11

INFINITE SEQUENCES AND SERIES

11.11

Applications of Taylor Polynomials

In this section, we will learn about:

Two types of applications of Taylor polynomials.

APPLICATIONS IN APPROXIMATING FUNCTIONS

First, we look at how they are used to approximate functions.

- Computer scientists like them because polynomials are the simplest of functions.

APPLICATIONS IN PHYSICS AND ENGINEERING

Then, we investigate how physicists and engineers use them in such fields as:

- Relativity
- Optics
- Blackbody radiation
- Electric dipoles
- Velocity of water waves
- Building highways across a desert

Suppose that *f*(*x*) is equal to the sum of its Taylor series at *a*:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

NOTATION $T_n(x)$

In Section 11.10, we introduced the notation $T_n(x)$ for the *n*th partial sum of this series.

- We called it the *n*th-degree Taylor polynomial of *f* at *a*.

Thus,

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$+ \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

APPROXIMATING FUNCTIONS

Since *f* is the sum of its Taylor series, we know that $T_n(x) \to f(x)$ as $n \to \infty$.

- Thus, T_n can be used as an approximation to *f* :

$$f(x) \approx T_n(x)$$

Notice that the first-degree Taylor polynomial

$$T_1(x) = f(a) + f'(a)(x - a)$$

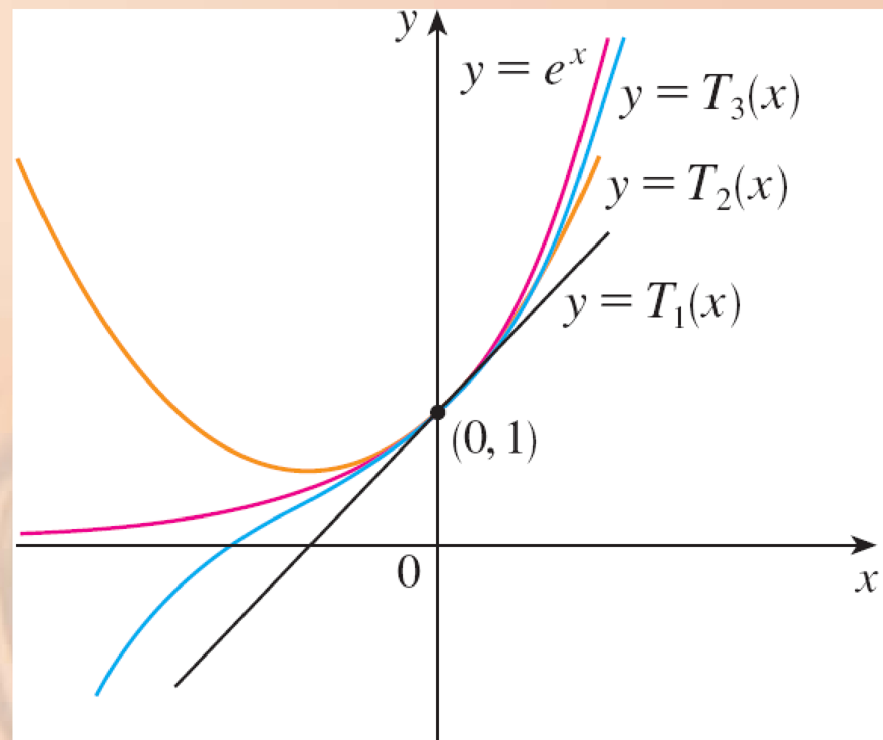
is the same as the linearization of *f* at *a* that we discussed in Section 3.10

Notice also that T_1 and its derivative have the same values at *a* that *f* and *f'* have.

- In general, it can be shown that the derivatives of T_n at *a* agree with those of *f* up to and including derivatives of order *n.*
- See Exercise 38.

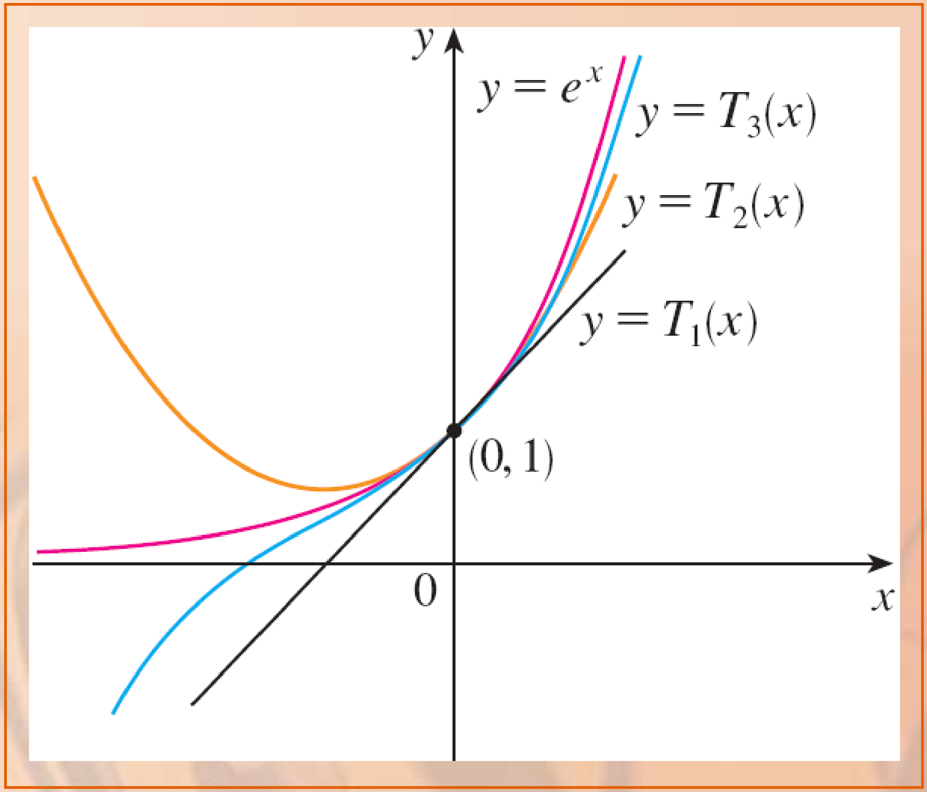
APPROXIMATING FUNCTIONS

To illustrate these ideas, let's take another look at the graphs of $y = e^x$ and its first few Taylor polynomials.



The graph of T_1 is the tangent line to $y = e^x$ at (0, 1).

- This tangent line is the best linear approximation to e^x near (0, 1).



