11

INFINITE SEQUENCES AND SERIES

Infinite sequences and series were introduced briefly in A Preview of Calculus in connection with Zeno's paradoxes and the decimal representation of numbers.

Their importance in calculus stems from Newton's idea of representing functions as sums of infinite series.

- For instance, in finding areas, he often integrated a function by first expressing it as a series and then integrating each term of the series.

We will pursue his idea in Section 11.10 in order to integrate such functions as e^{-x^2}.

- Recall that we have previously been unable to do this.

Many of the functions that arise in mathematical physics and chemistry, such as Bessel functions, are defined as sums of series.

- It is important to be familiar with the basic concepts of convergence of infinite sequences and series.

Physicists also use series in another way, as we will see in Section 11.11

- In studying fields as diverse as optics, special relativity, and electromagnetism, they analyze phenomena by replacing a function with the first few terms in the series that represents it.

11.1
Sequences

In this section, we will learn about:

Various concepts related to sequences.

SEQUENCE

A sequence can be thought of as a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

- The number a_1 is called the first term, a_2 is the second term, and in general a_n is the *n*th term.

We will deal exclusively with infinite sequences.

- So, each term a_n will have a successor a_{n+1}.

SEQUENCES

Notice that, for every positive integer n, there is a corresponding number a_n.

So, a sequence can be defined as:

- A function whose domain is the set of positive integers

However, we usually write a_n instead of the function notation $f(n)$ for the value of the function at the number n.

The sequence $\{ a_1, a_2, a_3, . . .\}$ is also denoted by:

$$\{a_n\} \quad \text{or} \quad \{a_n\}_{n=1}^{\infty}$$

Some sequences can be defined by giving a formula for the *n*th term.

Example 1

In the following examples, we give three descriptions of the sequence:

1. Using the preceding notation
2. Using the defining formula
3. Writing out the terms of the sequence

Example 1 a

Preceding Notation	Defining Formula	Terms of Sequence
$\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$	$a_n = \frac{n}{n+1}$	$\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots\right\}$

- In this and the subsequent examples, notice that *n* doesn't have to start at 1.

Example 1 b

Preceding Notation	Defining Formula	Terms of Sequence
$\left\{ \frac{(-1)^n (n+1)}{3^n} \right\}$	$a_n = \frac{(-1)^n (n+1)}{3^n}$	$\left\{ -\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \frac{5}{81}, \ldots, \frac{(-1)^n (n+1)}{3^n}, \ldots \right\}$

Example 1 c

Preceding Notation	Defining Formula	Terms of Sequence
$\left\{\sqrt{n-3}\right\}_{n=3}^{\infty}$	$a_n = \sqrt{n-3}$ $(n \geq 3)$	$\left\{0, 1, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n-3}, \ldots\right\}$

Example 1 d

Preceding Notation	Defining Formula	Terms of Sequence
$\left\{\cos\frac{n\pi}{6}\right\}_{n=0}^{\infty}$	$a_n = \cos\frac{n\pi}{6}$ $(n \geq 0)$	$\left\{1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, \ldots, \cos\frac{n\pi}{6}, \ldots\right\}$

Example 2

Find a formula for the general term a_n of the sequence

$$\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \ldots\right\}$$

assuming the pattern of the first few terms continues.

We are given that:

$$a_1 = \frac{3}{5} \qquad a_2 = -\frac{4}{25} \qquad a_3 = \frac{5}{125}$$

$$a_4 = -\frac{6}{625} \qquad a_5 = \frac{7}{3125}$$

Notice that the numerators of these fractions start with 3 and increase by 1 whenever we go to the next term.

- The second term has numerator 4 and the third term has numerator 5.
- In general, the *n*th term will have numerator $n+2$.

The denominators are the powers of 5.

- Thus, a_n has denominator 5^n.

The signs of the terms are alternately positive and negative.

- Hence, we need to multiply by a power of –1.

In Example 1 b, the factor $(–1)^n$ meant we started with a negative term.

Here, we want to start with a positive term.

Thus, we use $(-1)^{n-1}$ or $(-1)^{n+1}$.

- Therefore,

$$a_n = (-1)^{n-1} \frac{n+2}{5^n}$$

We now look at some sequences that don't have a simple defining equation.

The sequence $\{p_n\}$, where p_n is the population of the world as of January 1 in the year n

If we let a_n be the digit in the nth decimal place of the number e, then $\{a_n\}$ is a well-defined sequence whose first few terms are:

$$\{7, 1, 8, 2, 8, 1, 8, 2, 8, 4, 5, \ldots\}$$

The Fibonacci sequence $\{f_n\}$ is defined recursively by the conditions

$$f_1 = 1 \qquad f_2 = 1 \qquad f_n = f_{n-1} + f_{n-2} \qquad n \geq 3$$

- Each is the sum of the two preceding term terms.
- The first few terms are:

  {1, 1, 2, 3, 5, 8, 13, 21, …}

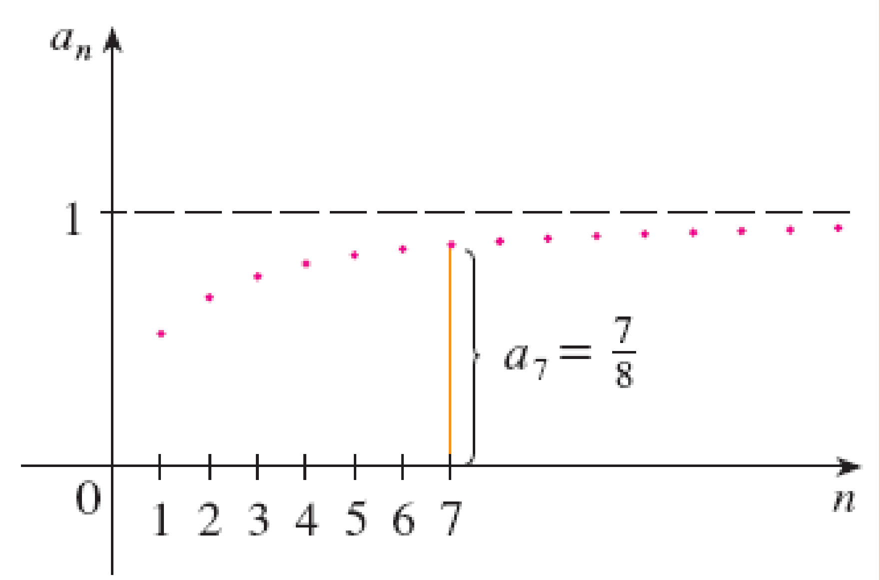
This sequence arose when the 13th-century Italian mathematician Fibonacci solved a problem concerning the breeding of rabbits.

