10

PARAMETRIC EQUATIONS AND POLAR COORDINATES

In Section 10.5, we defined the parabola in terms of a focus and directrix.

However, we defined the ellipse and hyperbola in terms of two foci.

10.6

Conic Sections in Polar Coordinates

In this section, we will:

Define the parabola, ellipse, and hyperbola all in terms of a focus and directrix.

CONIC SECTIONS IN POLAR COORDINATES

If we place the focus at the origin, then a conic section has a simple polar equation.

- This provides a convenient description of the motion of planets, satellites, and comets.

Let F be a fixed point (called the focus) and l be a fixed line (called the directrix) in a plane.

Let e be a fixed positive number (called the eccentricity.)

The set of all points *P* in the plane such that

$$\frac{|PF|}{|Pl|} = e$$

(that is, the ratio of the distance from *F* to the distance from l is the constant *e*) is a conic section.

The conic is:

a) an ellipse if $e < 1$.

b) a parabola if $e = 1$.

c) a hyperbola if $e > 1$.

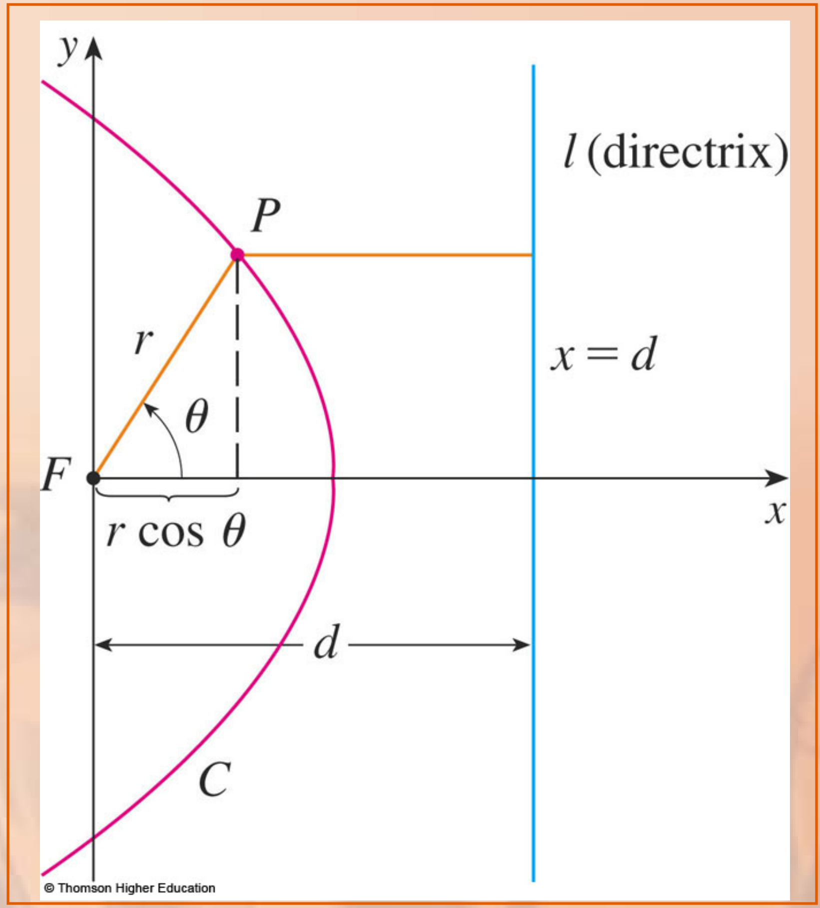
Notice that if the eccentricity is $e = 1$, then

$$|PF| = |Pl|$$

- Hence, the given condition simply becomes the definition of a parabola as given in Section 10.5

Let us place the focus *F* at the origin and the directrix parallel to the *y*-axis and *d* units to the right.

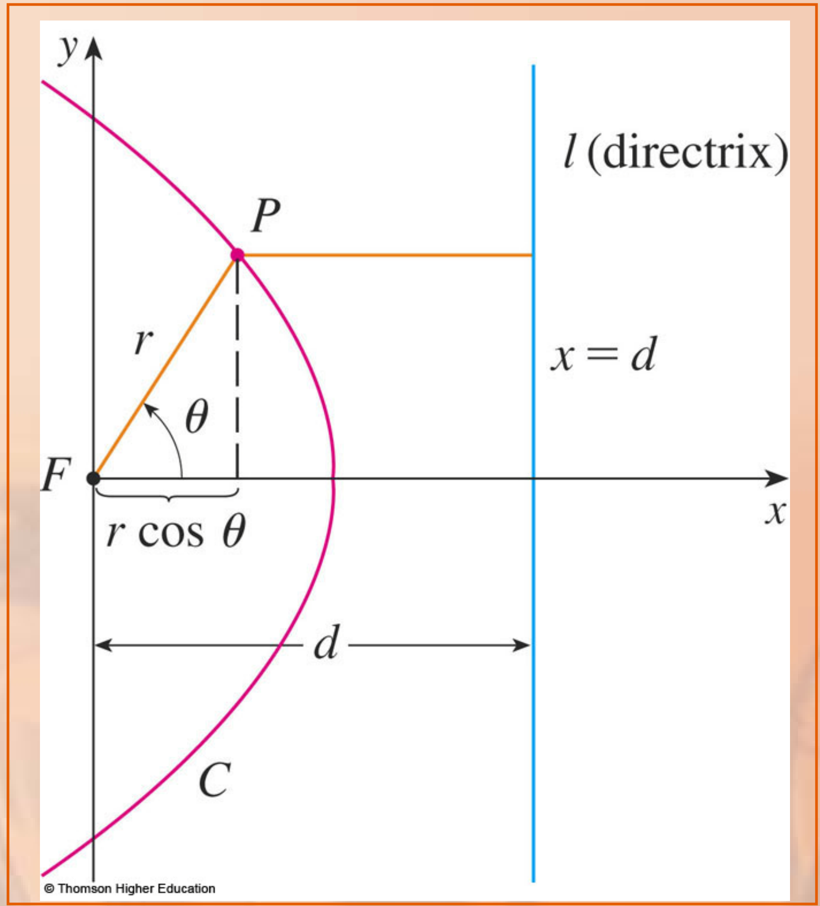
Thus, the directrix has equation $x = d$ and is perpendicular to the polar axis.