APPROXIMATING FUNCTIONS

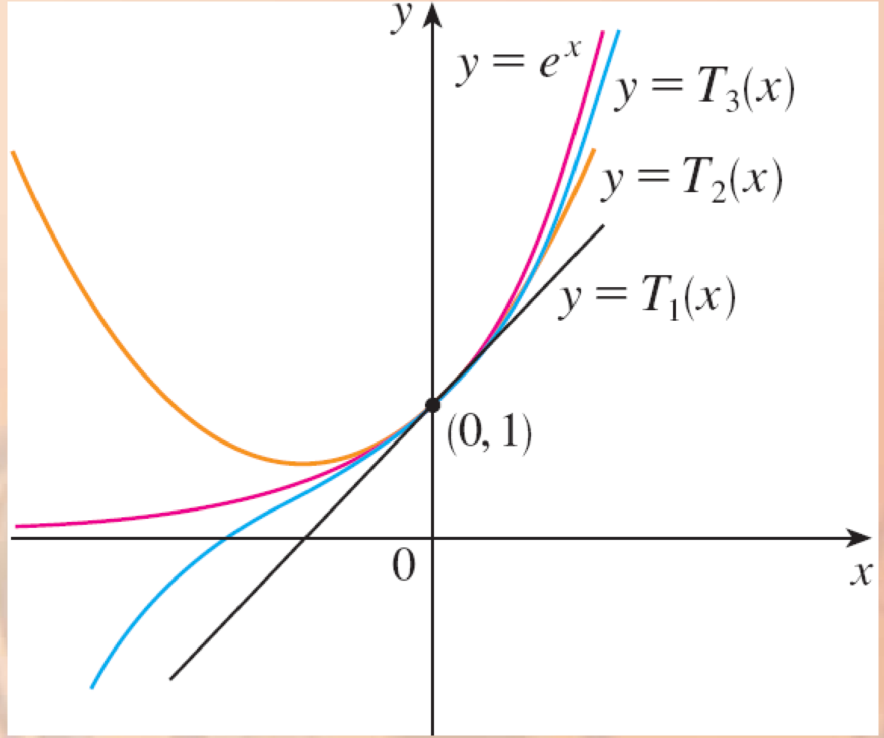
The graph of T_2 is the parabola

$$y = 1 + x + x^2/2$$

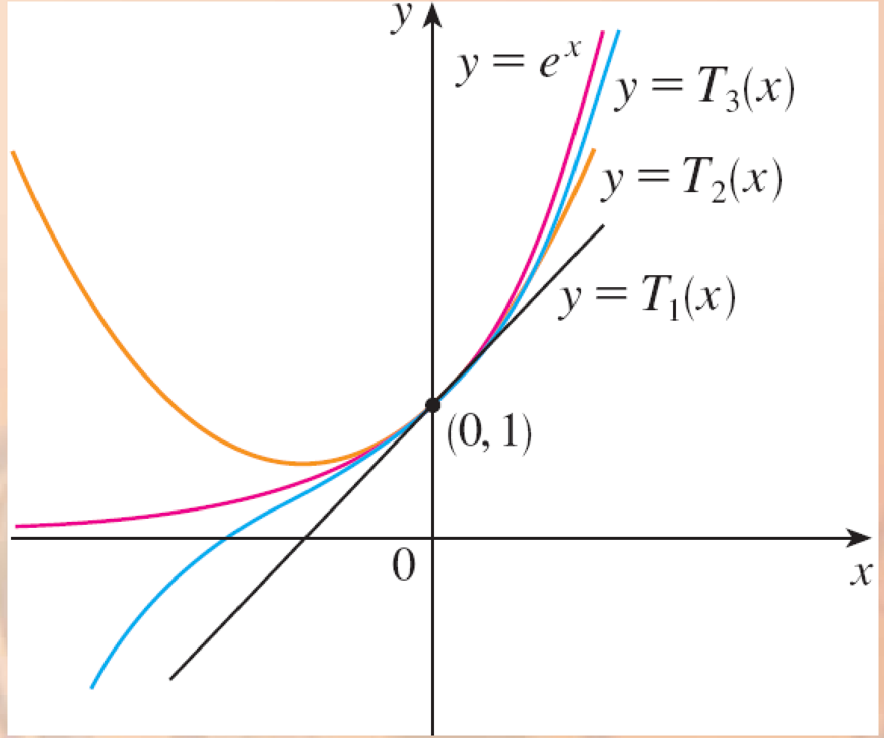
The graph of T_3 is the cubic curve

$$y = 1 + x + x^2/2 + x^3/6$$

- This is a closer fit to the curve $y = e^x$ than T_2.



The next Taylor polynomial would be an even better approximation, and so on.



The values in the table give a numerical demonstration of the convergence of the Taylor polynomials $T_n(x)$ to the function $y = e^x$.

	$x = 0.2$	$x = 3.0$
$T_2(x)$	1.220000	8.500000
$T_4(x)$	1.221400	16.375000
$T_6(x)$	1.221403	19.412500
$T_8(x)$	1.221403	20.009152
$T_{10}(x)$	1.221403	20.079665
e^x	1.221403	20.085537

APPROXIMATING FUNCTIONS

When $x = 0.2$, the convergence is very rapid.

When $x = 3$, however, it is somewhat slower.

- The farther x is from 0, the more slowly $T_n(x)$ converges to e^x.

	$x = 0.2$	$x = 3.0$
$T_2(x)$	1.220000	8.500000
$T_4(x)$	1.221400	16.375000
$T_6(x)$	1.221403	19.412500
$T_8(x)$	1.221403	20.009152
$T_{10}(x)$	1.221403	20.079665
e^x	1.221403	20.085537

APPROXIMATING FUNCTIONS

When using a Taylor polynomial T_n to approximate a function *f*, we have to ask these questions:

- How good an approximation is it?
- How large should we take *n* to be to achieve a desired accuracy?

APPROXIMATING FUNCTIONS

To answer these questions, we need to look at the absolute value of the remainder:

$$|R_n(x)| = |f(x) - T_n(x)|$$

METHODS FOR ESTIMATING ERROR

There are three possible methods for estimating the size of the error.

METHOD 1

If a graphing device is available, we can use it to graph $|R_n(x)|$ and thereby estimate the error.

METHOD 2

If the series happens to be an alternating series, we can use the Alternating Series Estimation Theorem.

METHOD 3

In all cases, we can use Taylor's Inequality (Theorem 9 in Section 11.10), which states that, if $|f^{(n+1)}(x)| \le M$, then

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$

a. Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a = 8$.

b. How accurate is this approximation when $7 \leq x \leq 9$?