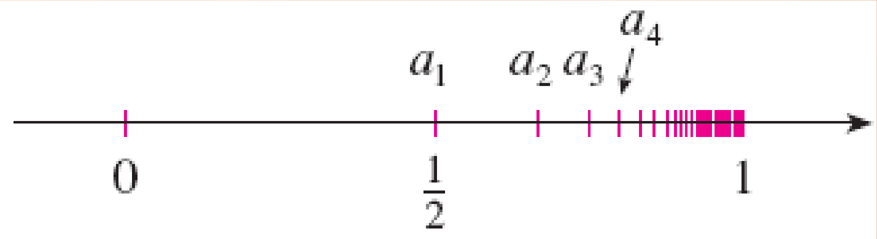
- See Exercise 71.

SEQUENCES

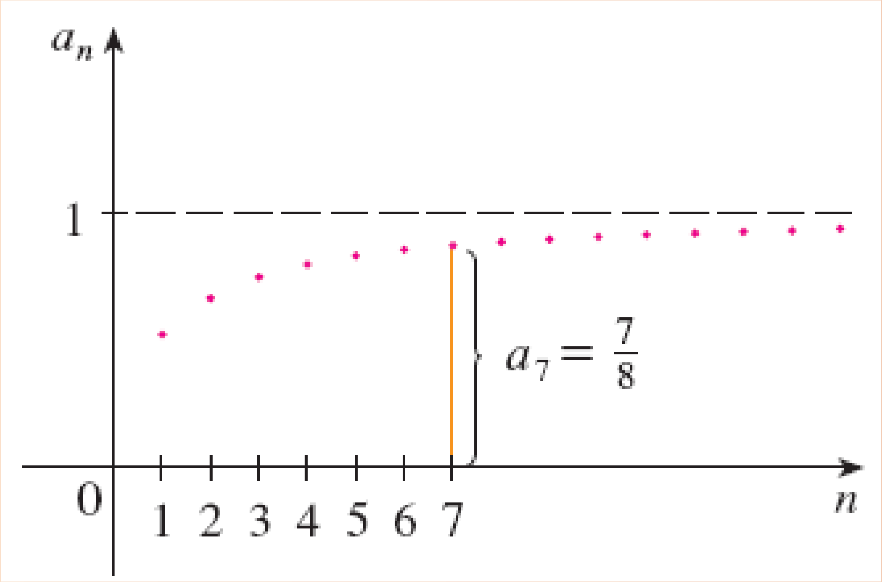
A sequence such as that in Example 1 a [$a_n = n/(n + 1)$] can be pictured either by:

- Plotting its terms on a number line
- Plotting its graph





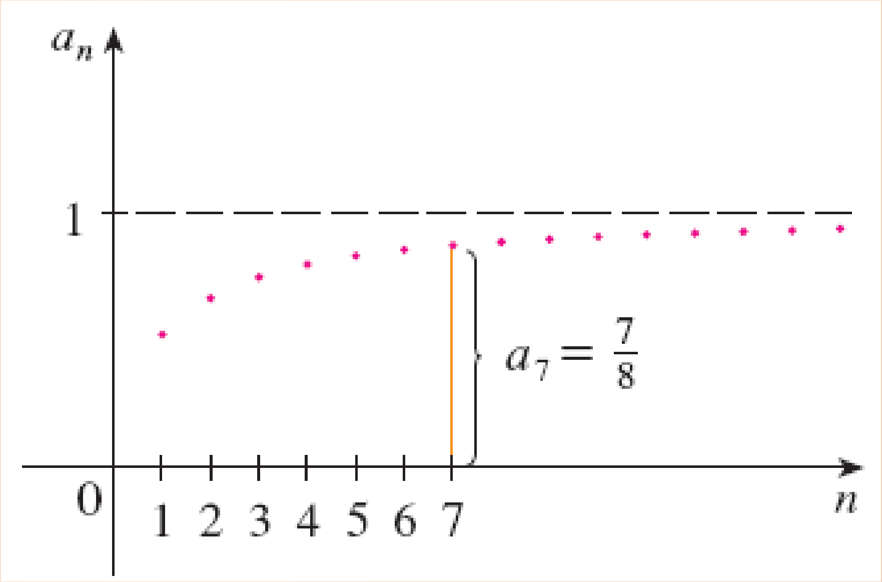
Note that, since a sequence is a function whose domain is the set of positive integers,

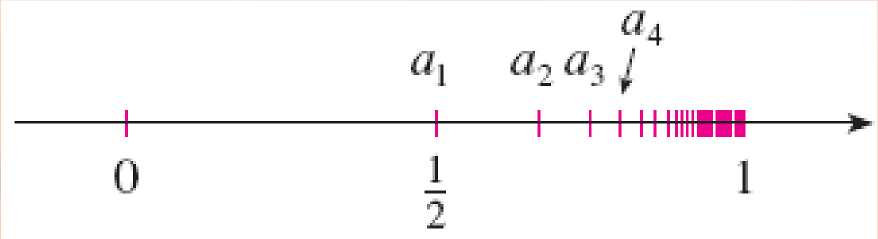


its graph consists of isolated points with coordinates

$$(1, a_1) \quad (2, a_2) \quad (3, a_3) \quad \dots \quad (n, a_n) \quad \dots$$

From either figure, it appears that the terms of the sequence $a_n = n/(n + 1)$ are approaching 1 as n becomes large.





In fact, the difference $1 - \dfrac{n}{n+1} = \dfrac{1}{n+1}$

can be made as small as we like by taking *n* sufficiently large.

- We indicate this by writing

$$\lim_{n \to \infty} \frac{n}{n+1} = 1$$

In general, the notation

$$\lim_{n\to\infty} a_n = L$$

means that the terms of the sequence $\{a_n\}$ approach L as n becomes large.

Notice that the following definition of the limit of a sequence is very similar to the definition of a limit of a function at infinity given in Section 2.6