If the point P has polar coordinates (r, θ), we see that:

$$|PF| = r$$

$$|Pl| = d - r\cos\theta$$



Thus, the condition $|PF| / |Pl| = e$,

or $|PF| = e|Pl|$, becomes:

$$r = e(d - r \cos \theta)$$

Squaring both sides of the polar equation and convert to rectangular coordinates, we get

$$x^2 + y^2 = e^2(d - x)^2 = e^2(d^2 - 2dx + x^2)$$

or

$$(1 - e^2)x^2 + 2de^2x + y^2 = e^2d^2$$

After completing the square, we have:

$$\left(x+\frac{e^2d}{1-e^2}\right)^2+\frac{y^2}{1-e^2}=\frac{e^2d^2}{(1-e^2)^2}$$

If $e < 1$, we recognize Equation 3 as the equation of an ellipse.

In fact, it is of the form

$$\frac{(x-h)^2}{a^2}+\frac{y^2}{b^2}=1$$

where

$$h=-\frac{e^2d}{1-e^2} \qquad a^2=\frac{e^2d^2}{(1-e^2)^2} \qquad b^2=\frac{e^2d^2}{1-e^2}$$

In Section 10.5, we found that the foci of an ellipse are at a distance *c* from the center, where

$$c^2 = a^2 - b^2 = \frac{e^4 d^2}{(1-e^2)^2}$$

This shows that

$$c = \frac{e^2 d}{1 - e^2} = -h$$

Also, it confirms that the focus as defined in Theorem 1 means the same as the focus defined in Section 10.5

It also follows from Equations 4 and 5 that the eccentricity is given by:

$$e = \frac{c}{a}$$

If $e > 1$, then $1 - e^2 < 0$ and we see that Equation 3 represents a hyperbola.

Just as we did before, we could rewrite Equation 3 in the form

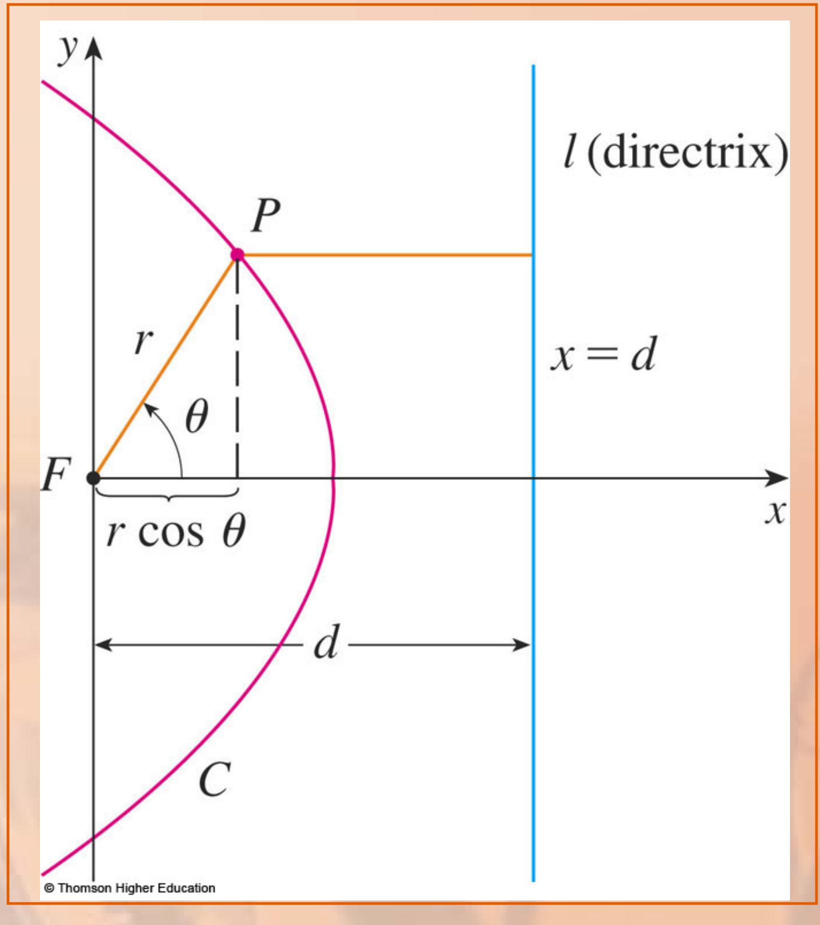
$$\frac{(x-h)^2}{a^2}-\frac{y^2}{b^2}=1$$

Thus, we see that:

$$e=\frac{c}{a} \quad \text{where} \quad c^2=a^2+b^2$$

By solving Equation 2 for *r*, we see that the polar equation of the conic shown here can be written as:

$$r = \frac{ed}{1 + e\cos\theta}$$



If the directrix is chosen to be to the left of the focus as $x = -d$, or parallel to the polar axis as $y = \pm d$, then the polar equation of the conic is given by the following theorem.