APPROXIMATING FUNCTIONS

Example 1 a

$$f(x) = \sqrt[3]{x} = x^{1/3}$$

$$f'(x) = \tfrac{1}{3} x^{-2/3}$$

$$f''(x) = -\tfrac{2}{9} x^{-5/3}$$

$$f'''(x) = \tfrac{10}{27} x^{-8/3}$$

$$f(8) = 2$$

$$f'(8) = \tfrac{1}{12}$$

$$f''(8) = -\tfrac{1}{144}$$

Hence, the second-degree Taylor polynomial is:

$$T_2(x) = f(8) + \frac{f'(8)}{1!}(x-8) + \frac{f''(8)}{2!}(x-8)^2$$

$$= 2 + \tfrac{1}{12}(x-8) - \tfrac{1}{288}(x-8)^2$$

The desired approximation is:

$$\sqrt[3]{x} \approx T_2(x)$$

$$= 2 + \tfrac{1}{12}(x-8) - \tfrac{1}{288}(x-8)^2$$

The Taylor series is not alternating when $x < 8$.

- Thus, we can't use the Alternating Series Estimation Theorem here.

However, we can use Taylor's Inequality with *n* = 2 and *a* = 8:

$$|R_2(x)| \le \frac{M}{3!}|x-8|^3$$

where $f'''(x) \le M$.

Since $x \geq 7$, we have $x^{8/3} \geq 7^{8/3}$,

and so:

$$f'''(x) = \frac{10}{27} \cdot \frac{1}{x^{8/3}} \leq \frac{10}{27} \cdot \frac{1}{7^{8/3}} < 0.0021$$

- Hence, we can take $M = 0.0021$

Also, $7 \le x \le 9$.

So, $-1 \le x - 8 \le 1$ and $|x - 8| \le 1$.

- Then, Taylor's Inequality gives:

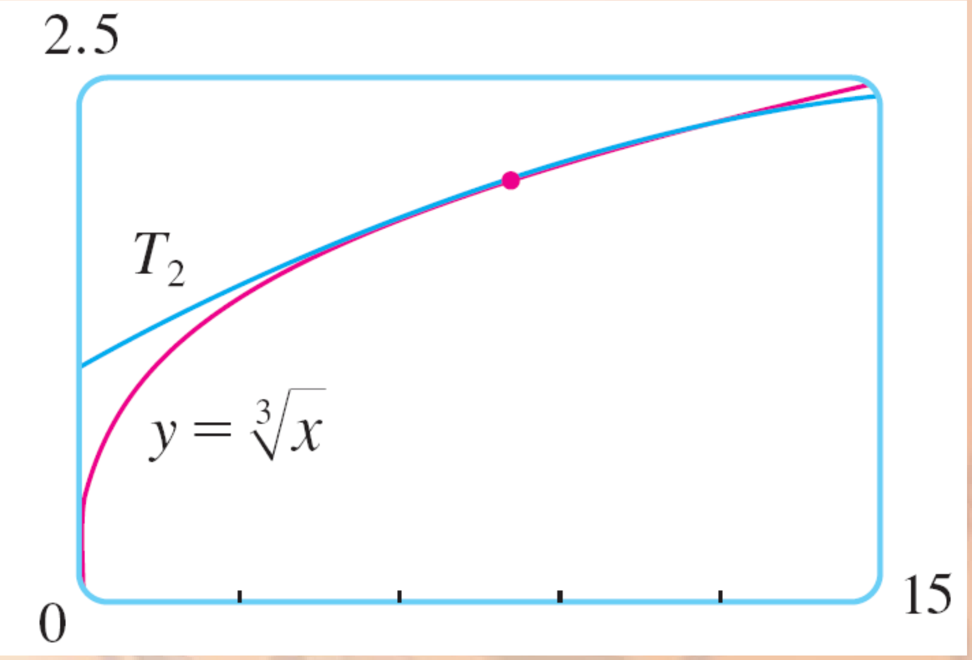
$$|R_2(x)| \le \frac{0.0021}{3!} \cdot 1^3 = \frac{0.0021}{6} < 0.0004$$

- Thus, if $7 \le x \le 9$, the approximation in part a is accurate to within 0.0004

Let’s use a graphing device to check the calculation in Example 1.

APPROXIMATING FUNCTIONS

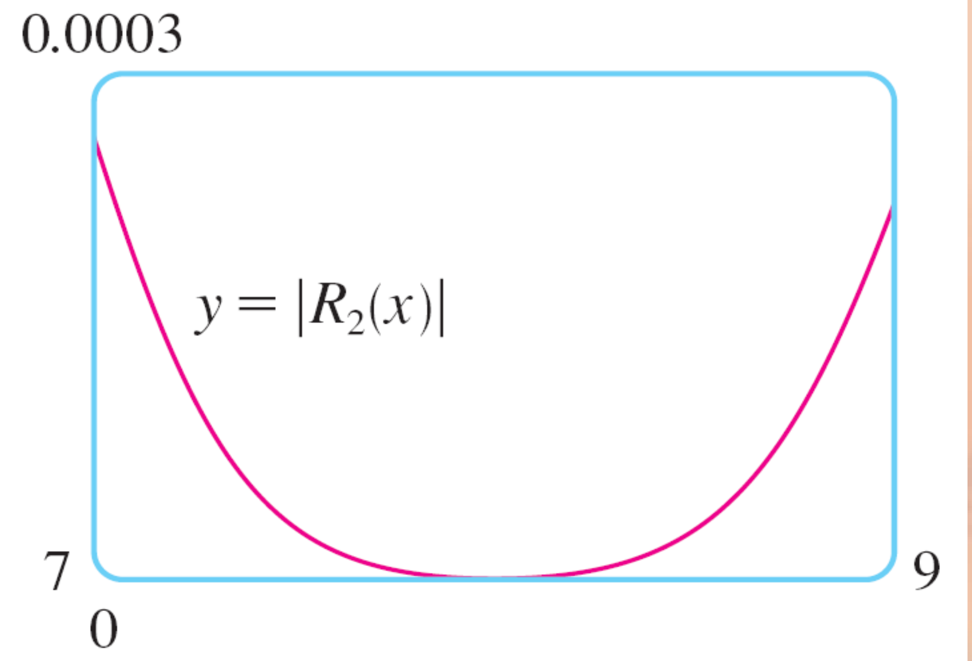
The figure shows that the graphs of $y = \sqrt[3]{x}$ and $y = T_2(x)$ are very close to each other when x is near 8.



This figure shows the graph of $|R_2(x)|$ computed from the expression

$$|R_2(x)| = |\sqrt[3]{x} - T_2(x)|$$

- We see that $|R_2(x)| < 0.0003$ when $7 \le x \le 9$



Thus, in this case, the error estimate from graphical methods is slightly better than the error estimate from Taylor's Inequality.

a. What is the maximum error possible in using the approximation

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

when $-0.3 \leq x \leq 0.3$?

- Use this approximation to find sin 12° correct to six decimal places.

b. For what values of *x* is this approximation accurate to within 0.00005?

Notice that the Maclaurin series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

alternates for all nonzero values of *x* and the successive terms decrease in size as $|x| < 1$.

- So, we can use the Alternating Series Estimation Theorem.

