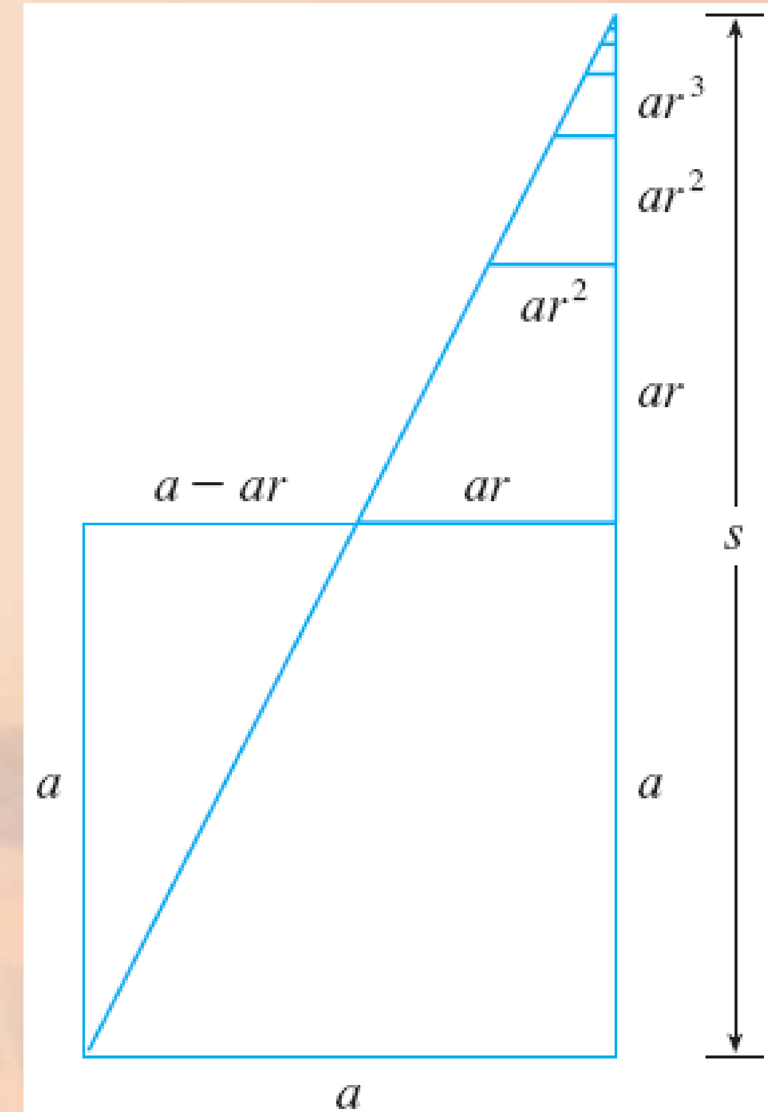
- Thus, when $|r| < 1$, the series is convergent and its sum is $a/(1-r)$.

If $r \leq -1$ or $r > 1$, the sequence $\{r^n\}$ is divergent by Result 9 in Section 11.1

So, by Equation 3, $\lim_{n \to \infty} s_n$ does not exist.

- Hence, the series diverges in those cases.

The figure provides a geometric demonstration of the result in Example 1.



