11

INFINITE SEQUENCES AND SERIES

In section 11.9, we were able to find power series representations for a certain restricted class of functions.

Here, we investigate more general problems.

- Which functions have power series representations?
- How can we find such representations?

11.10

Taylor and Maclaurin Series

In this section, we will learn:

How to find the Taylor and Maclaurin Series of a function and to multiply and divide a power series.

We start by supposing that *f* is any function that can be represented by a power series

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \ldots \qquad |x-a| < R$$

Let's try to determine what the coefficients c_n must be in terms of f.

- To begin, notice that, if we put $x = a$ in Equation 1, then all terms after the first one are 0 and we get:

$$f(a) = c_0$$

By Theorem 2 in Section 11.9, we can differentiate the series in Equation 1 term by term:

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \dots \qquad |x-a| < R$$

Substitution of $x = a$ in Equation 2 gives:

$$f'(a) = c_1$$

Equation 3

Now, we differentiate both sides of Equation 2 and obtain:

$$f''(x) = 2c_2 + 2 \cdot 3c_3(x-a) + 3 \cdot 4c_4(x-a)^2 + \ldots \quad |x-a| < R$$

Again, we put $x = a$ in Equation 3.

- The result is:

$$f''(a) = 2c_2$$

Let's apply the procedure one more time.

 Equation 4

Differentiation of the series in Equation 3 gives:

$$f'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x-a) + 3 \cdot 4 \cdot 5c_5(x-a)^2 \ldots \quad |x-a| < R$$

Then, substitution of $x = a$ in Equation 4 gives:

$$f'''(a) = 2 \cdot 3c_3 = 3!c_3$$

By now, you can see the pattern.

- If we continue to differentiate and substitute $x = a$, we obtain:

$$f^{(n)}(a) = 2 \cdot 3 \cdot 4 \cdot \cdots \cdot nc_n = n!c_n$$

Solving the equation for the *n*th coefficient c_{n}, we get:

$$c_n = \frac{f^{(n)}(a)}{n!}$$

The formula remains valid even for $n = 0$ if we adopt the conventions that $0! = 1$ and $f^{(0)} = (f)$.

- Thus, we have proved the following theorem.

Theorem 5

If *f* has a power series representation (expansion) at *a*, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad |x-a| < R$$

then its coefficients are given by:

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Equation 6

Substituting this formula for c_n back into the series, we see that if f has a power series expansion at a, then it must be of the following form.

Equation 6

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots$$

The series in Equation 6 is called the Taylor series of the function *f* at *a* (or about *a* or centered at *a*).

For the special case *a* = 0, the Taylor series becomes:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots$$

This case arises frequently enough that it is given the special name Maclaurin series.

TAYLOR & MACLAURIN SERIES

The Taylor series is named after the English mathematician Brook Taylor (1685–1731).

The Maclaurin series is named for the Scottish mathematician Colin Maclaurin (1698–1746).

- This is despite the fact that the Maclaurin series is really just a special case of the Taylor series.

MACLAURIN SERIES

Maclaurin series are named after Colin Maclaurin because he popularized them in his calculus textbook Treatise of Fluxions published in 1742.

Note

We have shown that if, *f* can be represented as a power series about *a*, then *f* is equal to the sum of its Taylor series.

- However, there exist functions that are not equal to the sum of their Taylor series.
- An example is given in Exercise 70.

Find the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence.

Example 1

If $f(x) = e^x$, then $f^{(n)}(x) = e^x$.

So, $f^{(n)}(0) = e^0 = 1$ for all n.

- Hence, the Taylor series for f at 0 (that is, the Maclaurin series) is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

To find the radius of convergence, we let $a_n = x^n/n!$

- Then, $$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \frac{|x|}{n+1} \to 0 < 1$$
- So, by the Ratio Test, the series converges for all x and the radius of convergence is $R = \infty$.

The conclusion we can draw from Theorem 5 and Example 1 is:

- If e^x has a power series expansion at 0, then

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

So, how can we determine whether e^x does have a power series representation?

Let's investigate the more general question:

- Under what circumstances is a function equal to the sum of its Taylor series?

In other words, if *f* has derivatives of all orders, when is the following true?

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

As with any convergent series, this means that $f(x)$ is the limit of the sequence of partial sums.