The error in approximating sin *x* by the first three terms of its Maclaurin series is at most $\left|\frac{x^7}{7!}\right| = \frac{|x|^7}{5040}$

- If $-0.3 \leq x \leq 0.3$, then $|x| \leq 0.3$
- So, the error is smaller than $\frac{(0.3)^7}{5040} \approx 4.3 \times 10^{-8}$

To find sin 12°, we first convert to radian measure.

$$\sin 12^\circ = \sin\left(\frac{12\pi}{180}\right)$$

$$= \sin\left(\frac{\pi}{15}\right) \approx \frac{\pi}{15} - \left(\frac{\pi}{15}\right)^3 \frac{1}{3!} + \left(\frac{\pi}{15}\right)^5 \frac{1}{5!}$$

$$\approx 0.20791169$$

- Correct to six decimal places, sin 12° ≈ 0.207912

The error will be smaller than 0.00005 if:

$$\frac{|x^7|}{5040} < 0.00005$$

- Solving this inequality for *x*, we get:

$$|x|^7 < 0.252 \quad \text{or} \quad |x| < (0.252)^{1/7} \approx 0.821$$

- The given approximation is accurate to within 0.00005 when $|x| < 0.82$

What if we use Taylor's Inequality to solve Example 2?

Since $f^{(7)}(x) = -\cos x$, we have

$|f^{(7)}(x)| \leq 1$, and so

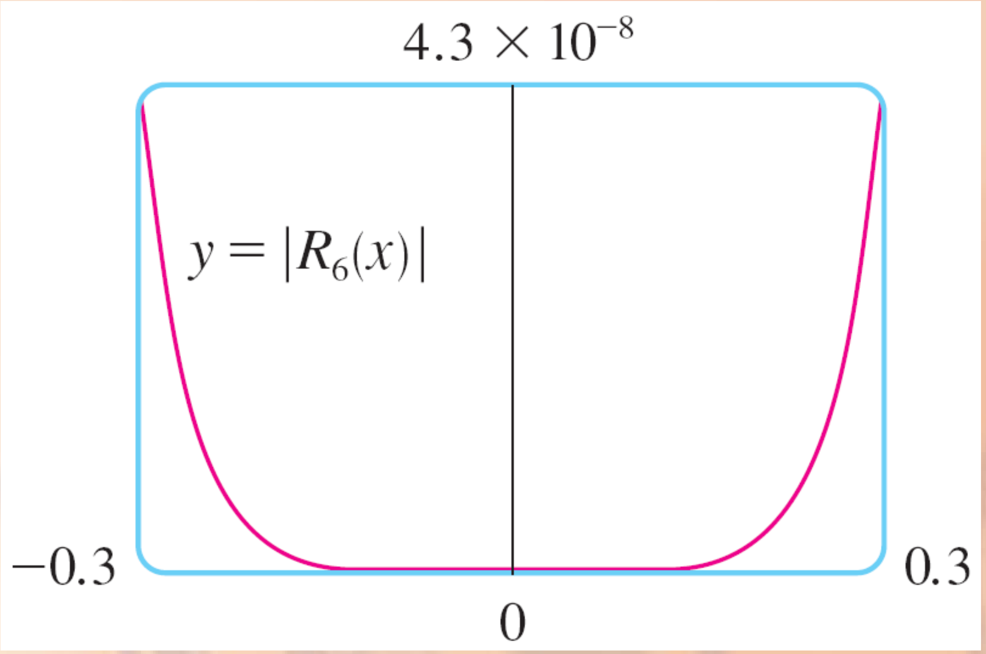
$$|R_6(x)| \leq \frac{1}{7!}|x|^7$$

- Thus, we get the same estimates as with the Alternating Series Estimation Theorem.

What about graphical methods?

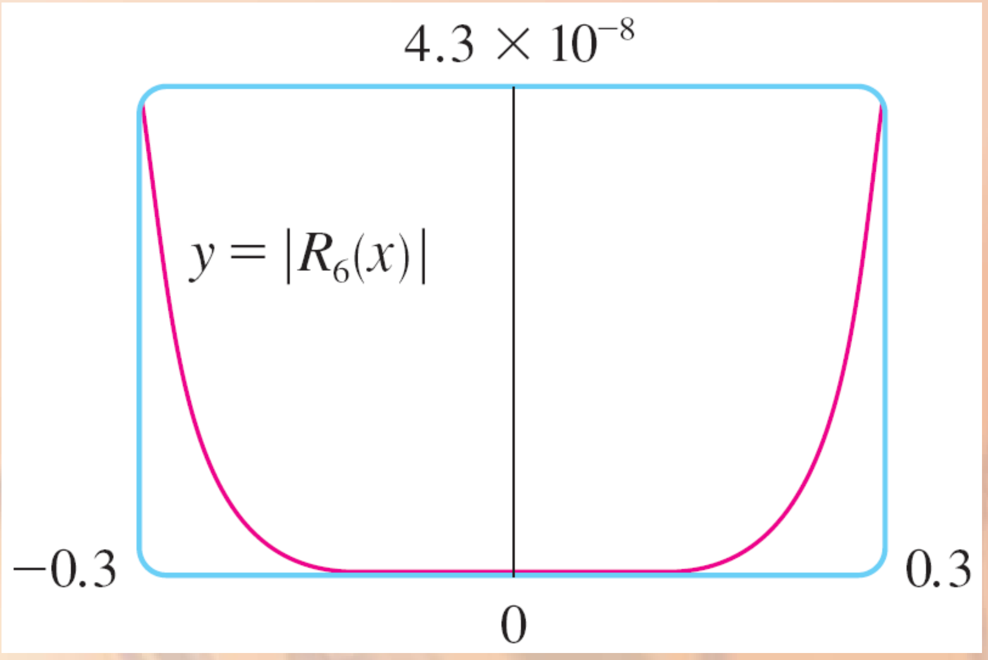
The figure shows the graph of

$$|R_6(x)| = |\sin x - (x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5)|$$



We see that $|R_6(x)| < 4.3 \times 10^{-8}$ when $|x| \le 0.3$

- This is the same estimate that we obtained in Example 2.