LIMIT OF A SEQUENCE

Definition 1

A sequence $\{a_n\}$ has the limit L, and we write

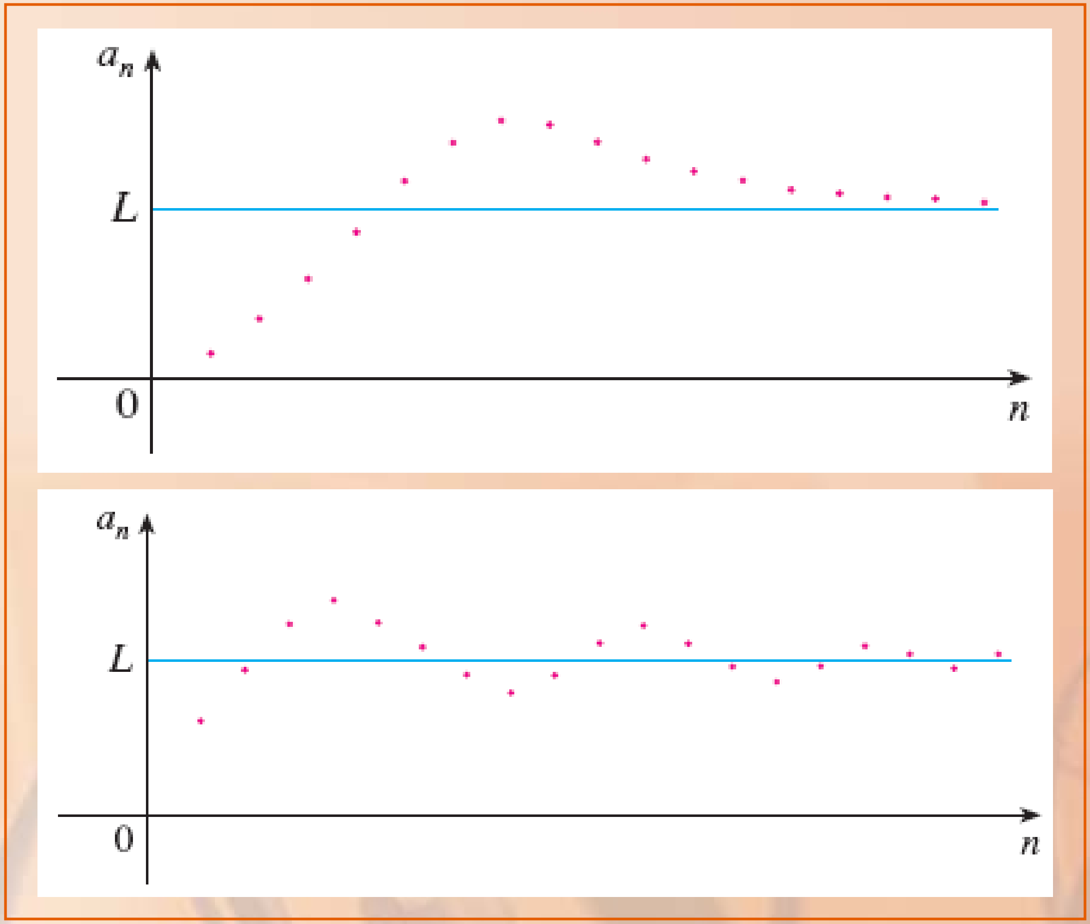
$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like, by taking n sufficiently large.

- If $\lim_{n\to\infty} a_n$ exists, the sequence converges (or is convergent).
- Otherwise, it diverges (or is divergent).

LIMIT OF A SEQUENCE

Here, Definition 1 is illustrated by showing the graphs of two sequences that have the limit *L*.



A more precise version of Definition 1 is as follows.

Definition 2

A sequence $\{a_n\}$ has the limit L, and we write

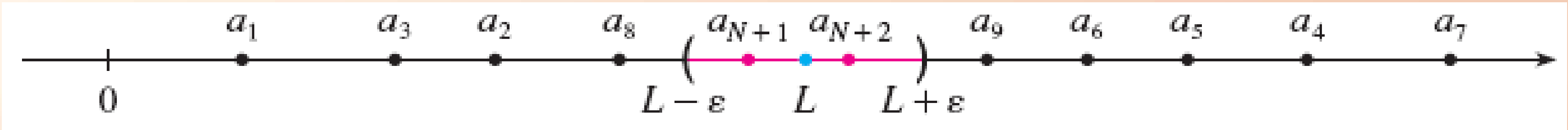
$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if for every $\varepsilon > 0$ there is a corresponding integer N such that:

$$\text{if} \qquad n > N \qquad \text{then} \qquad |a_n - L| < \varepsilon$$

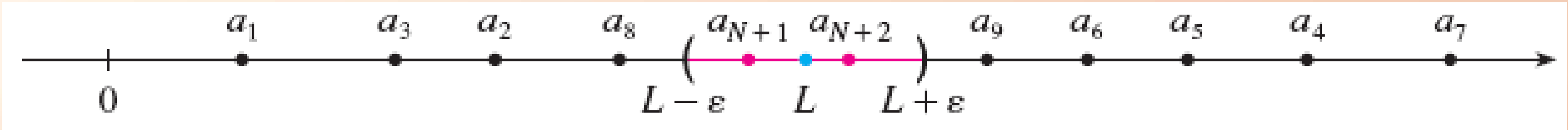
LIMIT OF A SEQUENCE

Definition 2 is illustrated by the figure, in which the terms a_1, a_2, a_3, . . . are plotted on a number line.



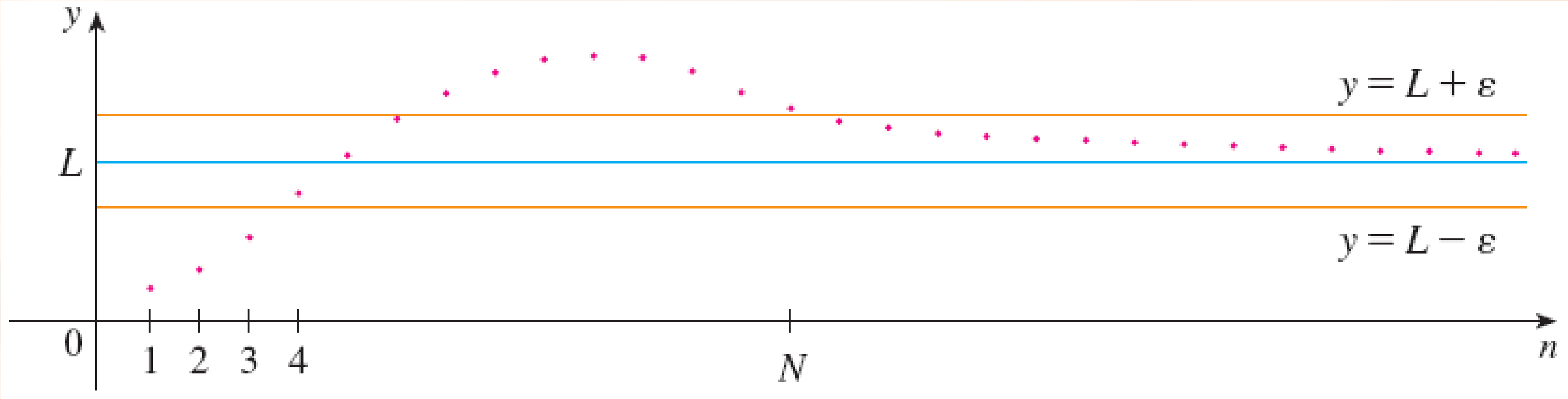
LIMIT OF A SEQUENCE

No matter how small an interval ($L - \varepsilon$, $L + \varepsilon$) is chosen, there exists an N such that all terms of the sequence from a_{N+1} onward must lie in that interval.



Another illustration of Definition 2 is given here.