Theorem 6

A polar equation of the form

$$r = \frac{ed}{1 \pm e\cos\theta} \quad \text{or} \quad r = \frac{ed}{1 \pm e\sin\theta}$$

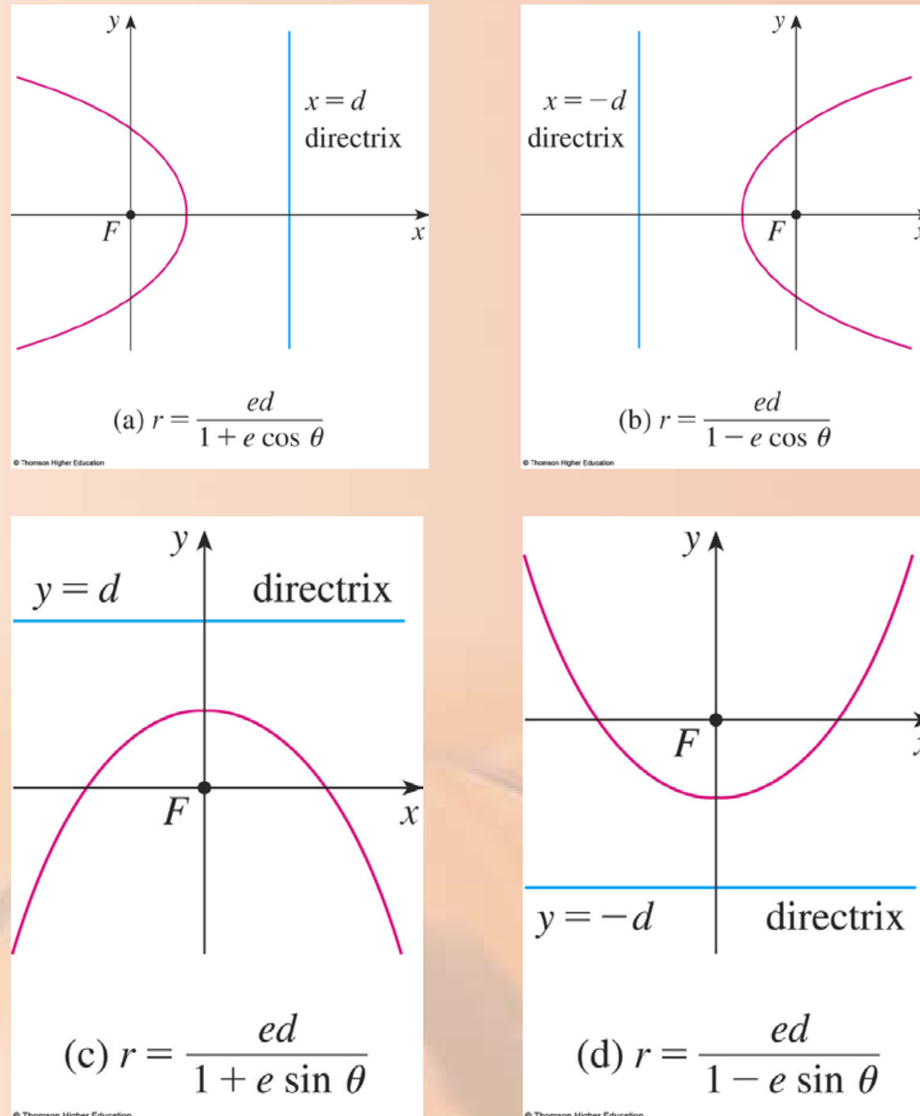
represents a conic section with eccentricity *e*.

The conic is:

- An ellipse if $e < 1$.
- A parabola if $e = 1$.
- A hyperbola if $e > 1$.

Theorem 6

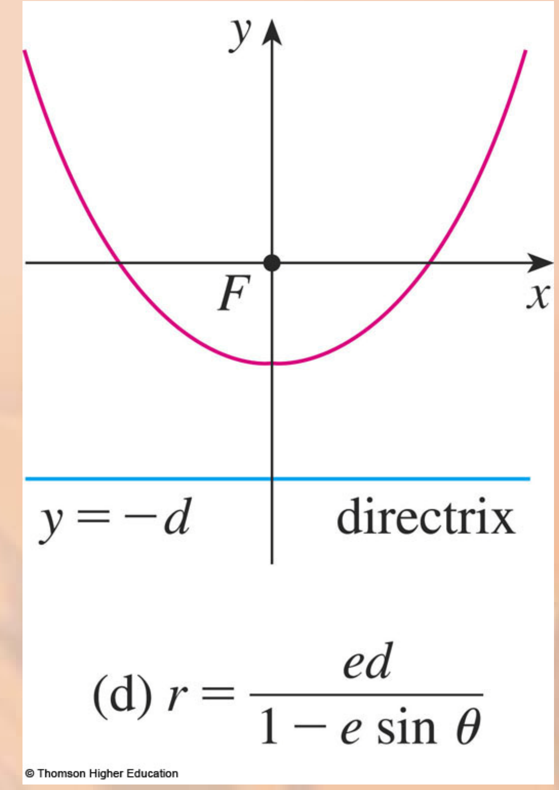
The theorem is illustrated here.



Find a polar equation for a parabola that has its focus at the origin and whose directrix is the line $y = -6$.

Using Theorem 6 with $e = 1$ and $d = 6$, and using part (d) of the earlier figure, we see that the equation of the parabola is:

$$r = \frac{6}{1 - \sin\theta}$$



A conic is given by the polar equation

$$r = \frac{10}{3 - 2\cos\theta}$$

- Find the eccentricity.
- Identify the conic.
- Locate the directrix.
- Sketch the conic.

Dividing numerator and denominator by 3, we write the equation as:

$$r = \frac{\frac{10}{3}}{1 - \frac{2}{3}\cos\theta}$$

- From Theorem 6, we see that this represents an ellipse with $e = 2/3$.