In the case of the Taylor series, the partial sums are:

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$+ \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

*n*TH-DEGREE TAYLOR POLYNOMIAL OF *f* AT *a*

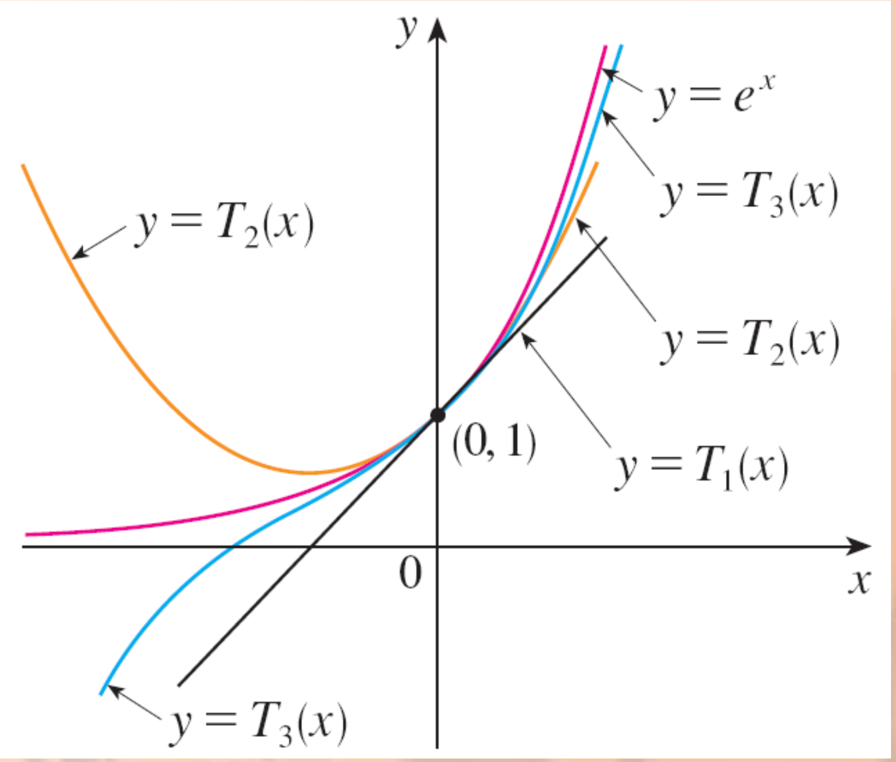
Notice that T_n is a polynomial of degree n called the nth-degree Taylor polynomial of f at a.

For instance, for the exponential function $f(x) = e^x$, the result of Example 1 shows that the Taylor polynomials at 0 (or Maclaurin polynomials) with n = 1, 2, and 3 are:

$$T_1(x) = 1 + x \qquad T_2(x) = 1 + x + \frac{x^2}{2!}$$

$$T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

The graphs of the exponential function and those three Taylor polynomials are drawn here.



In general, $f(x)$ is the sum of its Taylor series if:

$$f(x) = \lim_{n \to \infty} T_n(x)$$

REMAINDER OF TAYLOR SERIES

If we let $R_n(x) = f(x) - T_n(x)$

so that $f(x) = T_n(x) + R_n(x)$

then $R_n(x)$ is called the remainder of the Taylor series.

If we can somehow show that $\lim_{n\to\infty} R_n(x) = 0$, then it follows that:

$$\begin{aligned}\lim_{n\to\infty} T_n(x) &= \lim_{n\to\infty}[f(x) - R_n(x)] \\ &= f(x) - \lim_{n\to\infty} R_n(x) \\ &= f(x)\end{aligned}$$

- Therefore, we have proved the following.

Theorem 8

If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a and

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x - a| < R$, then f is equal to the sum of its Taylor series on the interval $|x - a| < R$.

In trying to show that $\lim_{n \to \infty} R_n(x) = 0$ for a specific function *f*, we usually use the following fact.

TAYLOR'S INEQUALITY   Theorem 9

If $|f^{(n+1)}(x)| \leq M$ for $|x - a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x-a| \leq d$$

TAYLOR'S INEQUALITY

To see why this is true for $n = 1$, we assume that $|f''(x)| \leq M$.

- In particular, we have $f''(x) \leq M$.
- So, for $a \leq x \leq a + d$, we have:

$$\int_a^x f''(t)\,dt \leq \int_a^x M\,dt$$

TAYLOR'S INEQUALITY

- An antiderivative of f'' is f'.

- So, by Part 2 of the Fundamental Theorem of Calculus (FTC2), we have:

$$f'(x) - f'(a) \leq M(x - a)$$

or

$$f'(x) \leq f'(a) + M(x - a)$$

TAYLOR'S INEQUALITY

- Thus,

$$\int_a^x f'(t)\,dt \le \int_a^x [f'(a) + M(t-a)\,dt$$

$$f(x) - f(a) \le f'(a)(x-a) + M\frac{(x-a)^2}{2}$$

$$f(x) - f(a) - f'(a)(x-a) \le \frac{M}{2}(x-a)^2$$

TAYLOR'S INEQUALITY

- However,

$$R_1(x) = f(x) - T_1(x) = f(x) - f(a) - f'(a)(x - a)$$

- So,

$$R_1(x) \leq \frac{M}{2}(x-a)^2$$

TAYLOR'S INEQUALITY

- A similar argument, using $f''(x) \geq -M$, shows that:

$$R_1(x) \geq -\frac{M}{2}(x-a)^2$$

- So,

$$|R_1(x)| \leq \frac{M}{2}|x-a|^2$$

TAYLOR'S INEQUALITY

- We have assumed that $x > a$.