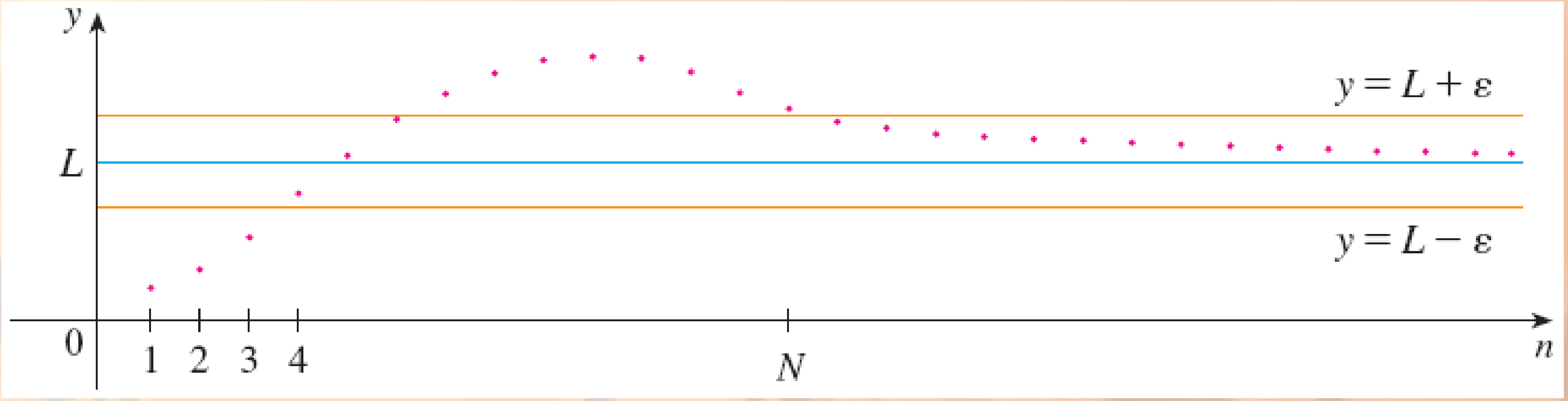
- The points on the graph of $\{a_n\}$ must lie between the horizontal lines $y = L + \varepsilon$ and $y = L - \varepsilon$ if $n > N$.



LIMIT OF A SEQUENCE

This picture must be valid no matter how small *ε* is chosen.

- Usually, however, a smaller *ε* requires a larger *N*.



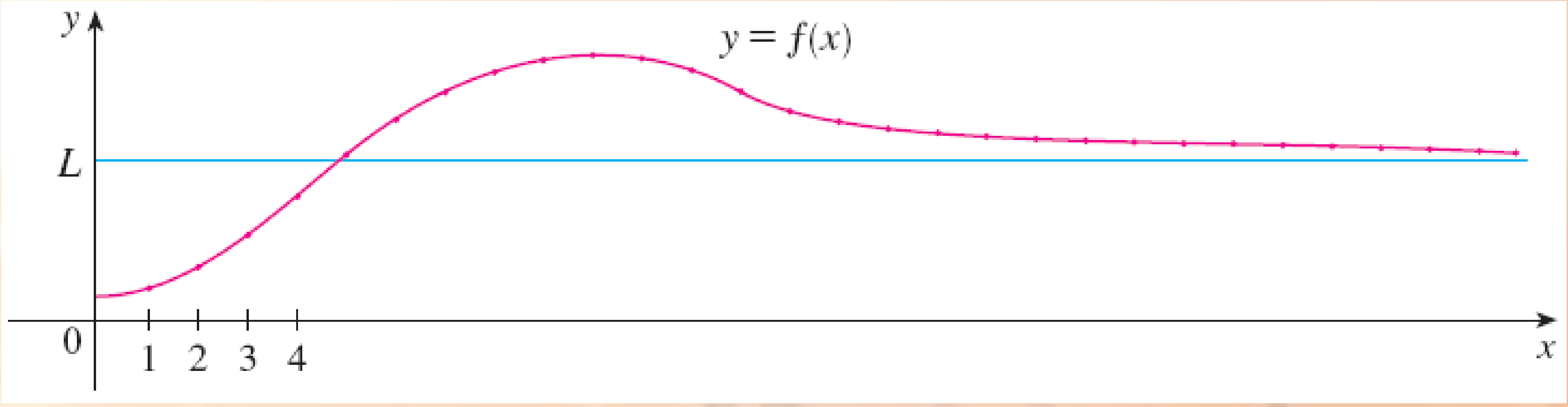
If you compare Definition 2 with Definition 7 in Section 2.6, you will see that the only difference between $\lim_{n\to\infty} a_n = L$ and $\lim_{x\to\infty} f(x) = L$ is that *n* is required to be an integer.

- Thus, we have the following theorem.

If $\lim_{x \to \infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then

$$\lim_{n \to \infty} a_n = L$$

Theorem 3 is illustrated here.



In particular, since we know that

$\lim_{x\to\infty}(1/x^r)=0$ when $r > 0$ (Theorem 5 in Section 2.6), we have:

$$\lim_{n\to\infty}\frac{1}{n^r}=0 \qquad \text{if } r>0$$

If a_n becomes large as n becomes large, we use the notation

$$\lim_{n\to\infty} a_n = \infty$$

- The following precise definition is similar to Definition 9 in Section 2.6

LIMIT OF A SEQUENCE

Definition 5

$\lim_{n\to\infty} a_n = \infty$ means that, for every positive number *M*, there is an integer *N* such that

if $n > N$ then $a_n > M$

If $\lim_{n\to\infty} a_n = \infty$, then the sequence $\{a_n\}$ is divergent, but in a special way.

- We say that $\{a_n\}$ diverges to ∞.

The Limit Laws given in Section 2.3 also hold for the limits of sequences and their proofs are similar.

LIMIT LAWS FOR SEQUENCES

Suppose $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant.

LIMIT LAWS FOR SEQUENCES

Then,

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n \qquad \lim_{n\to\infty} c = c$$

LIMIT LAWS FOR SEQUENCES

Also,

$$\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if} \quad \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^P = \left[\lim_{n\to\infty} a_n\right]^P \quad \text{if} \;\; p > 0 \text{ and } a_n > 0$$

The Squeeze Theorem can also be adapted for sequences, as follows.

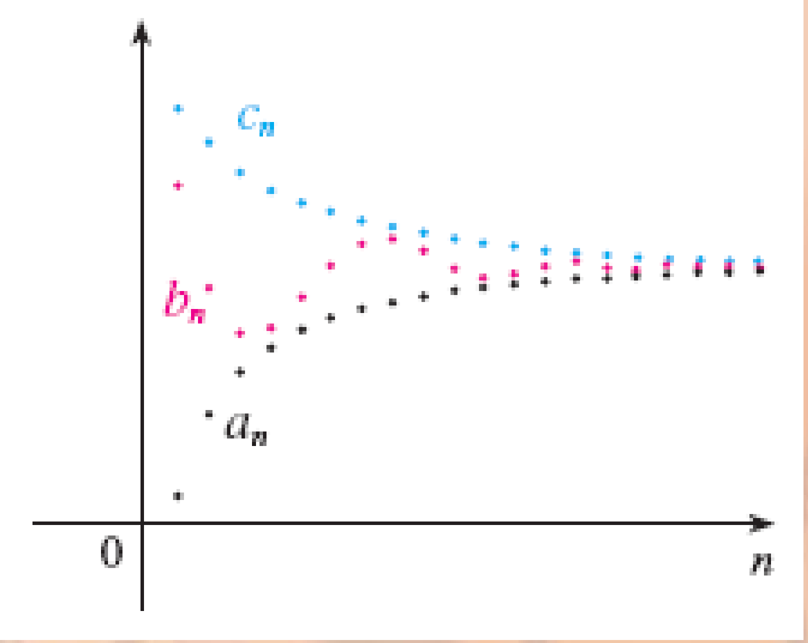
SQUEEZE THEOREM FOR SEQUENCES

If $a_n \leq b_n \leq c_n$ for $n \geq n_0$

and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$,

then

$$\lim_{n\to\infty} b_n = L$$



LIMITS OF SEQUENCES

Another useful fact about limits of sequences is given by the following theorem.