Since *ed* = 10/3, we have:

$$d = \frac{\frac{10}{3}}{e} = \frac{\frac{10}{3}}{\frac{2}{3}} = 5$$

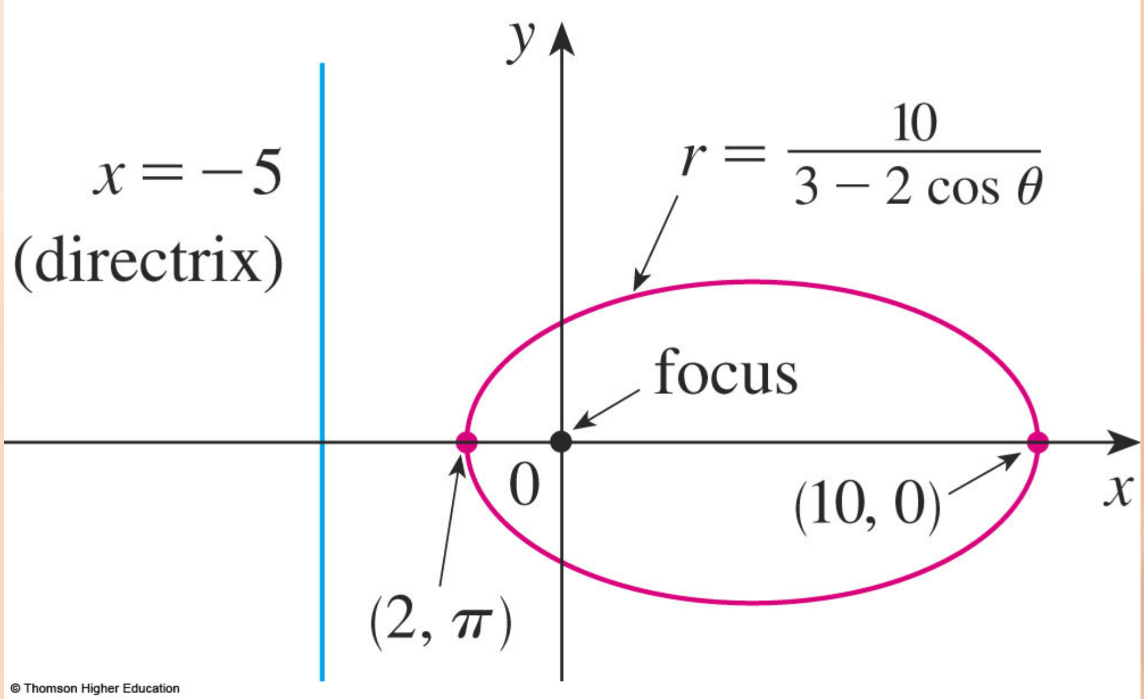
- So, the directrix has Cartesian equation $x = -5$.

When $\theta = 0$, $r = 10$.

When $\theta = \pi$, $r = 2$.

- Thus, the vertices have polar coordinates (10, 0) and (2, π).

The ellipse is sketched here.



Sketch the conic $r = \dfrac{12}{2 + 4\sin\theta}$

- Writing the equation in the form $r = \dfrac{6}{1 + 2\sin\theta}$ we see that the eccentricity is $e = 2$.
- Thus, the equation represents a hyperbola.

- Since $ed = 6$, $d = 3$ and the directrix has equation $y = 3$.
- The vertices occur when $\theta = \pi/2$ and $3\pi/2$; so, they are $(2, \pi/2)$ and $(–6, 3\pi/2) = (6, \pi/2)$.

It is also useful to plot the *x*-intercepts.

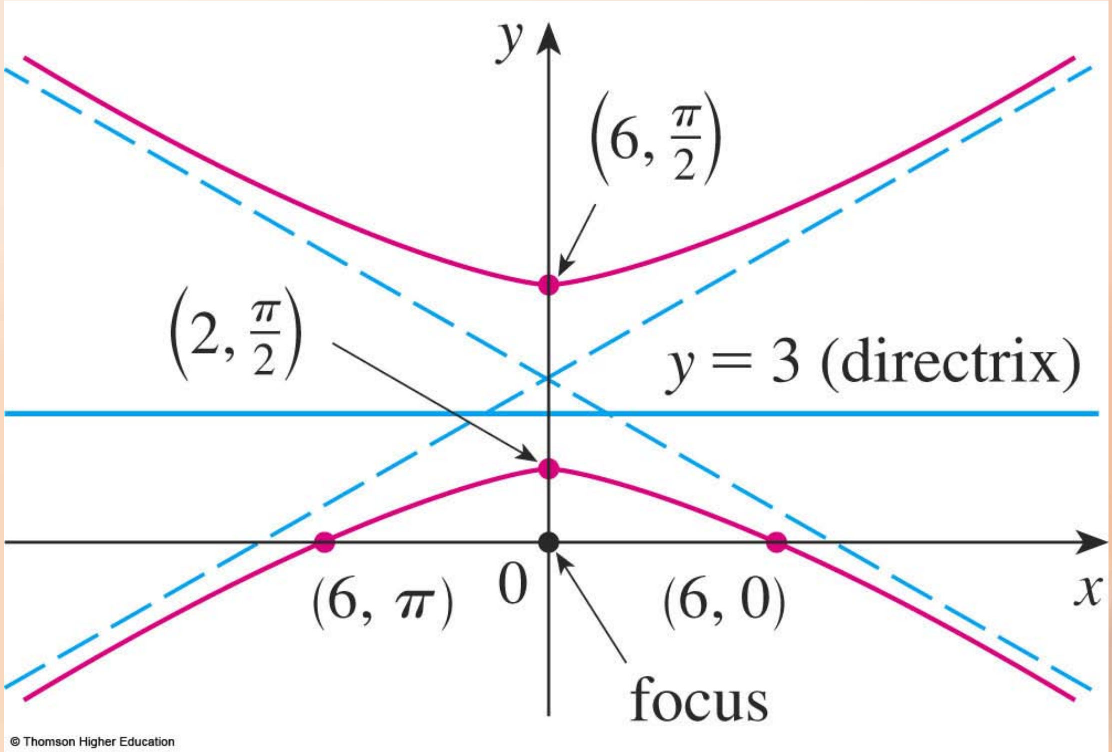
- These occur when $\theta = 0, \pi$.
- In both cases, $r = 6$.