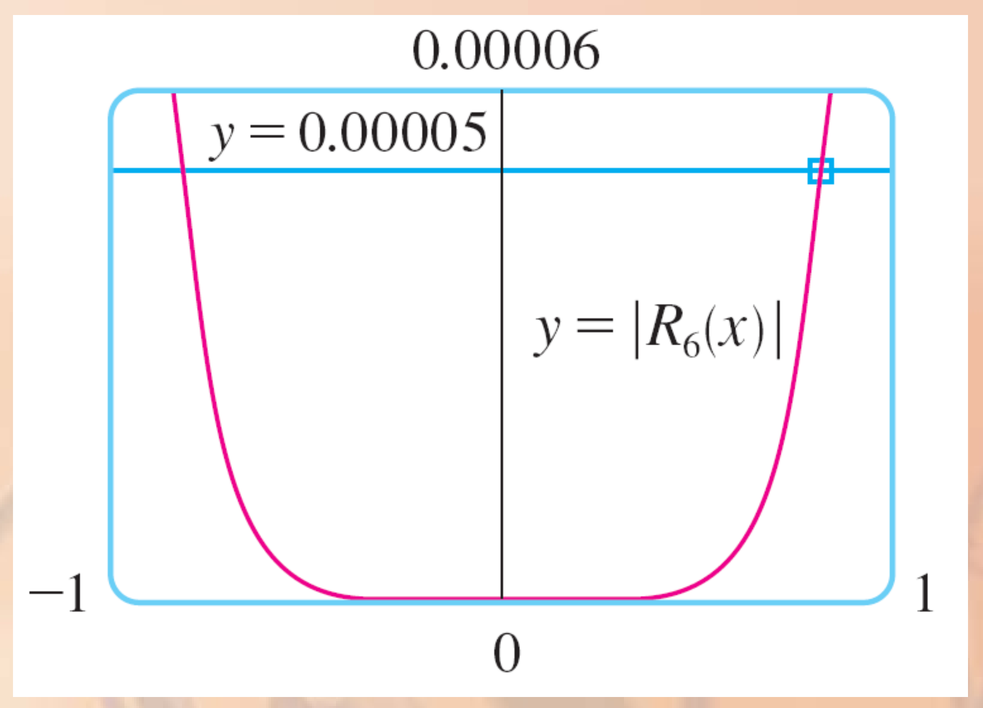


For part b, we want $|R_6(x)| < 0.00005$

- So, we graph both $y = |R_6(x)|$ and $y = 0.00005$, as follows.

By placing the cursor on the right intersection point, we find that the inequality is satisfied when $|x| < 0.82$

- Again, this is the same estimate that we obtained in the solution to Example 2.



If we had been asked to approximate sin 72° instead of sin 12° in Example 2, it would have been wise to use the Taylor polynomials at $a = \pi/3$ (instead of $a = 0$).

- They are better approximations to sin x for values of x close to $\pi/3$.

Notice that 72° is close to 60° (or $\pi/3$ radians).

- The derivatives of sin x are easy to compute at $\pi/3$.

The Maclaurin polynomial approximations

$$T_1(x) = x \qquad T_3(x) = x - \frac{x^3}{3!}$$

$$T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \qquad T_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

to the sine curve are graphed

in the following figure.

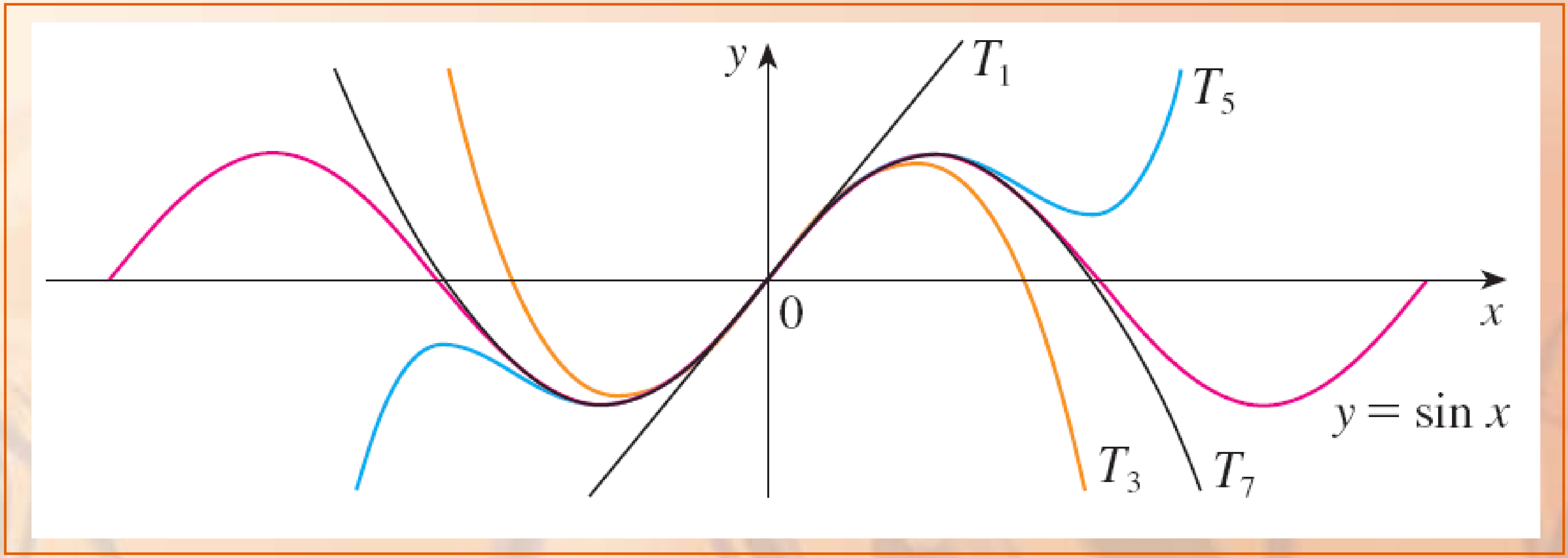
APPROXIMATING FUNCTIONS

$$T_1(x) = x$$

$$T_3(x) = x - \frac{x^3}{3!}$$

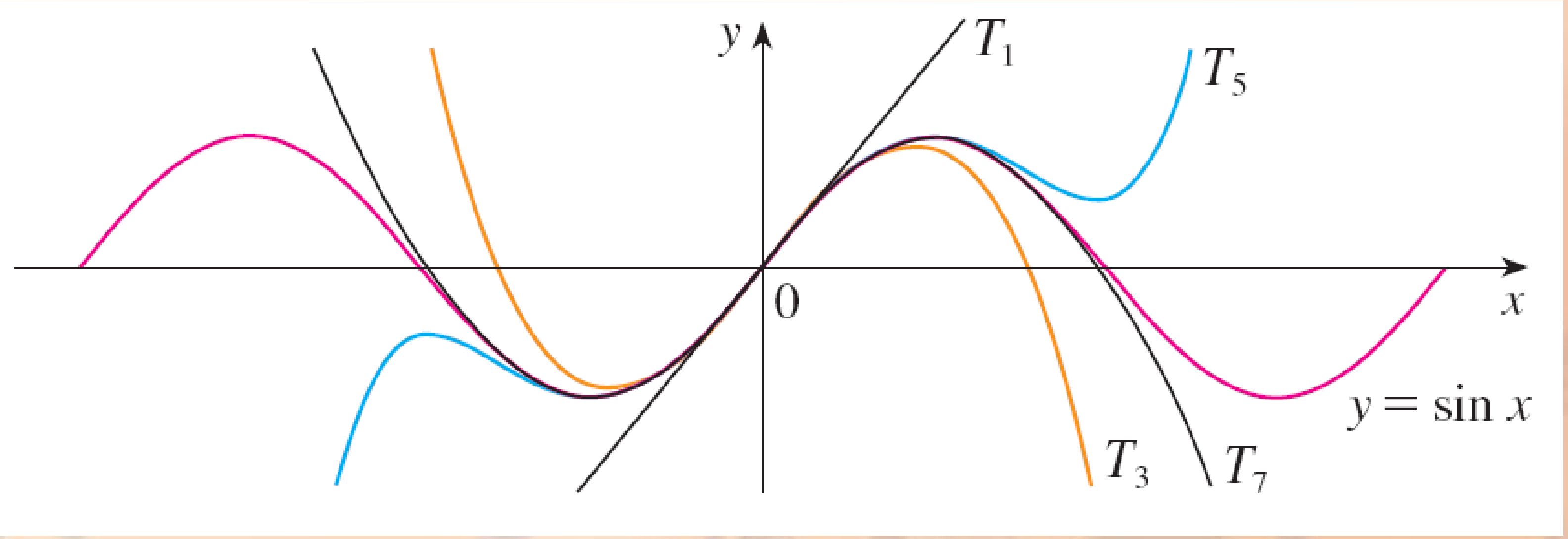
$$T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$T_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$