- The proof is left as Exercise 75.

Theorem 6

If $\lim_{n\to\infty} |a_n| = 0$

then $\lim_{n\to\infty} a_n = 0$

Find $\lim_{n\to\infty} \dfrac{n}{n+1}$

- The method is similar to the one we used in Section 2.6
- We divide the numerator and denominator by the highest power of *n* and then use the Limit Laws.

Thus,

$$\lim_{n\to\infty}\frac{n}{n+1}=\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}=\frac{\lim_{n\to\infty}1}{\lim_{n\to\infty}1+\lim_{n\to\infty}\frac{1}{n}}$$

$$=\frac{1}{1+0}=1$$

- Here, we used Equation 4 with $r = 1$.

Calculate $\lim_{n\to\infty} \frac{\ln n}{n}$

- Notice that both the numerator and denominator approach infinity as $n \to \infty$.

Here, we can’t apply l’Hospital’s Rule directly.

- It applies not to sequences but to functions of a real variable.

However, we can apply l'Hospital's Rule to the related function $f(x) = (\ln x)/x$ and obtain:

$$\lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{1/x}{1} = 0$$

Therefore, by Theorem 3, we have:

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0$$

 Example 6

Determine whether the sequence $a_n = (-1)^n$ is convergent or divergent.

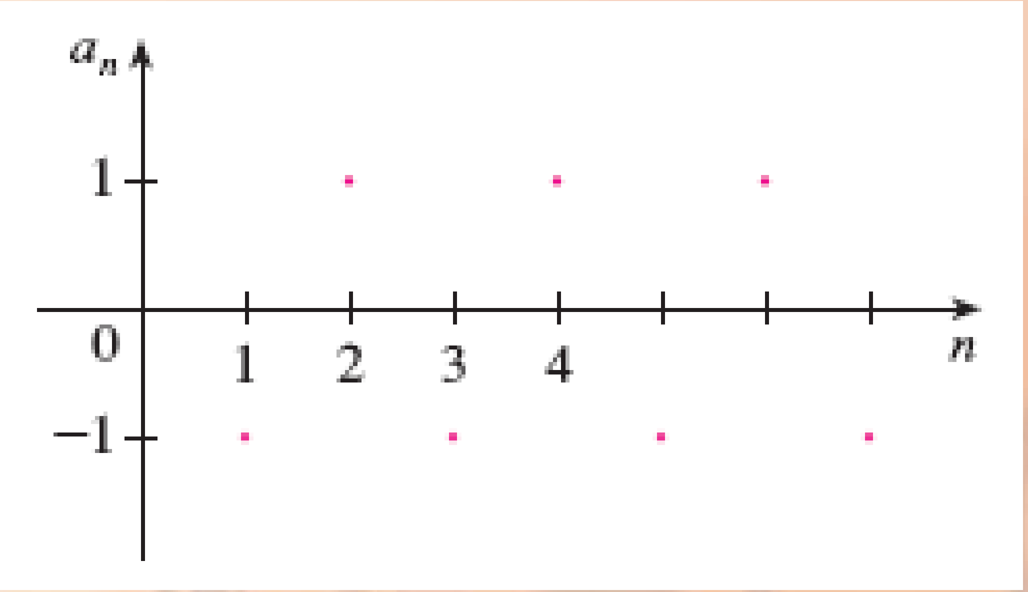
Example 6

If we write out the terms of the sequence, we obtain:

$$\{-1, 1, -1, 1, -1, 1, -1, \ldots\}$$

The graph of the sequence is shown.

- The terms oscillate between 1 and –1 infinitely often.
- Thus, a_n does not approach any number.



Thus, $\lim_{n\to\infty}(-1)^n$ does not exist.

That is, the sequence $\{(-1)^n\}$ is divergent.

Evaluate $\lim_{n\to\infty} \frac{(-1)^n}{n}$ if it exists.

- $\lim_{n\to\infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n\to\infty} \frac{1}{n} = 0$
- Thus, by Theorem 6, $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$

The following theorem says that, if we apply a continuous function to the terms of a convergent sequence, the result is also convergent.

If $\lim_{n\to\infty} a_n = L$ and the function *f* is continuous at *L*, then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

- The proof is left as Exercise 76.

Find $\lim_{n\to\infty} \sin(\pi / n)$

- The sine function is continuous at 0.
- Thus, Theorem 7 enables us to write:

$$\lim_{n\to\infty} \sin(\pi / n) = \sin\left(\lim_{n\to\infty}(\pi / n)\right)$$

$$= \sin 0 = 0$$

Example 9

Discuss the convergence of the sequence $a_n = n!/n^n$, where $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$

Both the numerator and denominator approach infinity as $n \to \infty$.

However, here, we have no corresponding function for use with l'Hospital's Rule.

- $x!$ is not defined when x is not an integer.

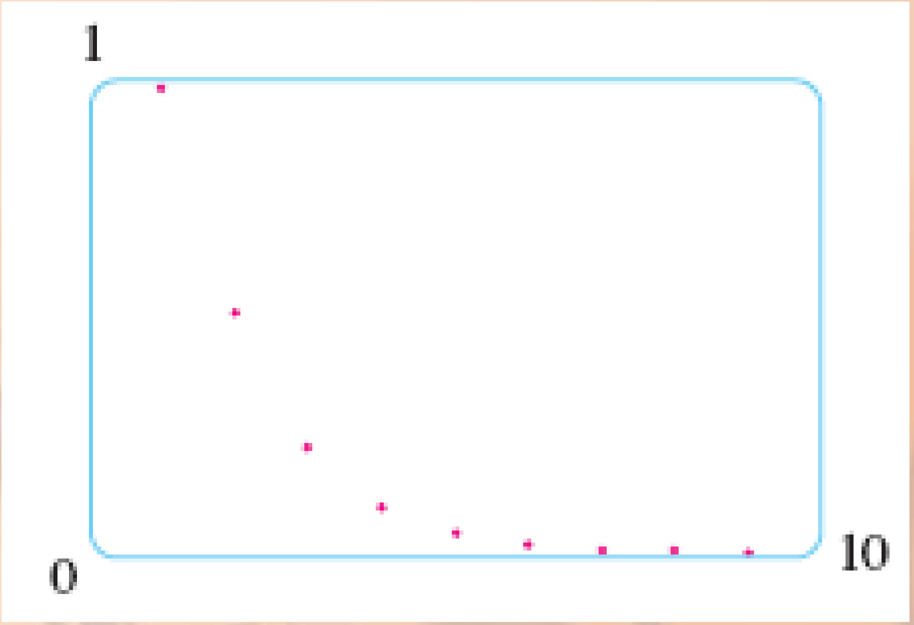
Let's write out a few terms to get a feeling for what happens to a_n as n gets large:

$$a_1 = 1 \qquad a_2 = \frac{1\cdot 2}{2\cdot 2} \qquad a_3 = \frac{1\cdot 2\cdot 3}{3\cdot 3\cdot 3}$$