- However, similar calculations show that this inequality is also true for $x < a$.

TAYLOR'S INEQUALITY

This proves Taylor's Inequality for the case where $n = 1$.

- The result for any n is proved in a similar way by integrating $n + 1$ times.
- See Exercise 69 for the case $n = 2$

Note

In Section 11.11, we will explore the use of Taylor's Inequality in approximating functions.

Our immediate use of it is in conjunction with Theorem 8.

TAYLOR'S INEQUALITY

In applying Theorems 8 and 9, it is often helpful to make use of the following fact.

TAYLOR'S INEQUALITY

Equation 10

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad \text{for every real number } x$$

- This is true because we know from Example 1 that the series $\sum x^n/n!$ converges for all x, and so its nth term approaches 0.

Prove that e^x is equal to the sum of its Maclaurin series.

- If $f(x) = e^x$, then $f^{(n+1)}(x) = e^x$ for all n.
- If d is any positive number and $|x| \le d$, then $|f^{(n+1)}(x)| = e^x \le e^d$.

So, Taylor's Inequality, with $a = 0$ and $M = e^d$, says that:

$$|R_n(x)| \leq \frac{e^d}{(n+1)!}|x|^{n+1} \qquad \text{for } |x| \leq d$$

- Notice that the same constant $M = e^d$ works for every value of n.

However, from Equation 10, we have:

$$\lim_{n\to\infty}\frac{e^d}{(n+1)!}|x|^{n+1} = e^d \lim_{n\to\infty}\frac{|x|^{n+1}}{(n+1)!} = 0$$

- It follows from the Squeeze Theorem that $\lim_{n\to\infty}|R_n(x)| = 0$ and so $\lim_{n\to\infty}R_n(x) = 0$

  for all values of *x*.

By Theorem 8, e^x is equal to the sum of its Maclaurin series, that is,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{for all } x$$

Equation 12

In particular, if we put $x = 1$ in Equation 11, we obtain the following expression for the number e as a sum of an infinite series:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

Find the Taylor series for $f(x) = e^x$ at $a = 2$.

- We have $f^{(n)}(2) = e^2$.
- So, putting $a = 2$ in the definition of a Taylor series (Equation 6), we get:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^2 = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^2$$

Again it can be verified, as in Example 1, that the radius of convergence is $R = \infty$.

As in Example 2, we can verify that $\lim_{n \to \infty} R_n(x) = 0$

Thus,

$$e^{x} = \sum_{n=0}^{\infty} \frac{e^{2}}{n!}(x-2)^{n} \qquad \text{for all } x$$

We have two power series expansions for e^x, the Maclaurin series in Equation 11 and the Taylor series in Equation 13.

- The first is better if we are interested in values of x near 0.
- The second is better if x is near 2.

Find the Maclaurin series for sin x and prove that it represents sin x for all x.

We arrange our computation in two columns:

$f(x) = \sin x$	$f(0) = 0$
$f'(x) = \cos x$	$f'(0) = 1$
$f''(x) = -\sin x$	$f''(0) = 0$
$f'''(x) = -\cos x$	$f'''(0) = -1$
$f^{(4)}(x) = \sin x$	$f^{(4)}(0) = 0$

As the derivatives repeat in a cycle of four, we can write the Maclaurin series as follows:

$$f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots$$

$$=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

Since $f^{(n+1)}(x)$ is $\pm\sin x$ or $\pm\cos x$,

we know that $|f^{(n+1)}(x)| \le 1$ for all x.

So, we can take M = 1 in Taylor's Inequality:

$$|R_n(x)| \le \frac{M}{(n+1)!}|x^{n+1}| = \frac{|x|^{n+1}}{(n+1)!}$$

Example 4

By Equation 10, the right side of that inequality approaches 0 as $n \to \infty$.

So, $|R_n(x)| \to 0$ by the Squeeze Theorem.

- It follows that $R_n(x) \to 0$ as $n \to \infty$.
- So, sin x is equal to the sum of its Maclaurin series by Theorem 8.