APPROXIMATING FUNCTIONS

You can see that as *n* increases, $T_n(x)$ is a good approximation to sin *x* on a larger and larger interval.



APPROXIMATING FUNCTIONS

One use of the type of calculation done in Examples 1 and 2 occurs in calculators and computers.

APPROXIMATING FUNCTIONS

For instance, a polynomial approximation is calculated (in many machines) when:

- You press the sin or e^x key on your calculator.
- A computer programmer uses a subroutine for a trigonometric or exponential or Bessel function.

APPROXIMATING FUNCTIONS

The polynomial is often a Taylor polynomial that has been modified so that the error is spread more evenly throughout an interval.

Taylor polynomials are also used frequently in physics.

APPLICATIONS TO PHYSICS

To gain insight into an equation, a physicist often simplifies a function by considering only the first two or three terms in its Taylor series.

- That is, the physicist uses a Taylor polynomial as an approximation to the function.
- Then, Taylor's Inequality can be used to gauge the accuracy of the approximation.

APPLICATIONS TO PHYSICS

The following example shows one way in which this idea is used in special relativity.

In Einstein's theory of special relativity, the mass of an object moving with velocity *v* is

$$m = \frac{m_0}{\sqrt{1 - v^2 / c^2}}$$

where:

- m_0 is the mass of the object when at rest.
- c is the speed of light.

The kinetic energy of the object is the difference between its total energy and its energy at rest:

$$K = mc^2 - m_0c^2$$

a. Show that, when v is very small compared with c, this expression for K agrees with classical Newtonian physics: $K = \frac{1}{2} m_0 v^2$

b. Use Taylor's Inequality to estimate the difference in these expressions for K when $|v| \leq 100$ ms.

Using the expressions given for *K* and *m*, we get:

$$K = mc^2 - m_0 c^2$$

$$= \frac{m_0 c^2}{\sqrt{1 - v^2 / c^2}} - m_0 c^2$$

$$= m_0 c^2 \left[\left(1 - \frac{v^2}{c^2} \right)^{-1/2} - 1 \right]$$

With $x = -v^2/c^2$, the Maclaurin series for $(1 + x)^{-1/2}$ is most easily computed as a binomial series with $k = -½$.

- Notice that $|x| < 1$ because $v < c$.

Therefore, we have:

$$(1+x)^{-1/2} = 1 - \tfrac{1}{2}x + \frac{\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)}{2!}x^2 + \frac{\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)\left(-\tfrac{5}{2}\right)}{3!}x^3 + \ldots$$

$$= 1 - \tfrac{1}{2}x + \tfrac{3}{8}x^2 - \tfrac{5}{16}x^3 + \ldots$$

Also, we have:

$$K = m_0 c^2 \left[\left(1 + \frac{1}{2} \frac{v^2}{c^2} + \frac{3}{8} \frac{v^4}{c^4} + \frac{5}{16} \frac{v^6}{c^6} + \ldots \right) - 1 \right]$$

$$= m_0 c^2 \left(\frac{1}{2} \frac{v^2}{c^2} + \frac{3}{8} \frac{v^4}{c^4} + \frac{5}{16} \frac{v^6}{c^6} + \ldots \right)$$

If v is much smaller than c, then all terms after the first are very small when compared with the first term.

- If we omit them, we get:

$$K \approx m_0 c^2 \left(\frac{1}{2} \frac{v^2}{c^2} \right) = \tfrac{1}{2} m_0 v^2$$

 Example 3 b

Let:

- $x = -v^2/c^2$
- $f(x) = m_0c^2[(1 + x)^{-1/2} - 1]$
- M is a number such that $|f''(x)| \leq M$

Then, we can use Taylor's Inequality to write:

$$|R_1(x)| \leq \frac{M}{2!} x^2$$

We have $f'''(x) = \tfrac{3}{4} m_0 c^2 (1+x)^{-5/2}$ and we are given that $|v| \leq 100$ m/s.

Thus,

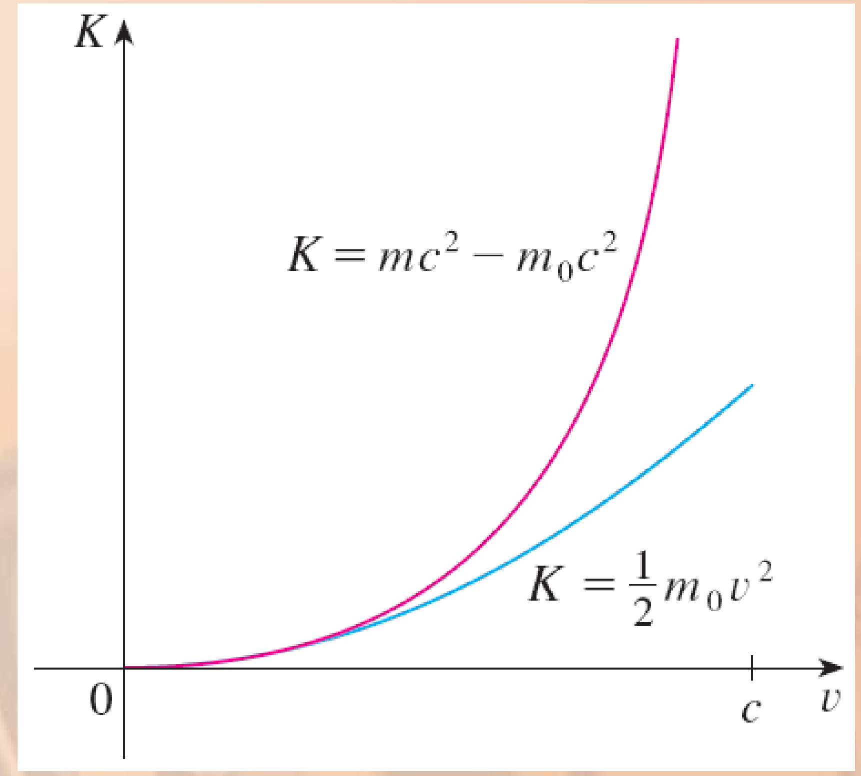
$$| f''(x) | - \frac{3m_0c^2}{4(1-v^2/c^2)^{5/2}} \leq \frac{3m_0c^2}{4(1-100^2/c^2)^{5/2}} (= M)$$