For additional accuracy, we could draw the asymptotes.

- Note that $r \to \pm\infty$ when $1 + 2 \sin \theta \to 0^+$ or 0^- and $1 + 2 \sin \theta = 0$ when $\sin \theta = -\frac{1}{2}$.

- Thus, the asymptotes are parallel to the rays $\theta = 7\pi/6$ and $\theta = 11\pi/6$.

The hyperbola is sketched here.



When rotating conic sections, we find it much more convenient to use polar equations than Cartesian equations.

- We use the fact (Exercise 77 in Section 10.3) that the graph of $r = f(\theta - \alpha)$ is the graph of $r = f(\theta)$ rotated counterclockwise about the origin through an angle α.

If the ellipse of Example 2 is rotated through an angle $\pi/4$ about the origin, find a polar equation and graph the resulting ellipse.

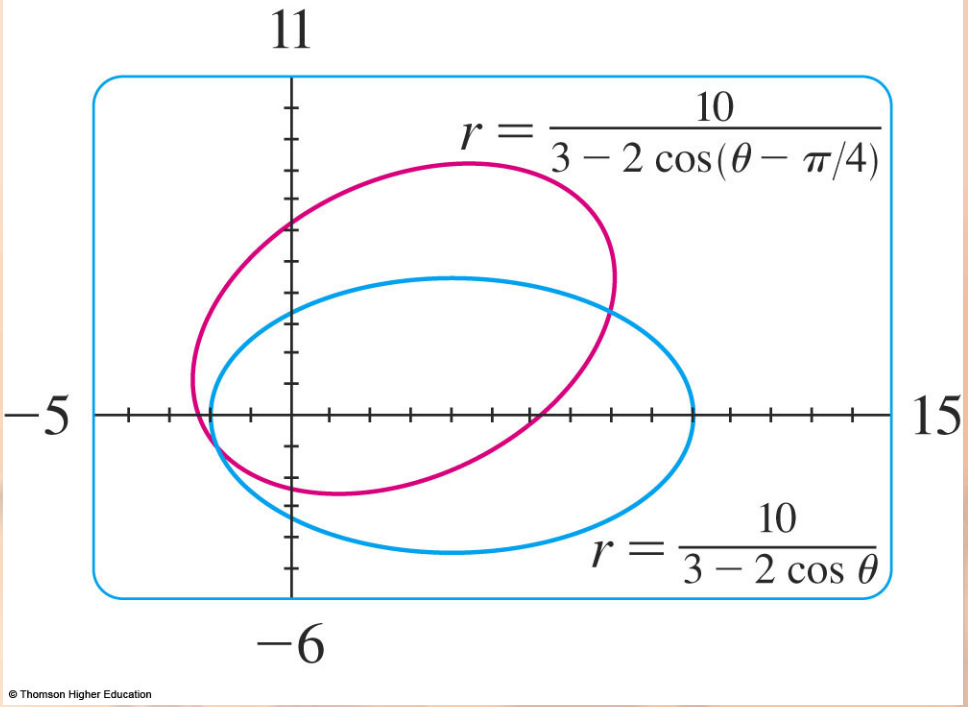
We get the equation of the rotated ellipse by replacing θ with $\theta - \pi/4$ in the equation given in Example 2.

- So, the new equation is:

$$r = \frac{10}{3 - 2\cos(\theta - \pi/4)}$$

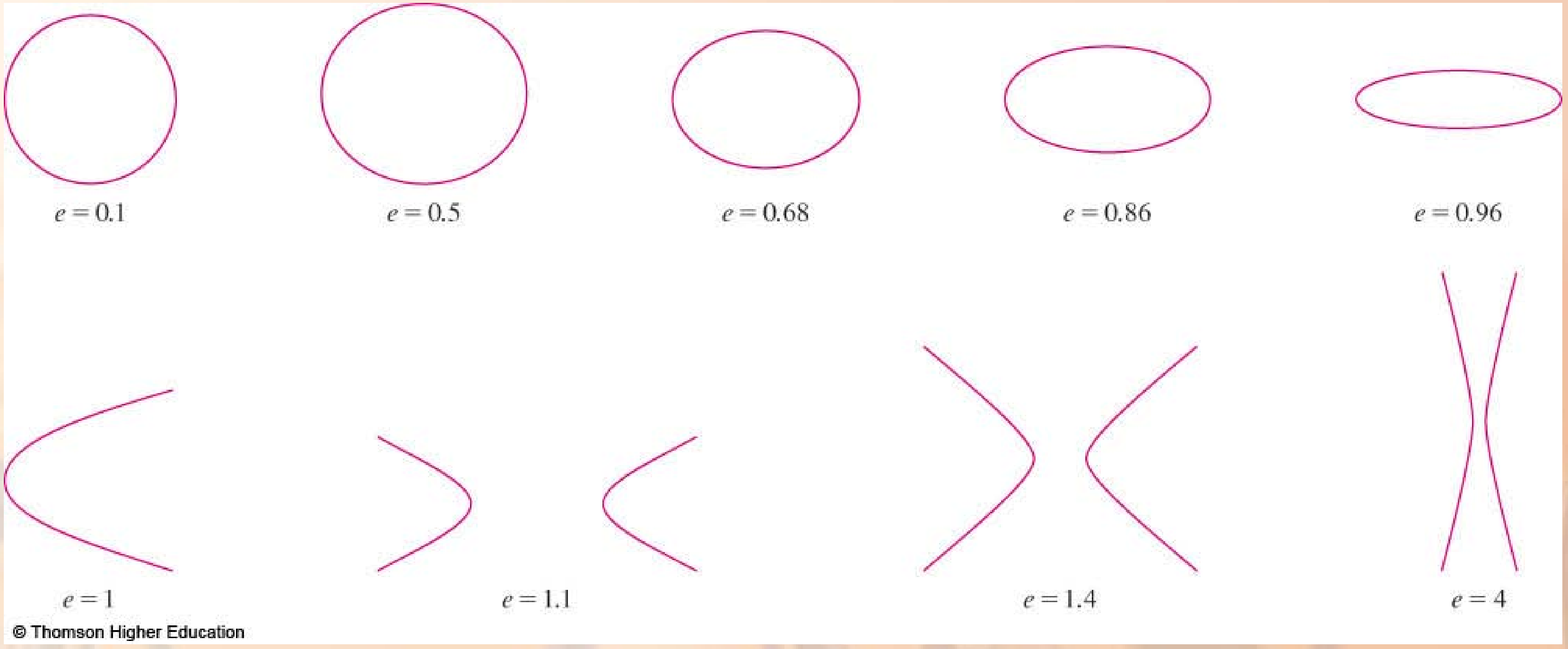
We use the equation to graph the rotated ellipse here.