Equation 15

We state the result of Example 4 for future reference.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

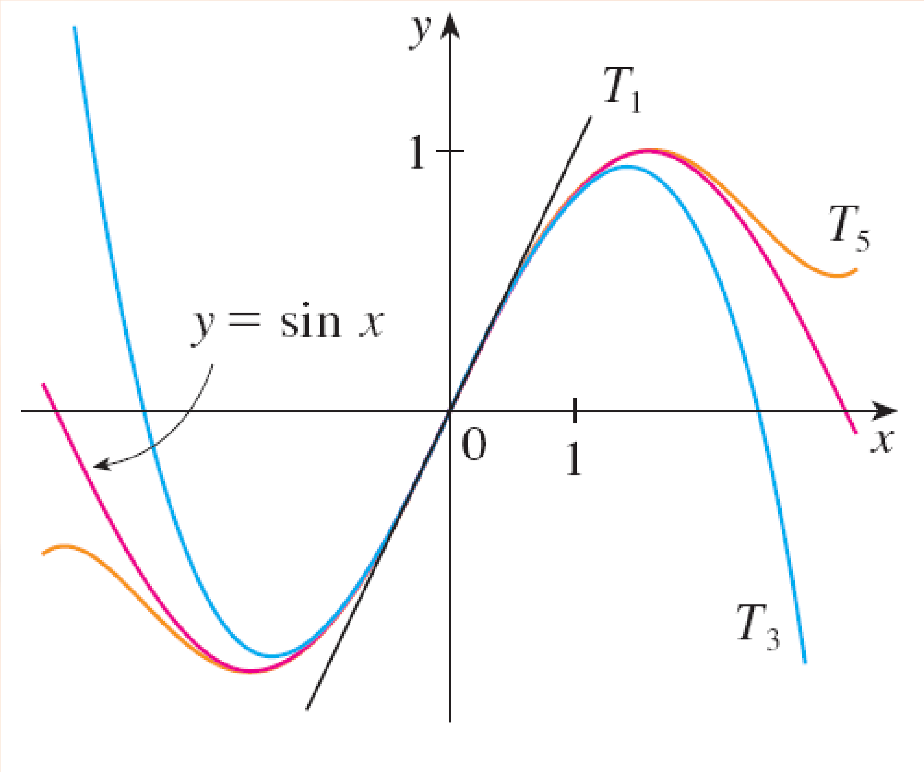
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \text{for all } x$$

The figure shows the graph of sin *x* together with its Taylor (or Maclaurin) polynomials

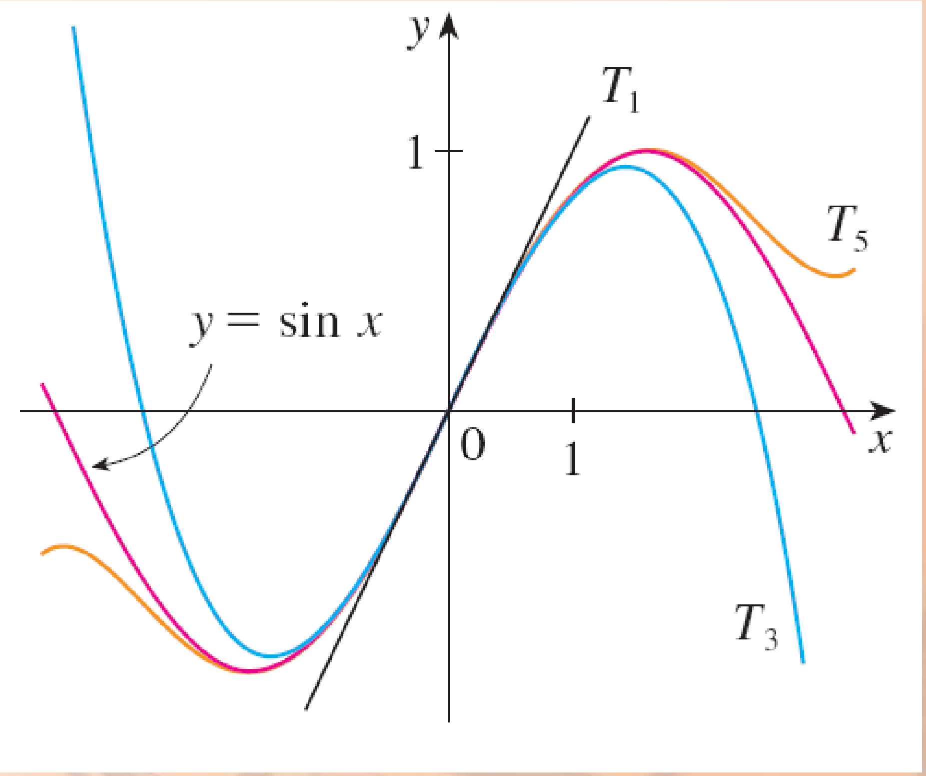
$$T_1(x) = x$$

$$T_3(x) = x - \frac{x^3}{3!}$$

$$T_5 = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$



Notice that, as *n* increases, $T_n(x)$ becomes a better approximation to sin *x*.