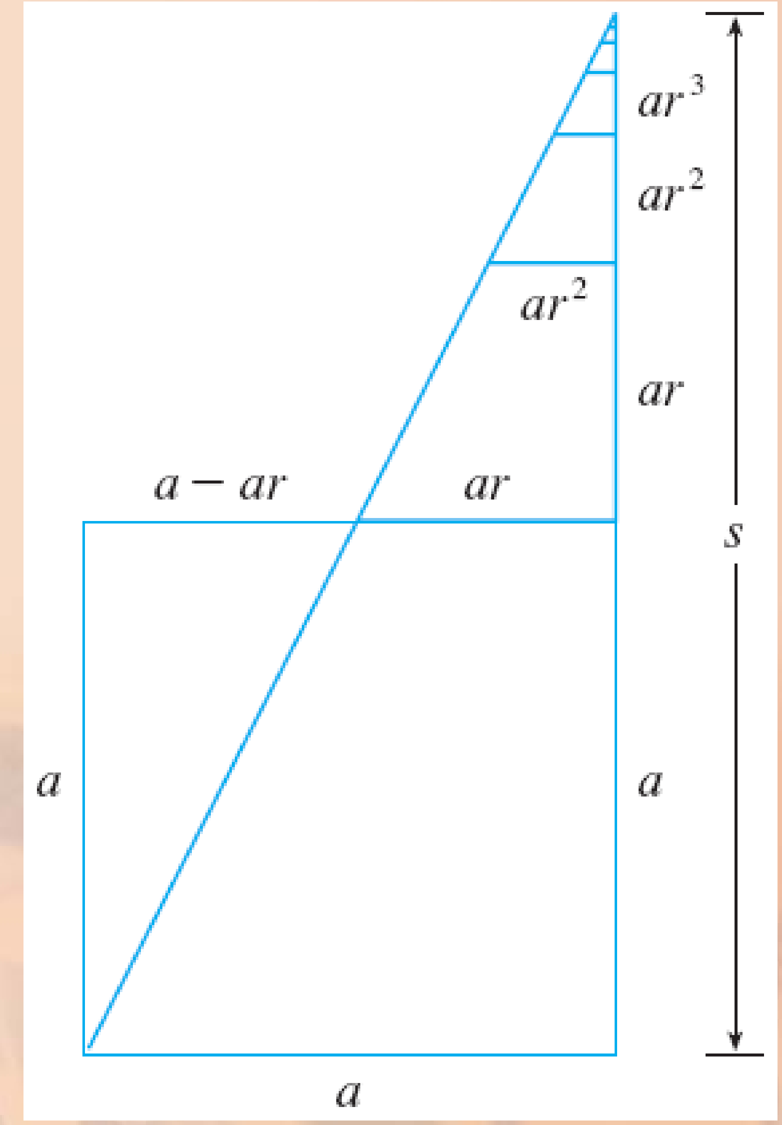
GEOMETRIC SERIES

If *s* is the sum of the series, then, by similar triangles,

$$\frac{s}{a} = \frac{a}{a - ar}$$

So,

$$s = \frac{a}{1 - r}$$



We summarize the results of Example 1 as follows.

The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$.

The sum of the series is:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad |r| < 1$$

If $|r| \geq 1$, the series is divergent.

Find the sum of the geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$$

- The first term is $a = 5$ and the common ratio is $r = -2/3$

- Since $|r| = 2/3 < 1$, the series is convergent by Result 4 and its sum is:

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots = \frac{5}{1 - \left(-\frac{2}{3}\right)}$$

$$= \frac{5}{\frac{5}{3}}$$

$$= 3$$

What do we really mean when we say that the sum of the series in Example 2 is 3?

- Of course, we can't literally add an infinite number of terms, one by one.

GEOMETRIC SERIES

However, according to Definition 2, the total sum is the limit of the sequence of partial sums.

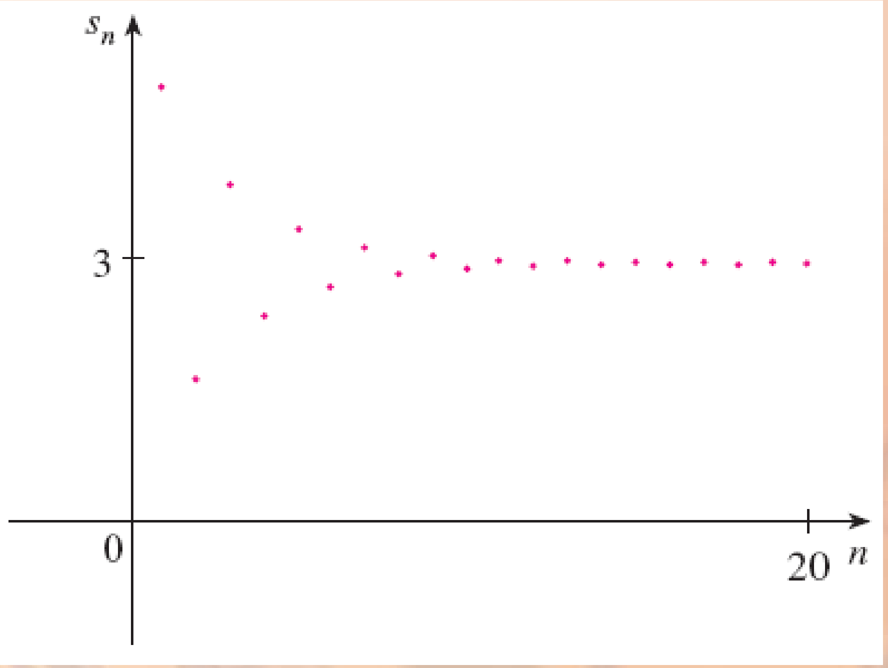
- So, by taking the sum of sufficiently many terms, we can get as close as we like to the number 3.