- Notice that the ellipse has been rotated about its left focus.



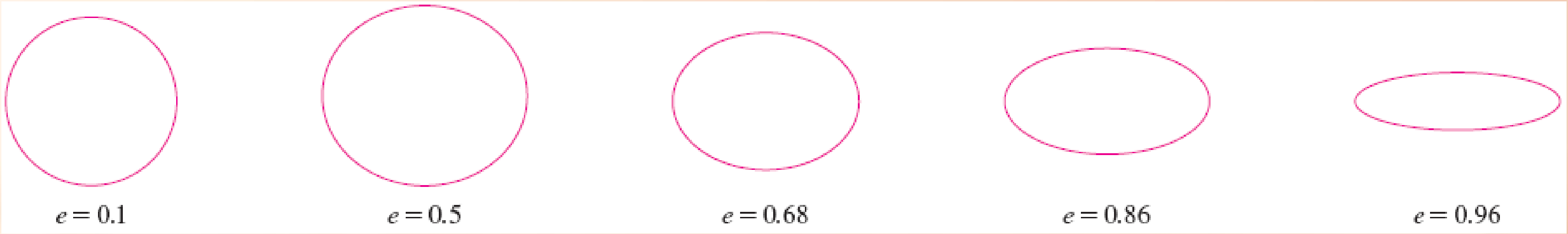
EFFECT OF ECCENTRICITY

Here, we use a computer to sketch a number of conics to demonstrate the effect of varying the eccentricity *e*.



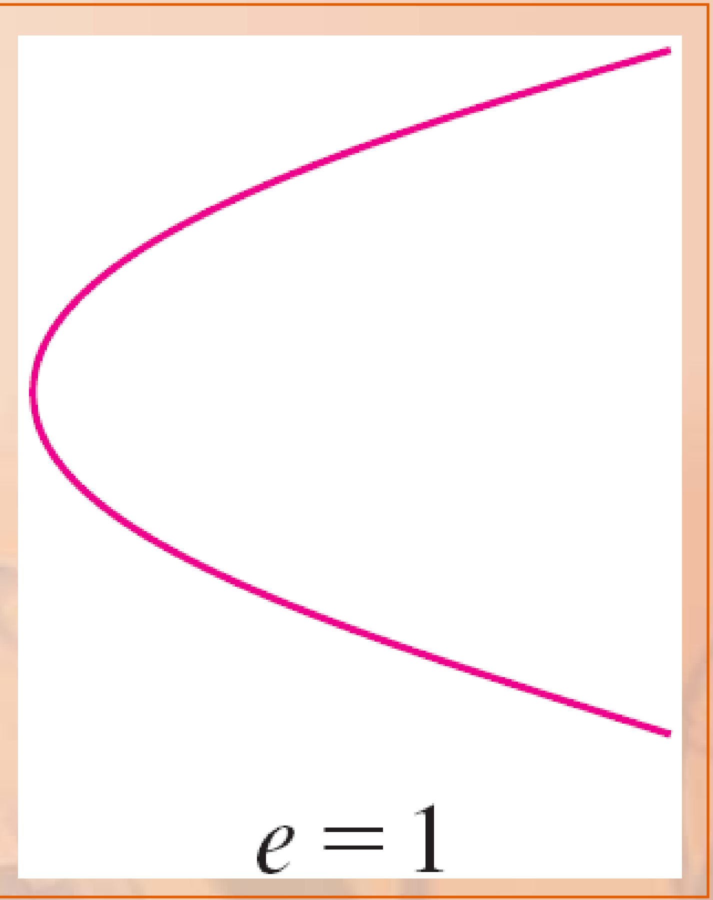
Notice that:

- When *e* is close to 0, the ellipse is nearly circular.
- As $e \to 1^-$, it becomes more elongated.



EFFECT OF ECCENTRICITY

When $e = 1$, of course, the conic is a parabola.



KEPLER’S LAWS

In 1609, the German mathematician and astronomer Johannes Kepler, on the basis of huge amounts of astronomical data, published the following three laws of planetary motion.

KEPLER'S FIRST LAW

A planet revolves around the sun in an elliptical orbit with the sun at one focus.

KEPLER'S SECOND LAW

The line joining the sun to

a planet sweeps out equal areas

in equal times.

KEPLER’S THIRD LAW

The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

KEPLER'S LAWS

Kepler formulated his laws in terms of the motion of planets around the sun.

- However, they apply equally well to the motion of moons, comets, satellites, and other bodies that orbit subject to a single gravitational force.

KEPLER'S LAWS

In Section 13.4, we will show how to deduce Kepler's Laws from Newton's Laws.