Find the Maclaurin series for cos *x*.

- We could proceed directly as in Example 4.
- However, it's easier to differentiate the Maclaurin series for sin *x* given by Equation 15, as follows.

Example 5

$$\cos x = \frac{d}{dx}(\sin x)$$

$$= \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \right)$$

$$= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

The Maclaurin series for sin x converges for all x.

- So, Theorem 2 in Section 11.9 tells us that the differentiated series for cos x also converges for all x.

Thus,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \text{for all } x$$

The Maclaurin series for e^x, sin x, and cos x that we found in Examples 2, 4, and 5 were discovered by Newton.

- These equations are remarkable because they say we know everything about each of these functions if we know all its derivatives at the single number 0.

Example 6

Find the Maclaurin series for the function

$f(x) = x \cos x$.

- Instead of computing derivatives and substituting in Equation 7, it's easier to multiply the series for cos *x* (Equation 16) by *x*:

$$x \cos x = x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1) \frac{x^{2n+1}}{(2n)!}$$

Example 7

Represent $f(x) = \sin x$ as the sum of its Taylor series centered at $\pi/3$.

Arranging our work in columns, we have:

$f(x) = \sin x$	$f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$
$f'(x) = \cos x$	$f'\left(\frac{\pi}{3}\right) = \frac{1}{2}$
$f''(x) = -\sin x$	$f''\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$
$f'''(x) = -\cos x$	$f'''\left(\frac{\pi}{3}\right) = -\frac{1}{2}$

That pattern repeats indefinitely.

Thus, the Taylor series at $\pi/3$ is:

$$f\left(\frac{\pi}{3}\right)+\frac{f'\left(\frac{\pi}{3}\right)}{1!}\left(x-\frac{\pi}{3}\right)+\frac{f''\left(\frac{\pi}{3}\right)}{2!}\left(x-\frac{\pi}{3}\right)^2$$

$$+\frac{f'''\left(\frac{\pi}{3}\right)}{3!}\left(x-\frac{\pi}{3}\right)^3+\cdots$$

$$=\frac{\sqrt{3}}{2}+\frac{1}{2\cdot 1!}\left(x-\frac{\pi}{3}\right)-\frac{\sqrt{3}}{2\cdot 2!}\left(x-\frac{\pi}{3}\right)^2-\frac{1}{2\cdot 3!}\left(x-\frac{\pi}{3}\right)^3+\cdots$$

The proof that this series represents sin x for all x is very similar to that in Example 4.

- Just replace x by $x - \pi/3$ in Equation 14.

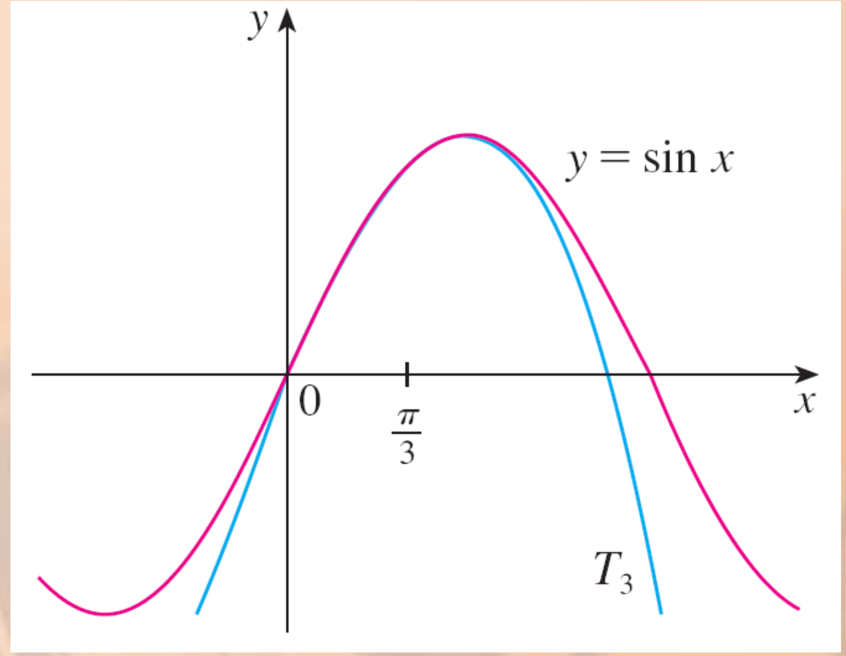
We can write the series in sigma notation if we separate the terms that contain $\sqrt{3}$:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{3}}{2(2n)!} \left(x - \frac{\pi}{3} \right)^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2(2n+1)!} \left(x - \frac{\pi}{3} \right)^{2n+1}$$