Thus, with $c = 3 \times 10^8$ m/s,

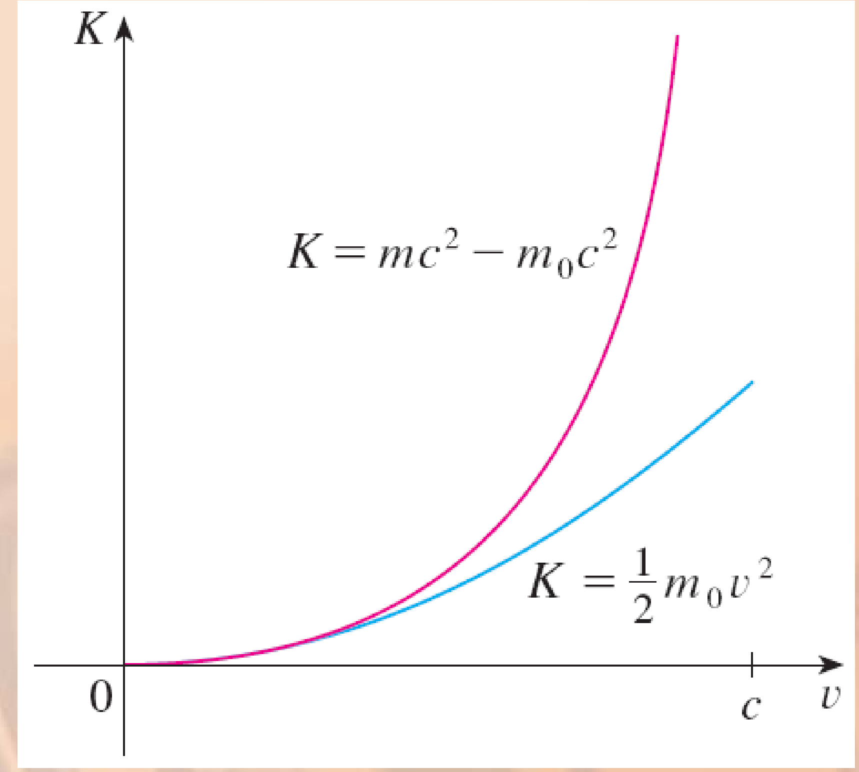
$$|R_1(x)| \le \frac{1}{2} \cdot \frac{3m_0c^2}{4(1-100^2/c^2)^{5/2}} \cdot \frac{100^4}{c^4} < (4.17 \times 10^{-10})m_0$$

- So, when $|v| \le 100$ m/s, the magnitude of the error in using the Newtonian expression for kinetic energy is at most $(4.2 \times 10^{-10})m_0$.

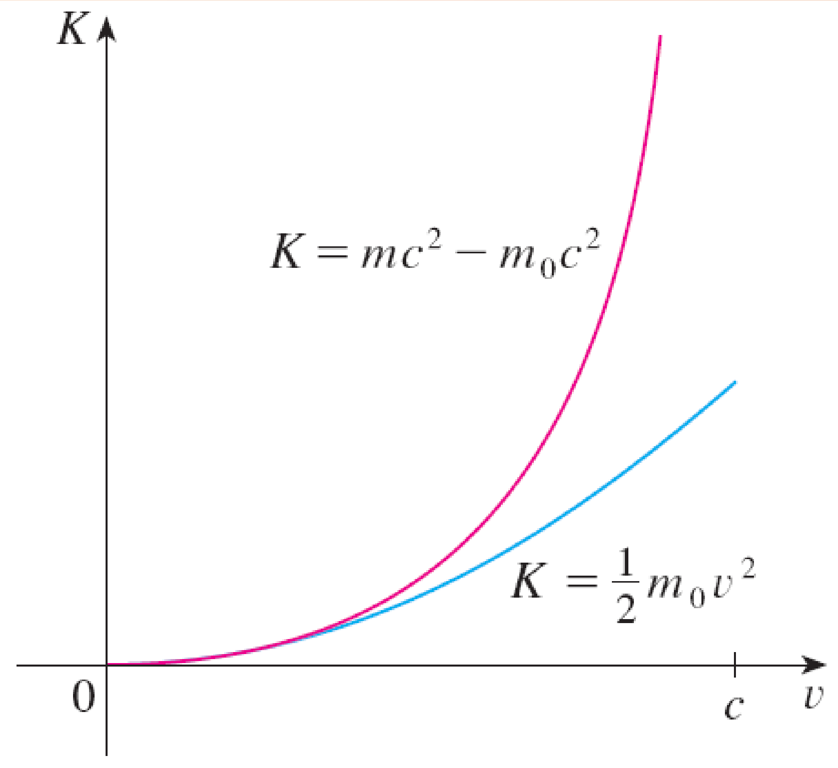
The upper curve in the figure is the graph of the expression for the kinetic energy *K* of an object with velocity in special relativity.



The lower curve shows the function used for *K* in classical Newtonian physics.