GEOMETRIC SERIES

The table shows the first ten partial sums s_n.

The graph shows how the sequence of partial sums approaches 3.

n	s_n
1	5.000000
2	1.666667
3	3.888889
4	2.407407
5	3.395062
6	2.736626
7	3.175583
8	2.882945
9	3.078037
10	2.947975



Is the series

$$\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$$

convergent or divergent?

Let's rewrite the *n*th term of the series

in the form ar^{n-1}:

$$\sum_{n=1}^{\infty} 2^{2n} 3^{1-n} = \sum_{n=1}^{\infty} (2^2)^n 3^{-(n-1)} = \sum_{n=1}^{\infty} \frac{4^n}{3^{n-1}} = \sum_{n=1}^{\infty} 4\left(\frac{4}{3}\right)^{n-1}$$

- We recognize this series as a geometric series with $a = 4$ and $r = 4/3$.
- Since $r > 1$, the series diverges by Result 4.

Write the number $2.3\overline{17} = 2.3171717\ldots$ as a ratio of integers.

- $2.3171717\ldots = 2.3 + \frac{17}{10^3} + \frac{17}{10^5} + \frac{17}{10^7} + \cdots$
- After the first term, we have a geometric series with $a = 17/10^3$ and $r = 1/10^2$.

- Therefore,

$$2.3\overline{17} = 2.3 + \frac{\frac{17}{10^3}}{1 - \frac{1}{10^2}} = 2.3 + \frac{\frac{17}{1000}}{\frac{99}{100}}$$

$$= \frac{23}{10} + \frac{17}{990}$$

$$= \frac{1147}{495}$$

Find the sum of the series $\sum_{n=0}^{\infty} x^n$ where $|x| < 1$.

- Notice that this series starts with $n = 0$.
- So, the first term is $x^0 = 1$.
- With series, we adopt the convention that $x^0 = 1$ even when $x = 0$.

Thus,

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

- This is a geometric series with $a = 1$ and $r = x$.

Since $|r| = |x| < 1$, it converges, and Result 4 gives:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

is convergent, and find its sum.

This is not a geometric series.

- So, we go back to the definition of a convergent series and compute the partial sums:

$$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)}$$

$$= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$$

We can simplify this expression if we use the partial fraction decomposition.

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$$

- See Section 7.4

Thus, we have:

$$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)}$$

$$= \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1} \right)$$

$$= \left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

$$= 1 - \frac{1}{n+1}$$

Thus,

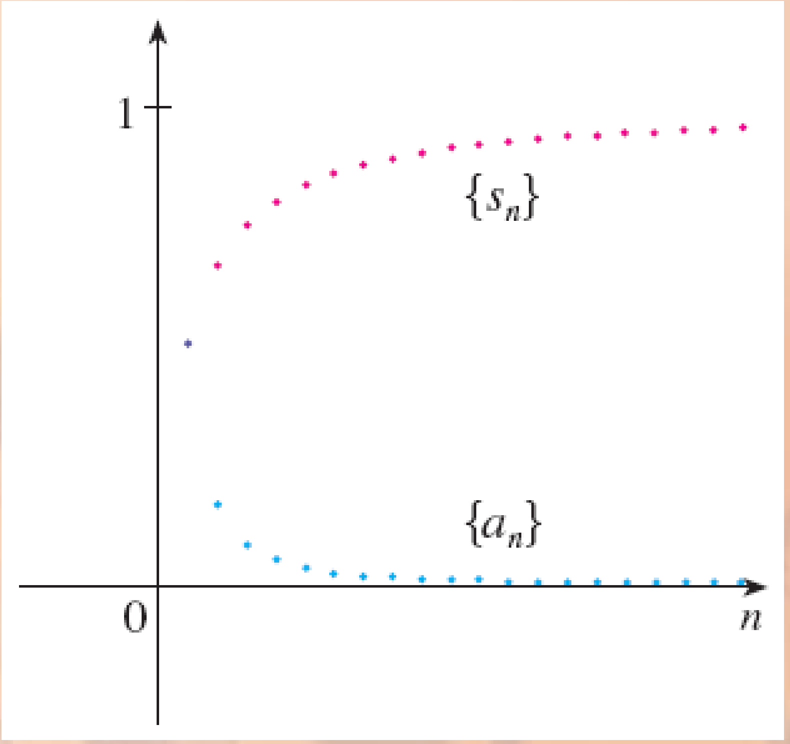
$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1-0=1$$

- Hence, the given series is convergent and

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=1$$

The figure illustrates Example 6 by showing the graphs of the sequence of terms $a_n = 1/[n(n + 1)]$ and the sequence $\{s_n\}$ of partial sums.

- Notice that $a_n \to 0$ and $s_n \to 1$.



Show that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is divergent.

For this particular series it's convenient to consider the partial sums $s_2, s_4, s_8, s_{16}, s_{32}, \ldots$ and show that they become large.

$$s_1 = 1$$

$$s_2 = 1 + \tfrac{1}{2}$$

$$s_4 = 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) > 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right)$$

$$= 1 + \tfrac{2}{2}$$

Similarly,

$$
\begin{aligned}
s_8 &= 1+\tfrac{1}{2}+\left(\tfrac{1}{3}+\tfrac{1}{4}\right)+\left(\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8}\right) \\
&> 1+\tfrac{1}{2}+\left(\tfrac{1}{4}+\tfrac{1}{4}\right)+\left(\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}\right) \\
&= 1+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2} \\
&= 1+\tfrac{3}{2}
\end{aligned}
$$

Similarly,

$$\begin{aligned}
s_{16} &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{9} + \cdots + \tfrac{1}{16}\right) \\
&> 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{16} + \cdots + \tfrac{1}{16}\right) \\
&= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} \\
&= 1 + \tfrac{4}{2}
\end{aligned}$$

Similarly, $s_{32} > 1 + 5/2$, $s_{64} > 1 + 6/2$, and, in general,

$$s_{2^n} > 1 + \frac{n}{2}$$

- This shows that $s_{2^n} \to \infty$ as $n \to \infty$, and so $\{s_n\}$ is divergent.
- Therefore, the harmonic series diverges.

HARMONIC SERIES

The method used in Example 7 for showing that the harmonic series diverges is due to the French scholar Nicole Oresme (1323–1382).

If the series $\sum_{n=1}^{\infty} a_n$ is convergent,

then $\lim_{n \to \infty} a_n = 0$

Let $$s_n = a_1 + a_2 + \cdots + a_n$$

Then, $$a_n = s_n - s_{n-1}$$

- Since $\Sigma\, a_n$ is convergent, the sequence $\{s_n\}$ is convergent.

Let $\lim_{n\to\infty} s_n = s$

Since n – 1 → ∞ as n → ∞,

we also have:

$$\lim_{n\to\infty} s_{n-1} = s$$

Therefore,

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty}\left(s_n - s_{n-1}\right) \\
&= \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} \\
&= s - s \\
&= 0
\end{aligned}$$

With any series $\Sigma\, a_n$ we associate two sequences:

- The sequence $\{s_n\}$ of its partial sums
- The sequence $\{a_n\}$ of its terms

If Σa_n is convergent, then

- The limit of the sequence $\{s_n\}$ is s (the sum of the series).
- The limit of the sequence $\{a_n\}$, as Theorem 6 asserts, is 0.

The converse of Theorem 6 is not true in general.