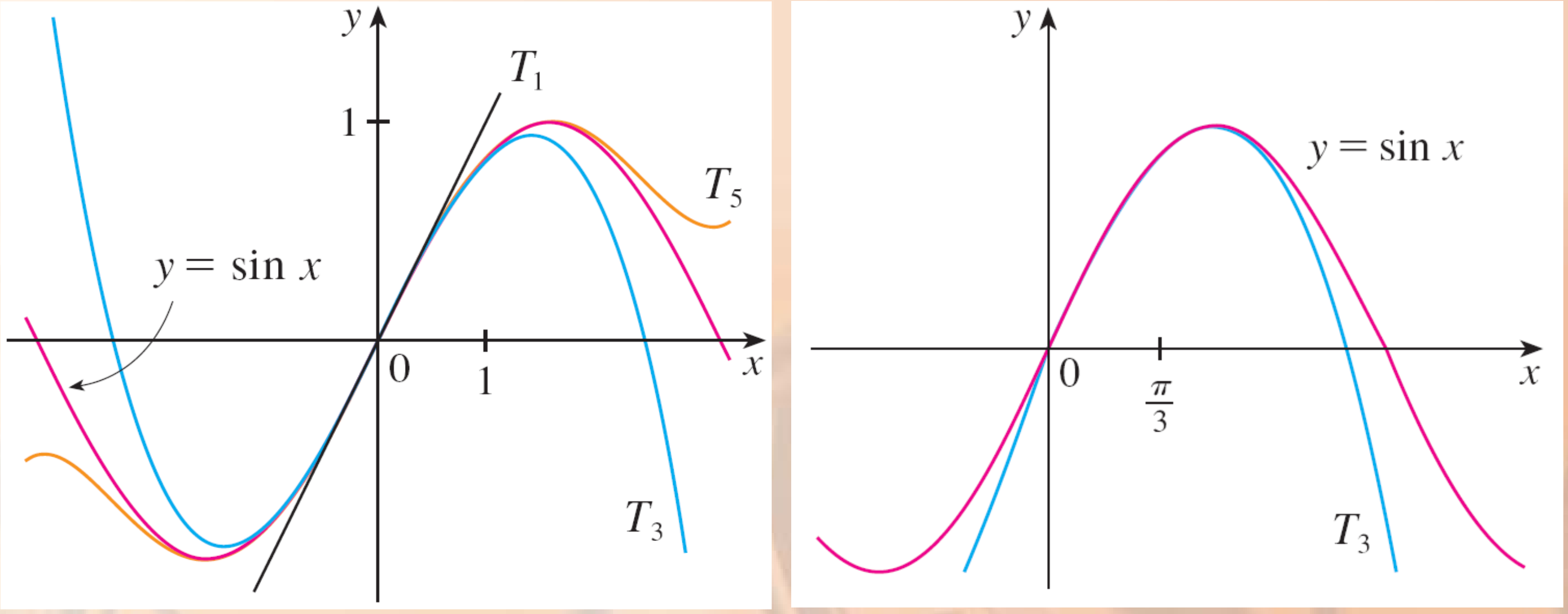
We have obtained two different series representations for sin x, the Maclaurin series in Example 4 and the Taylor series in Example 7.

- It is best to use the Maclaurin series for values of x near 0 and the Taylor series for x near $\pi/3$.

Notice that the third Taylor polynomial T_3 in the figure is a good approximation to sin *x* near $\pi/3$ but not as good near 0.



Compare it with the third Maclaurin polynomial T_3 in the earlier figure—where the opposite is true.



The power series that we obtained by indirect methods in Examples 5 and 6 and in Section 11.9 are indeed the Taylor or Maclaurin series of the given functions.

That is because Theorem 5 asserts that, no matter how a power series representation $f(x) = \sum c_n(x - a)^n$ is obtained, it is always true that $c_n = f^{(n)}(a)/n!$

- In other words, the coefficients are uniquely determined.

Find the Maclaurin series for $f(x) = (1 + x)^k$, where k is any real number.

Arranging our work in columns, we have:

$f(x) = (1+x)^k$	$f(0) = 1$
$f'(x) = k(1+x)^{k-1}$	$f'(0) = k$
$f''(x) = k(k-1)(1+x)^{k-2}$	$f''(0) = k(k-1)$
$f'''(x) = k(k-1)(k-2)(1+x)^{k-3}$	$f'''(0) = k(k-1)(k-2)$
.	.
.	.
.	.
$f^{(n)} = k(k-1)\cdots(k-n+1)(1+x)^{k-n}$	$f^{(n)}(0) = k(k-1)\cdots(k-n+1)$

Thus, the Maclaurin series of $f(x) = (1 + x)^k$ is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n$$

- This series is called the binomial series.