Here, we use Kepler's First Law, together with the polar equation of an ellipse, to calculate quantities of interest in astronomy.

For purposes of astronomical calculations, it's useful to express the equation of an ellipse in terms of its eccentricity *e* and its semimajor axis *a*.

We can write the distance from the focus to the directrix in terms of *a* if we use

Equation 4:

$$a^2 = \frac{e^2 d^2}{(1-e^2)^2} \Rightarrow d^2 = \frac{a^2(1-e^2)^2}{e^2}$$

$$\Rightarrow d = \frac{a(1-e^2)}{e}$$

- So, $ed = a(1 - e^2)$.

If the directrix is $x = a$, then the polar equation is:

$$r = \frac{ed}{1 + e\cos\theta} = \frac{a(1-e^2)}{1 + e\cos\theta}$$

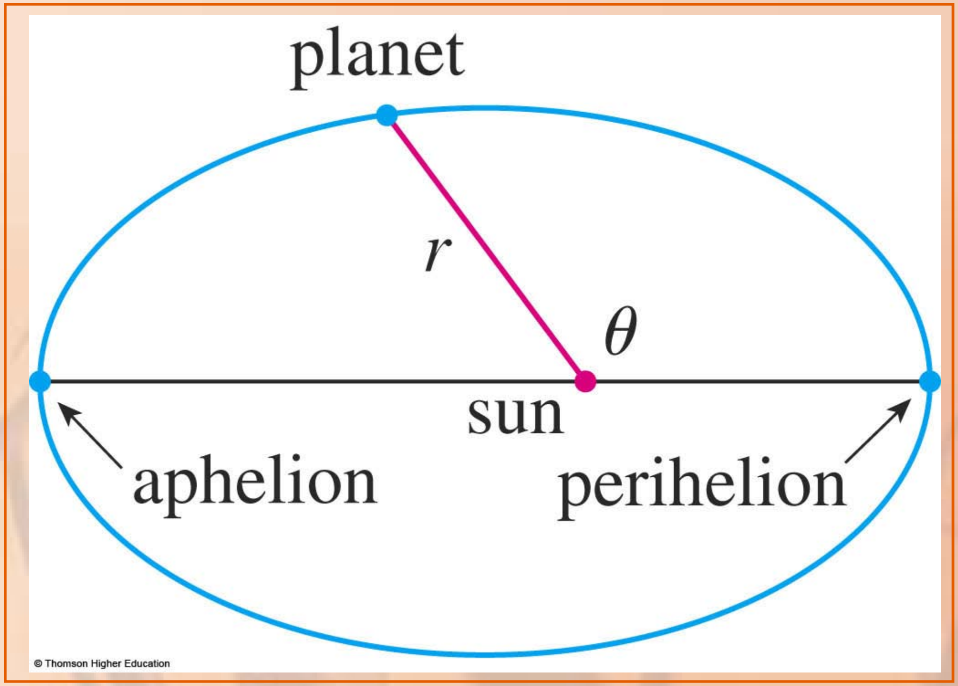
Equation 7

The polar equation of an ellipse with focus at the origin, semimajor axis *a*, eccentricity *e*, and directrix *x* = *d* can be written in the form

$$r = \frac{a(1-e^2)}{1+e\cos\theta}$$

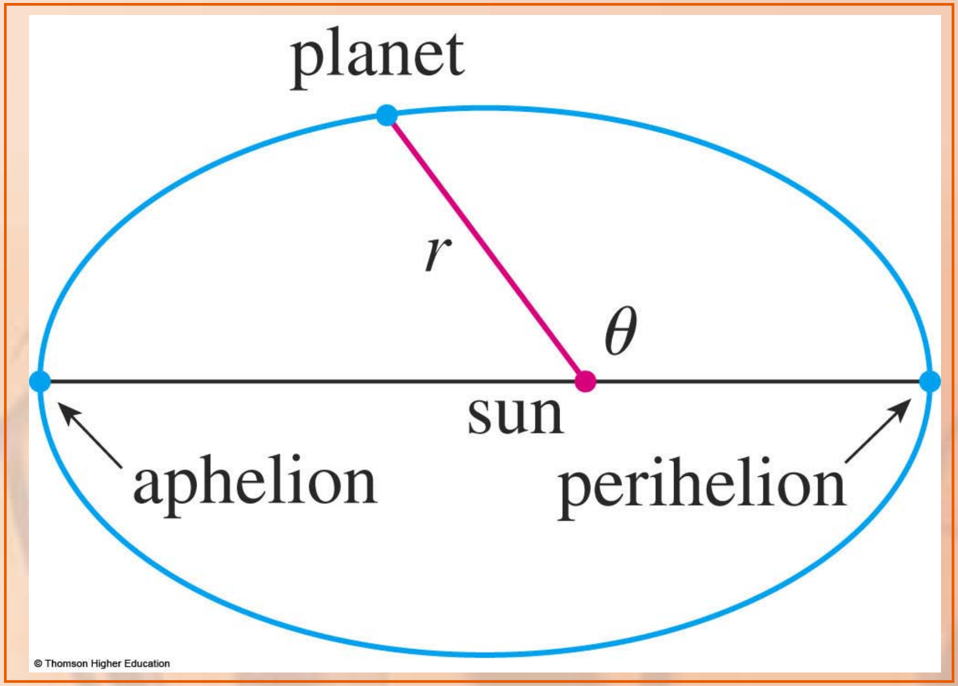
PERIHELION AND APHELION

The positions of a planet that are closest to and farthest from the sun are called its perihelion and aphelion, respectively, and correspond to the vertices of the ellipse.



PERIHELION AND APHELION DISTANCES

The distances from the sun to the perihelion and aphelion are called the perihelion distance and aphelion distance, respectively.