Therefore,

$$a_n = \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot n \cdot \cdots \cdot n}$$

From these expressions and the graph here, it appears that the terms are decreasing and perhaps approach 0.



To confirm this, observe from Equation 8 that

$$a_n = \frac{1}{n}\left(\frac{2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot \cdots \cdot n}\right)$$

- Notice that the expression in parentheses is at most 1 because the numerator is less than (or equal to) the denominator.

Thus, $0 < a_n \leq \dfrac{1}{n}$

- We know that $1/n \to 0$ as $n \to \infty$.
- Therefore $a_n \to 0$ as $n \to \infty$ by the Squeeze Theorem.

For what values of *r* is the sequence $\{r^n\}$ convergent?

- From Section 2.6 and the graphs of the exponential functions in Section 1.5, we know that $\lim_{x \to \infty} a^x = \infty$ for *a* > 1 and $\lim_{x \to \infty} a^x = 0$ for 0 < *a* < 1.

Thus, putting *a* = *r* and using Theorem 3, we have:

$$\lim_{n\to\infty} r^n = \begin{cases} \infty & \text{if } r > 1 \\ 0 & \text{if } 0 < r < 1 \end{cases}$$

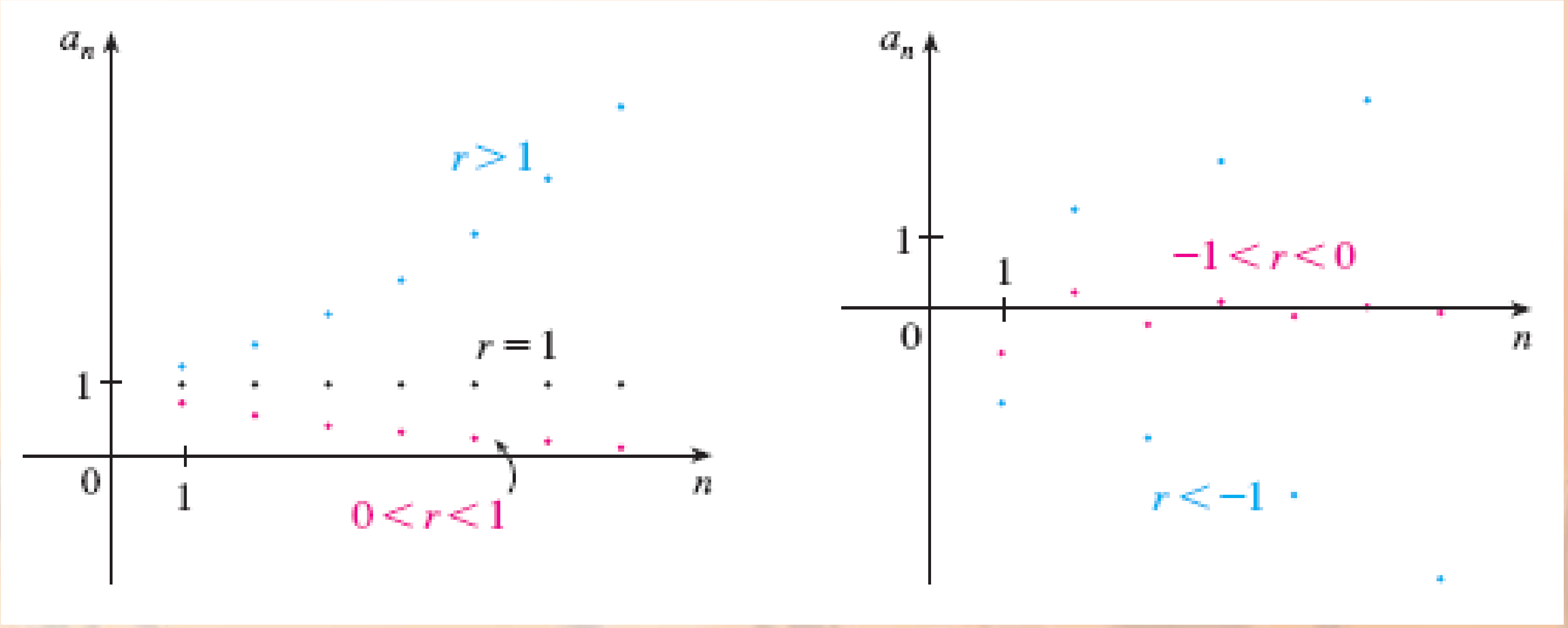
- It is obvious that

$$\lim_{n\to\infty} 1^n = 1 \quad \text{and} \quad \lim_{n\to\infty} 0^n = 0$$

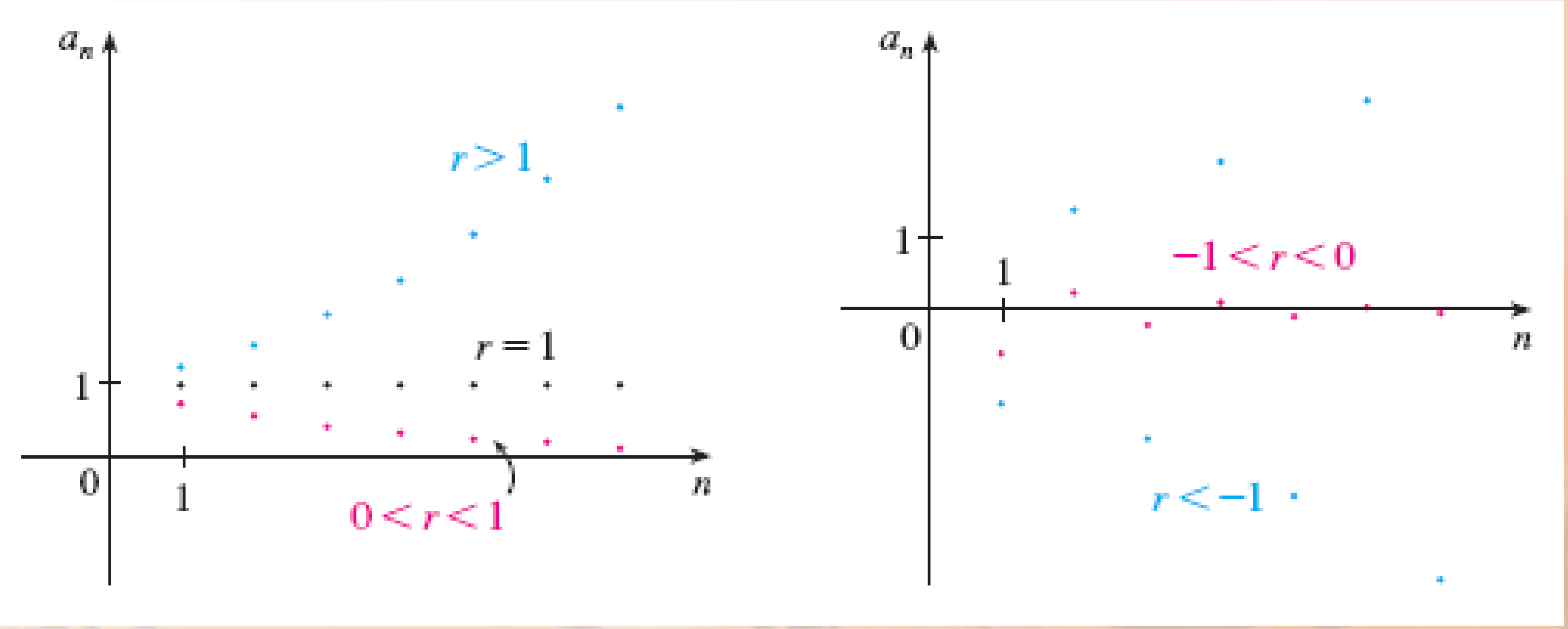
If $-1 < r < 0$, then $0 < |r| < 1$.

- Thus, $\lim_{n\to\infty} |r^n| = \lim_{n\to\infty} |r|^n = 0$
- Therefore, by Theorem 6, $\lim_{n\to\infty} r^n = 0$

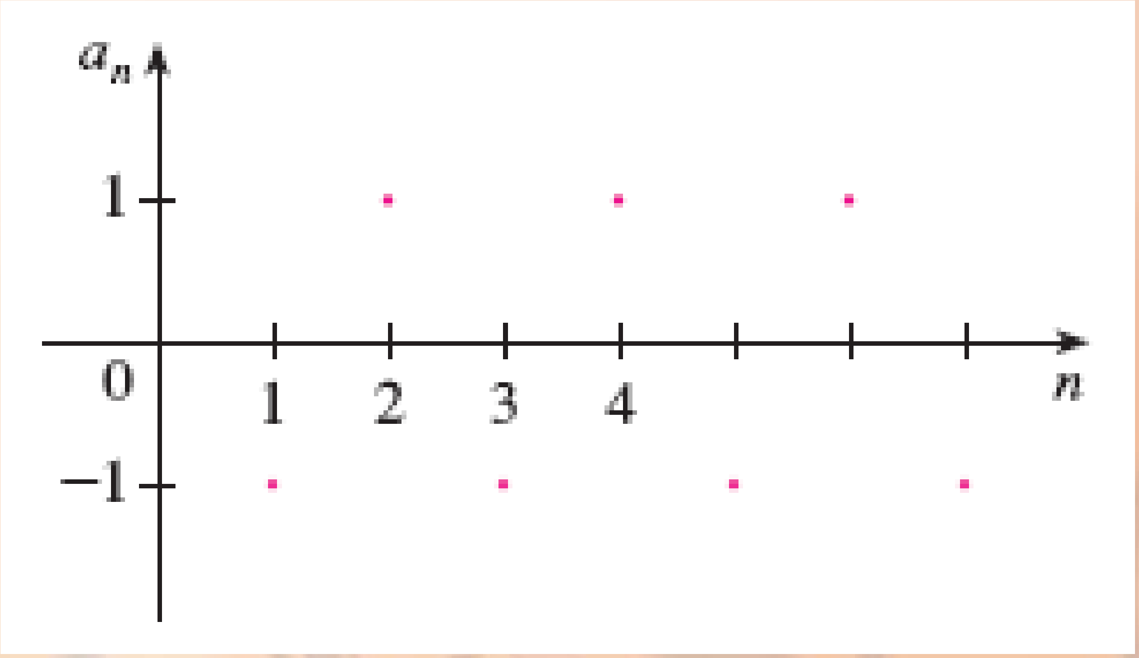
If $r \leq -1$, then $\{r^n\}$ diverges as in Example 6.



The figure shows the graphs for various values of *r*.



The case $r = -1$ was shown earlier.



The results of Example 10 are summarized for future use, as follows.

The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

A sequence $\{a_n\}$ is called:

- Increasing, if $a_n < a_{n+1}$ for all $n \geq 1$, that is, $a_1 < a_2 < a_3 < \cdot \cdot \cdot$
- Decreasing, if $a_n > a_{n+1}$ for all $n \geq 1$
- Monotonic, if it is either increasing or decreasing

The sequence $\left\{\frac{3}{n+5}\right\}$ is decreasing because

$$\frac{3}{n+5} > \frac{3}{(n+1)+5} = \frac{3}{n+6}$$

and so $a_n > a_{n+1}$ for all $n \geq 1$.

Show that the sequence

$$a_n = \frac{n}{n^2+1}$$

is decreasing.

We must show that $a_{n+1} < a_n$,

that is,

$$\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1}$$

This inequality is equivalent to the one we get by cross-multiplication:

$$\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1} \Leftrightarrow (n+1)(n^2+1) < n[(n+1)^2+1]$$

$$\Leftrightarrow n^3+n^2+n+1 < n^3+2n^2+2n$$

$$\Leftrightarrow 1 < n^2+n$$

Since $n \geq 1$, we know that the inequality $n^2 + n > 1$ is true.

- Therefore, $a_{n+1} < a_n$.
- Hence, $\{a_n\}$ is decreasing.

Consider the function $f(x) = \dfrac{x}{x^2+1}$:

$$f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2}$$

$$= \frac{1-x^2}{(x^2+1)^2} < 0 \qquad \text{whenever } x^2 > 1$$

Thus, f is decreasing on $(1, \infty)$.

- Hence, $f(n) > f(n + 1)$.
- Therefore, $\{a_n\}$ is decreasing.

A sequence $\{a_n\}$ is bounded:

- Above, if there is a number M such that $a_n \leq M$ for all $n \geq 1$
- Below, if there is a number m such that $m \leq a_n$ for all $n \geq 1$
- If it is bounded above and below

For instance,

- The sequence $a_n = n$ is bounded below ($a_n > 0$) but not above.
- The sequence $a_n = n/(n+1)$ is bounded because $0 < a_n < 1$ for all n.

BOUNDED SEQUENCES

We know that not every bounded sequence is convergent.

- For instance, the sequence $a_n = (-1)^n$ satisfies $-1 \le a_n \le 1$ but is divergent from Example 6.