 Example 8

If its *n*th term is a_n, then

$$\left|\frac{a_{n+1}}{a_n}\right|$$

$$= \left|\frac{k(k-1)\cdots(k-n+1)(k-n)x^{n+1}}{(n+1)!} \cdot \frac{n!}{k(k-1)\cdots(k-n+1)x^n}\right|$$

$$= \frac{|k-n|}{n+1}|x| = \frac{\left|1-\frac{k}{n}\right|}{1+\frac{1}{n}}|x| \to |x| \qquad \text{as } n \to \infty$$

Therefore, by the Ratio Test,

the binomial series converges if $|x| < 1$

and diverges if $|x| > 1$.

BINOMIAL COEFFICIENTS.

The traditional notation for the coefficients in the binomial series is:

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

- These numbers are called the binomial coefficients.

The following theorem states that $(1 + x)^k$ is equal to the sum of its Maclaurin series.

- It is possible to prove this by showing that the remainder term $R_n(x)$ approaches 0.
- That, however, turns out to be quite difficult.
- The proof outlined in Exercise 71 is much easier.

THE BINOMIAL SERIES

Theorem 17

If *k* is any real number and $|x| < 1$,

then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

$$= 1 + kx + \frac{k(k-1)}{2!} x^2$$

$$+ \frac{k(k-1)(k-2)}{3!} x^3 + \cdots$$

Though the binomial series always converges when $|x| < 1$, the question of whether or not it converges at the endpoints, ± 1, depends on the value of k.

- It turns out that the series converges at 1 if $-1 < k \leq 0$ and at both endpoints if $k \geq 0$.

Notice that, if k is a positive integer and $n > k$, then the expression for $\binom{k}{n}$ contains a factor $(k - k)$.

So, $\binom{k}{n} = 0$ for $n > k$.

- This means that the series terminates and reduces to the ordinary Binomial Theorem when k is a positive integer.

Example 9

Find the Maclaurin series

for the function $$f(x) = \frac{1}{\sqrt{4-x}}$$

and its radius of convergence.

We write *f*(*x*) in a form where we can use the binomial series:

$$\frac{1}{\sqrt{4-x}} = \frac{1}{\sqrt{4\left(1-\frac{x}{4}\right)}}$$

$$= \frac{1}{2\sqrt{1-\frac{x}{4}}} = \frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2}$$

Using the binomial series with $k = -\frac{1}{2}$ and with x replaced by $-x/4$, we have:

$$\frac{1}{\sqrt{4-x}}$$

$$=\frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2}$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(-\frac{x}{4}\right)^{n}$$

$$= \frac{1}{2}\left[1 + \left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right) + \frac{\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)}{2!}\left(-\frac{x}{4}\right)^2\right.$$

$$+ \frac{\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)\left(-\tfrac{5}{2}\right)}{3!}\left(-\frac{x}{4}\right)^3$$

$$\left. + \cdots + \frac{\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)\left(-\tfrac{5}{2}\right)\cdots\left(-\tfrac{1}{2} - n + 1\right)}{n!}\left(-\frac{x}{4}\right)^n + \cdots\right]$$

$$= \frac{1}{2}\left[1 + \frac{1}{8}x + \frac{1\cdot 3}{2!8^2}x^2 + \frac{1\cdot 3\cdot 5}{3!8^3}x^3 + \right.$$

$$\left. \cdots + \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{n!8^n}x^n + \cdots \right]$$

- We know from Theorem 17 that this series converges when $|-x/4| < 1$, that is, $|x| < 4$.
- So, the radius of convergence is $R = 4$.

For future reference, we collect some important Maclaurin series that we have derived in this section and Section 11.9, in the following table.

IMPORTANT MACLAURIN SERIES **Table 1**

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

IMPORTANT MACLAURIN SERIES **Table 1**

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

IMPORTANT MACLAURIN SERIES Table 1

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

USES OF TAYLOR SERIES

One reason Taylor series are important is that they enable us to integrate functions that we couldn’t previously handle.

USES OF TAYLOR SERIES

In fact, in the introduction to this chapter, we mentioned that Newton often integrated functions by first expressing them as power series and then integrating the series term by term.

USES OF TAYLOR SERIES

The function $f(x) = e^{x^2}$ can't be integrated by techniques discussed so far.