PERIHELION AND APHELION

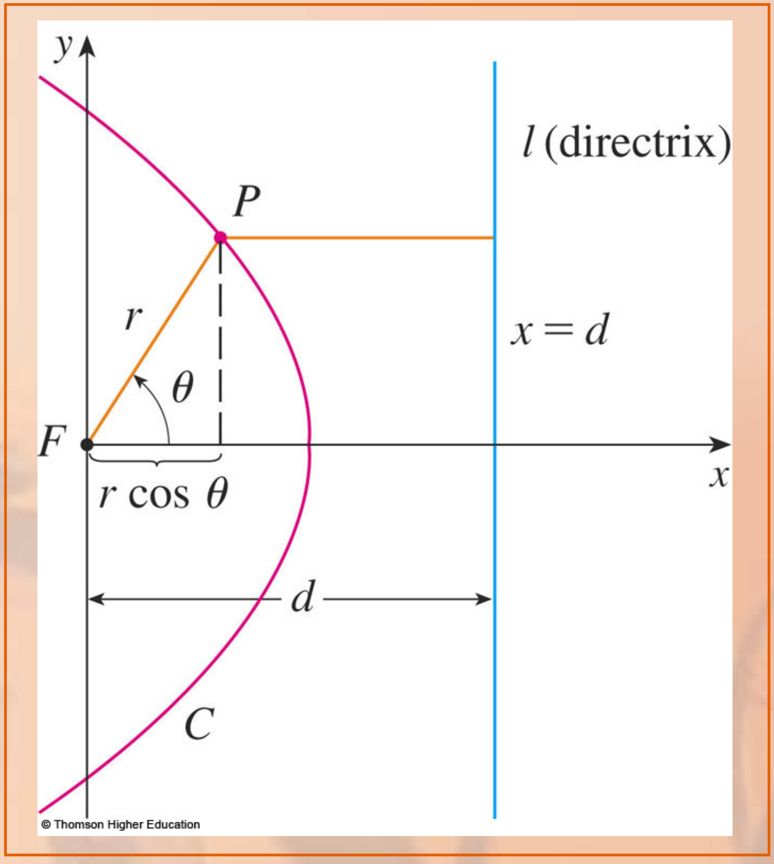
In this figure, the sun is at the focus *F.*

So, at perihelion, we have $\theta = 0$ and, from Equation 7,

$$r = \frac{a(1-e^2)}{1+e\cos 0}$$

$$= \frac{a(1-e)(1+e)}{1+e}$$

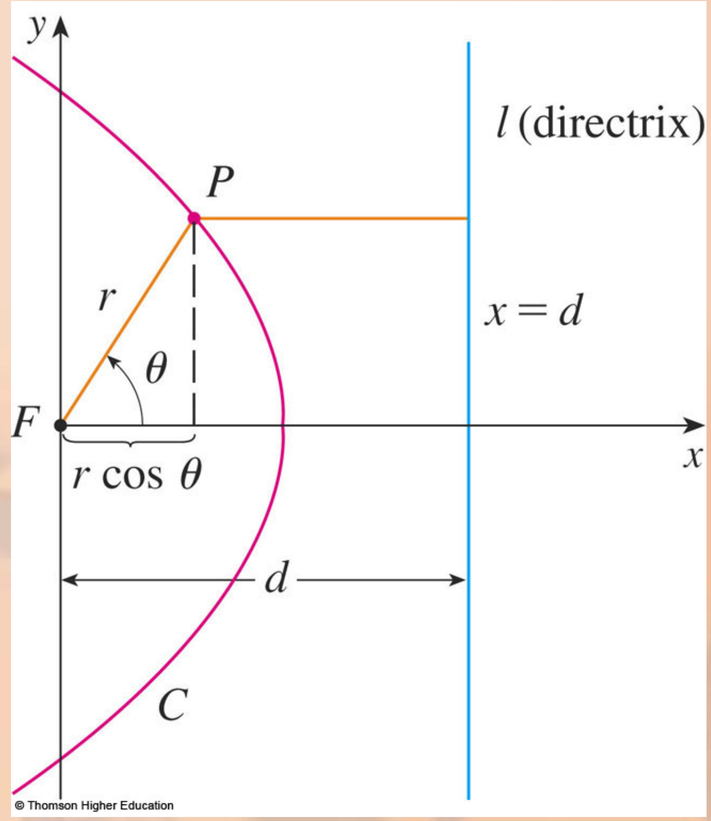
$$= a(1-e)$$



PERIHELION AND APHELION

Similarly, at aphelion,

$$\theta = \pi \text{ and } r = a(1 + e)$$



PERIHELION AND APHELION

Equation 8

The perihelion distance from a planet to the sun is

$$a(1 - e)$$

and the aphelion distance is

$$a(1 + e)$$

a. Find an approximate polar equation for the elliptical orbit of the earth around the sun (at one focus) given that:

- The eccentricity is about 0.017
- The length of the major axis is about 2.99×10^8 km.

b. Find the distance from the earth to the sun at perihelion and at aphelion.

Example 5 a

The length of the major axis is:

$$2a = 2.99 \times 10^8$$

So, $a = 1.495 \times 10^8$.

We are given that *e* = 0.017

- So, from Equation 7, an equation of the earth's orbit around the sun is:

$$r = \frac{a(1-e^2)}{1+e\cos\theta} = \frac{(1.495\times 10^8)[1-(0.017)^2]}{1+0.017\cos\theta}$$

- Approximately, $r = \dfrac{1.49\times 10^8}{1+0.017\cos\theta}$

From Equation 8, the perihelion distance from the earth to the sun is:

$$a(1-e) \approx (1.495\times 10^{8})(1-0.017)$$
$$\approx 1.47\times 10^{8}\ \text{km}$$

Similarly, the aphelion distance is:

$$a(1+e) \approx (1.495 \times 10^8)(1+0.017)$$
$$\approx 1.52 \times 10^8 \text{ km}$$