BOUNDED SEQUENCES

Similarly, not every monotonic sequence is convergent ($a_n = n \to \infty$).

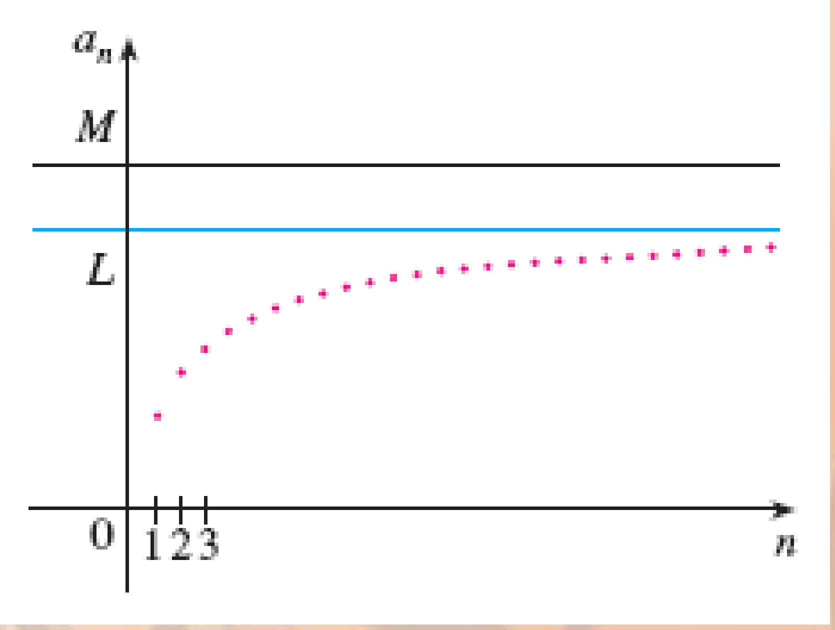
BOUNDED SEQUENCES

However, if a sequence is both bounded and monotonic, then it must be convergent.

- This fact is proved as Theorem 12.
- However, intuitively, you can understand why it is true by looking at the following figure.

BOUNDED SEQUENCES

If $\{a_n\}$ is increasing and $a_n \leq M$ for all n, then the terms are forced to crowd together and approach some number L.



BOUNDED SEQUENCES

The proof of Theorem 12 is based on the Completeness Axiom for the set $\mathbb{R}$ of real numbers.

- This states that, if *S* is a nonempty set of real numbers that has an upper bound *M* ($x \le M$ for all *x* in *S*), then *S* has a least upper bound *b*.

This means:

- b is an upper bound for S.
- However, if M is any other upper bound, then $b \leq M$.

BOUNDED SEQUENCES

The Completeness Axiom is

an expression of the fact that there is

no gap or hole in the real number line.

Every bounded, monotonic sequence is convergent.

Suppose $\{a_n\}$ is an increasing sequence.

- Since $\{a_n\}$ is bounded, the set $S = \{a_n | n \geq 1\}$ has an upper bound.
- By the Completeness Axiom, it has a least upper bound L.

Theorem 12—Proof

Given $\varepsilon > 0$, $L - \varepsilon$ is not an upper bound for S (since L is the least upper bound).

- Therefore,

  $a_N > L - \varepsilon$ for some integer N

However, the sequence is increasing.

So, $a_n \geq a_N$ for every $n > N$.

- Thus, if $n > N$, we have $a_n > L - \varepsilon$
- Since $a_n \leq L$, thus $0 \leq L - a_n < \varepsilon$

Thus,

$$|L - a_n| < \varepsilon \qquad \text{whenever } n > N$$

Therefore,

$$\lim_{n\to\infty} a_n = L$$

Theorem 12—Proof

A similar proof (using the greatest lower bound) works if $\{a_n\}$ is decreasing.

MONOTONIC SEQ. THEOREM

The proof of Theorem 12 shows that a sequence that is increasing and bounded above is convergent.

- Likewise, a decreasing sequence that is bounded below is convergent.

MONOTONIC SEQ. THEOREM

This fact is used many times in dealing with infinite series.

Investigate the sequence $\{a_n\}$ defined by the recurrence relation

$$a_1 = 2 \qquad a_{n+1} = \tfrac{1}{2}(a_n + 6) \qquad \text{for } n = 1, 2, 3, \ldots$$

We begin by computing the first several terms:

$a_1 = 2$ $\quad a_2 = \frac{1}{2}(2+6) = 4$ $\quad a_3 = \frac{1}{2}(4+6) = 5$

$a_4 = \frac{1}{2}(5+6) = 5.5$ $\quad a_5 = 5.75$ $\quad a_6 = 5.875$

$a_7 = 5.9375$ $\quad a_8 = 5.96875$ $\quad a_9 = 5.984375$

- These initial terms suggest the sequence is increasing and the terms are approaching 6.

To confirm that the sequence is increasing, we use mathematical induction to show that $a_{n+1} > a_n$ for all $n \geq 1$.

- Mathematical induction is often used in dealing with recursive sequences.

That is true for $n = 1$ because

$a_2 = 4 > a_1$.

If we assume that it is true for $n = k$, we have:

$$a_{k+1} > a_k$$

- Hence, $a_{k+1} + 6 > a_k + 6$

  and $\frac{1}{2}(a_{k+1} + 6) > \frac{1}{2}(a_k + 6)$

- Thus, $a_{k+2} > a_{k+1}$

We have deduced that $a_{n+1} > a_n$ is true for $n = k + 1$.

Therefore, the inequality is true for all n by induction.

Next, we verify that $\{a_n\}$ is bounded by showing that $a_n < 6$ for all n.

- Since the sequence is increasing, we already know that it has a lower bound: $a_n \geq a_1 = 2$ for all n

We know that $a_1 < 6$.

So, the assertion is true for $n = 1$.

Suppose it is true for *n* = *k*.

- Then, $a_k < 6$
- Thus, $a_k + 6 < 12$

  and $\frac{1}{2}(a_k + 6) < \frac{1}{2}(12) = 6$
- Hence, $a_{k+1} < 6$

This shows, by mathematical induction, that $a_n < 6$ for all n.

Since the sequence $\{a_n\}$ is increasing and bounded, Theorem 12 guarantees that it has a limit.

- However, the theorem doesn't tell us what the value of the limit is.

Nevertheless, now that we know $L = \lim_{n\to\infty} a_n$ exists, we can use the recurrence relation to write:

$$\begin{aligned}\lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty} \tfrac{1}{2}(a_n + 6)\\ &= \tfrac{1}{2}(\lim_{n\to\infty} a_n + 6)\\ &= \tfrac{1}{2}(L + 6)\end{aligned}$$

Since $a_n \to L$, it follows that $a_{n+1} \to L$, too (as $n \to \infty$, $n + 1 \to \infty$ too).

- Thus, we have:

$$L = \tfrac{1}{2}(L + 6)$$

- Solving this equation for L, we get $L = 6$, as predicted.