- Its antiderivative is not an elementary function (see Section 7.5).
- In the following example, we use Newton's idea to integrate this function.

a. Evaluate $\int e^{-x^2} dx$ as an infinite series.

b. Evaluate $\int_0^1 e^{-x^2} dx$ correct to within an error of 0.001

First, we find the Maclaurin series for $f(x) = e^{-x^2}$

- It is possible to use the direct method.
- However, let's find it simply by replacing x with $-x^2$ in the series for e^x given in Table 1.

Thus, for all values of *x*,

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

$$= 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots$$

Now, we integrate term by term:

$$\int e^{-x^2}\,dx = \int\left(1-\frac{x^2}{1!}+\frac{x^4}{2!}-\frac{x^6}{3!}+\cdots+(-1)^n\frac{x^{2n}}{n!}+\cdots\right)dx$$

$$= C + x - \frac{x^3}{3\cdot 1!}+\frac{x^5}{5\cdot 2!}-\frac{x^7}{7\cdot 3!}$$

$$+\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)n!}+\cdots$$

- This series converges for all *x* because the original series for e^{-x^2} converges for all *x*.

The FTC gives:

$$\int_0^1 x^{-x^2}\,dx = \left[x - \frac{x^3}{3\cdot 1!} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \frac{x^9}{9\cdot 4!} - \cdots \right]_0^1$$

$$= 1 - \tfrac{1}{3} + \tfrac{1}{10} - \tfrac{1}{42} + \tfrac{1}{216} - \cdots$$

$$\approx 1 - \tfrac{1}{3} + \tfrac{1}{10} - \tfrac{1}{42} + \tfrac{1}{216}$$

$$\approx 0.7475$$

The Alternating Series Estimation Theorem shows that the error involved in this approximation is less than

$$\frac{1}{11 \cdot 5!} = \frac{1}{1320} < 0.001$$

USES OF TAYLOR SERIES

Another use of Taylor series is illustrated in the next example.

- The limit could be found with l’Hospital’s Rule.
- Instead, we use a series.

Evaluate $$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$$

- Using the Maclaurin series for e^x, we have the following result.

Example 11

$$\lim_{x\to 0}\frac{e^x-1-x}{x^2}=\lim_{x\to 0}\frac{\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)-1-x}{x^2}$$

$$=\lim_{x\to 0}\frac{\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}\cdots}{x^2}$$

$$=\lim_{x\to 0}\left(\frac{1}{2}+\frac{x}{3!}+\frac{x^2}{4!}+\frac{x^3}{5!}\right)=\frac{1}{2}$$

- This is because power series are continuous functions.

MULTIPLICATION AND DIVISION OF POWER SERIES

If power series are added or subtracted, they behave like polynomials.

- Theorem 8 in Section 11.2 shows this.
- In fact, as the following example shows, they can also be multiplied and divided like polynomials.

In the example, we find only the first few terms.

- The calculations for the later terms become tedious.
- The initial terms are the most important ones.

 Example 12

Find the first three nonzero terms in the Maclaurin series for:

a. $e^x \sin x$

b. $\tan x$

Using the Maclaurin series for e^x and sin x in Table 1, we have:

$$e^x \sin x = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3!} + \cdots\right)$$

We multiply these expressions, collecting like terms just as for polynomials:

$$
\begin{array}{rl}
 & 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots \\
\times & x \qquad\qquad - \frac{1}{6}x^3 + \cdots \\
\hline
 & x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \cdots \\
+ & - \frac{1}{6}x^3 - \frac{1}{6}x^4 + \cdots \\
 & x + x^2 + \frac{1}{3}x^3 + \cdots
\end{array}
$$

Thus,

$$e^x \sin x = x + x^2 + \frac{1}{3}x^3 + \cdots$$

Using the Maclaurin series in Table 1, we have:

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots}{1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots}$$

We use a procedure like long division:

$$
\begin{array}{r|l}
 & x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots \\
\hline
1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots & x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots \\
 & x - \frac{1}{2}x^3 + \frac{1}{24}x^5 - \cdots \\
\cline{2-2}
 & \frac{1}{3}x^3 - \frac{1}{30}x^5 - \cdots \\
 & \frac{1}{3}x^3 - \frac{1}{6}x^5 - \cdots \\
\cline{2-2}
 & \frac{1}{6}x^5 - \cdots
\end{array}
$$

Thus,

$$\tan x = x + \tfrac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$$

Although we have not attempted to justify the formal manipulations used in Example 12, they are legitimate.

MULTIPLICATION AND DIVISION

There is a theorem that states the following:

- Suppose both $f(x) = \Sigma c_n x^n$ and $g(x) = \Sigma b_n x^n$ converge for $|x| < R$ and the series are multiplied as if they were polynomials.
- Then, the resulting series also converges for $|x| < R$ and represents $f(x)g(x)$.

For division, we require $b_0 \neq 0$.

- The resulting series converges for sufficiently small *x*.