- If $\lim_{n\to\infty} a_n = 0$, we cannot conclude that $\Sigma\, a_n$ is convergent.

Observe that, for the harmonic series $\Sigma\, 1/n$, we have $a_n = 1/n \to 0$ as $n \to \infty$.

- However, we showed in Example 7 that $\Sigma\, 1/n$ is divergent.

THE TEST FOR DIVERGENCE **Test 7**

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$,

then the series $\sum_{n=1}^{\infty} a_n$

is divergent.

TEST FOR DIVERGENCE

The Test for Divergence follows from Theorem 6.

- If the series is not divergent, then it is convergent.
- Thus, $\lim_{n \to \infty} a_n = 0$

Example 8

Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

- $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{5n^2+4} = \lim_{n\to\infty} \frac{1}{5+4/n^2} = \frac{1}{5} \neq 0$$
- So, the series diverges by the Test for Divergence.

If we find that $\lim_{n \to \infty} a_n \neq 0$, we know that $\Sigma\, a_n$ is divergent.

If we find that $\lim_{n \to \infty} a_n = 0$, we know nothing about the convergence or divergence of $\Sigma\, a_n$.

Remember the warning in Note 2:

- If $\lim_{n\to\infty} a_n = 0$, the series $\Sigma\, a_n$ might converge or diverge.

If $\Sigma\, a_n$ and $\Sigma\, b_n$ are convergent series, then so are the series $\Sigma\, ca_n$ (where c is a constant), $\Sigma\,(a_n + b_n)$, and $\Sigma\,(a_n - b_n)$, and

$$\text{i. } \sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$$

$$\text{ii. } \sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

$$\text{iii. } \sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

These properties of convergent series follow from the corresponding Limit Laws for Sequences in Section 11.1

- For instance, we prove part ii of Theorem 8 as follows.

THEOREM 8 ii—PROOF

Let

$$s_n = \sum_{i=1}^{n} a_i \qquad s = \sum_{n=1}^{\infty} a_n$$

$$t_n = \sum_{i=1}^{n} b_i \qquad t = \sum_{n=1}^{\infty} b_n$$

THEOREM 8 ii—PROOF

The *n*th partial sum for the series

$\Sigma (a_n + b_n)$ is:

$$u_n = \sum_{i=1}^{n} (a_i + b_i)$$

THEOREM 8 ii—PROOF

Using Equation 10 in Section 5.2,

we have:

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \sum_{i=1}^{n} (a_i + b_i)$$

$$= \lim_{n\to\infty} \left(\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} a_i + \lim_{n\to\infty} \sum_{i=1}^{n} b_i$$

$$= \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n = s + t$$

THEOREM 8 ii—PROOF

Hence, Σ $(a_n + b_n)$ is convergent, and its sum is:

$$\sum_{n=1}^{\infty}(a_n + b_n) = s + t$$

$$= \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

Find the sum of the series

$$\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)}+\frac{1}{2^n}\right)$$

- The series $\Sigma\ 1/2^n$ is a geometric series with a = ½ and r = ½.
- Hence,

$$\sum_{n=1}^{\infty}\frac{1}{2^n}=\frac{\frac{1}{2}}{1-\frac{1}{2}}=1$$

- In Example 6, we found that: $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=1$

- So, by Theorem 8, the given series is convergent and

$$\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)}+\frac{1}{2^n}\right)=3\sum_{n=1}^{\infty}\frac{1}{n(n+1)}+\sum_{n=1}^{\infty}\frac{1}{2^n}$$
$$=3\cdot 1+1$$
$$=4$$

Note 4

A finite number of terms doesn’t affect the convergence or divergence of a series.

For instance, suppose that we were able to show that the series $\sum_{n=4}^{\infty} \frac{n}{n^3+1}$ is convergent.

- Since $\sum_{n=1}^{\infty} \frac{n}{n^3+1} = \frac{1}{2} + \frac{2}{9} + \frac{3}{28} + \sum_{n=4}^{\infty} \frac{n}{n^3+1}$

  it follows that the entire series $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$ is convergent.

Similarly, if it is known that the series $\sum_{n=N+1}^{\infty} a_n$ converges, then the full series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$$

is also convergent.