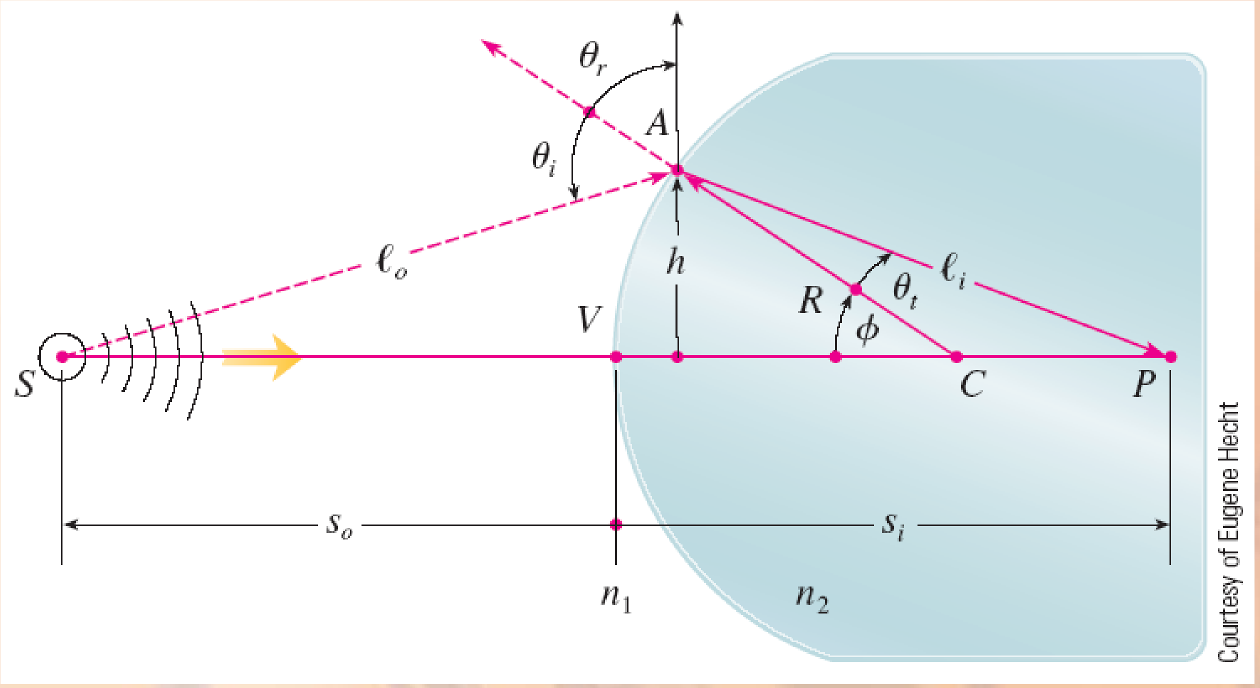
When *v* is much smaller than the speed of light, the curves are practically identical.



OPTICS

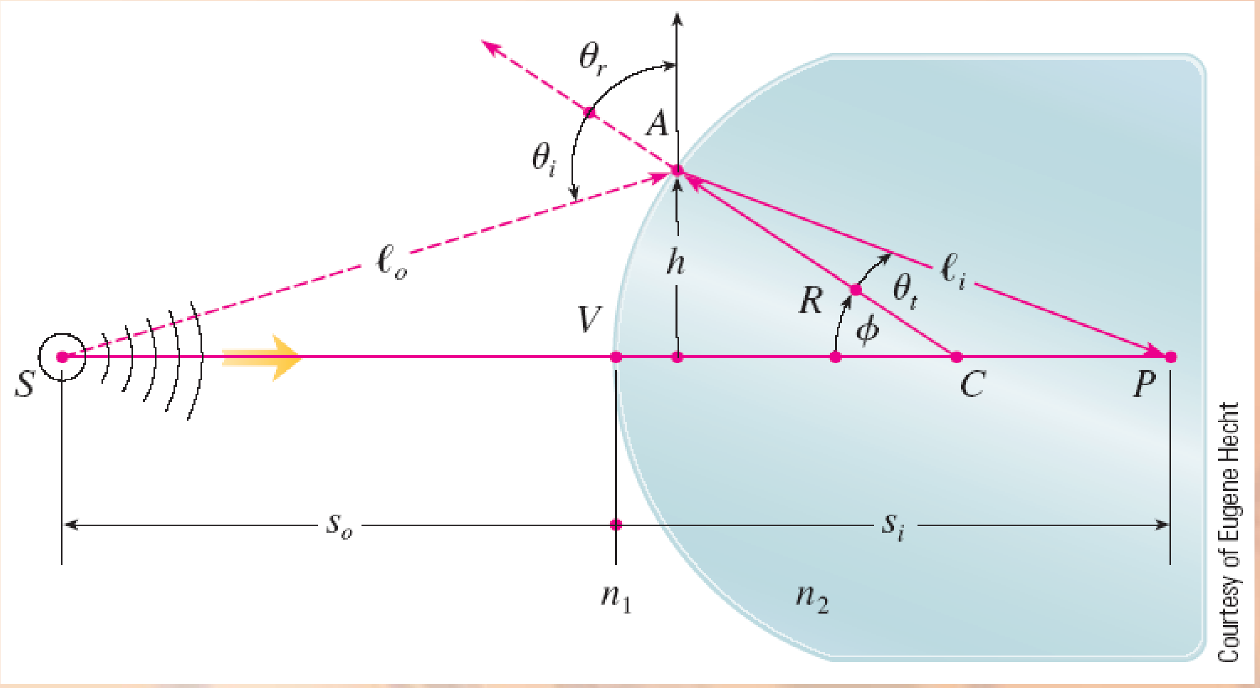
Another application to physics occurs in optics.

This figure is adapted from *Optics*, 4th ed., by Eugene Hecht.



It depicts a wave from the point source *S* meeting a spherical interface of radius *R* centered at *C*.

- The ray *SA* is refracted toward *P*.



Using Fermat's principle that light travels so as to minimize the time taken, Hecht derives the equation

$$\frac{n_1}{\ell_o}+\frac{n_2}{\ell_i}=\frac{1}{R}\left(\frac{n_2 s_i}{\ell_i}-\frac{n_1 s_o}{\ell_o}\right)$$

where:

- n_1 and n_2 are indexes of refraction.
- ℓ_o, ℓ_i, s_o, and s_i are the distances indicated in the figure.

By the Law of Cosines, applied to triangles *ACS* and *ACP*, we have:

$$\ell_o = \sqrt{R^2 + (s_o + R)^2 - 2R(s_o + R)\cos\phi}$$

$$\ell_i = \sqrt{R^2 + (s_i + R)^2 - 2R(s_i + R)\cos\phi}$$

As Equation 1 is cumbersome to work with, Gauss, in 1841, simplified it by using the linear approximation cos ø ≈ 1 for small values of ø.

- This amounts to using the Taylor polynomial of degree 1.

Then, Equation 1 becomes the following simpler equation:

$$\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R}$$

GAUSSIAN OPTICS

The resulting optical theory is known as Gaussian optics, or first-order optics.

- It has become the basic theoretical tool used to design lenses.

A more accurate theory is obtained by approximating cos ø by its Taylor polynomial of degree 3.

- This is the same as the Taylor polynomial of degree 2.

This takes into account rays for which *ø* is not so small—rays that strike the surface at greater distances *h* above the axis.

We use this approximation to derive the more accurate equation

$$\frac{n_1}{s_o}+\frac{n_2}{s_i}$$

$$=\frac{n_2-n_1}{R}+h^2\left[\frac{n_1}{2s_o}\left(\frac{1}{s_o}+\frac{1}{R}\right)^2+\frac{n_2}{2s_i}\left(\frac{1}{R}-\frac{1}{s_i}\right)^2\right]$$

THIRD-ORDER OPTICS

The resulting optical theory is known as third-order optics.