10

PARAMETRIC EQUATIONS AND POLAR COORDINATES

We have seen how to represent curves by parametric equations.

Now, we apply the methods of calculus to these parametric curves.

10.2

Calculus with Parametric Curves

In this section, we will:

Use parametric curves to obtain formulas for tangents, areas, arc lengths, and surface areas.

TANGENTS

In Section 10.1, we saw that some curves defined by parametric equations $x = f(t)$ and $y = g(t)$ can also be expressed—by eliminating the parameter—in the form $y = F(x)$.

- See Exercise 67 for general conditions under which this is possible.

If we substitute $x = f(t)$ and $y = g(t)$ in the equation $y = F(x)$, we get:

$$g(t) = F(f(t))$$

- So, if g, F, and f are differentiable, the Chain Rule gives:

$$g'(t) = F'(f(t))f'(t) = F'(x)f'(t)$$

If $f'(t) \neq 0$, we can solve for $F'(x)$:

$$F'(x) = \frac{g'(t)}{f'(t)}$$

TANGENTS

The slope of the tangent to the curve $y = F(x)$ at $(x, F(x))$ is $F'(x)$.

Thus, Equation 1 enables us to find tangents to parametric curves without having to eliminate the parameter.

Equation 2

Using Leibniz notation, we can rewrite Equation 1 in an easily remembered form:

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} \qquad \text{if} \quad \frac{dx}{dt} \neq 0$$

TANGENTS

If we think of a parametric curve as being traced out by a moving particle, then

- *dy*/*dt* and *dx*/*dt* are the vertical and horizontal velocities of the particle.

- Formula 2 says that the slope of the tangent is the ratio of these velocities.

From Equation 2, we can see that the curve has:

- A horizontal tangent when $dy/dt = 0$ (provided $dx/dt \neq 0$).
- A vertical tangent when $dx/dt = 0$ (provided $dy/dt \neq 0$).

This information is useful for sketching parametric curves.

As we know from Chapter 4, it is also useful to consider d^2y/dx^2.

This can be found by replacing y by dy/dx in Equation 2:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

A curve C is defined by the parametric equations $x = t^2$, $y = t^3 - 3t$.

a. Show that C has two tangents at the point (3, 0) and find their equations.

b. Find the points on C where the tangent is horizontal or vertical.

c. Determine where the curve is concave upward or downward.

d. Sketch the curve.

Notice that $y = t^3 - 3t = t(t^2 - 3) = 0$ when $t = 0$ or $t = \pm\sqrt{3}$.

- Thus, the point (3, 0) on *C* arises from two values of the parameter: $t = \sqrt{3}$ and $t = -\sqrt{3}$
- This indicates that *C* crosses itself at (3, 0).

Since

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t} = \frac{3}{2}\left(t - \frac{1}{t}\right)$$

the slope of the tangent when $t = \pm\sqrt{3}$ is:

$$dy/dx = \pm 6/(2\sqrt{3}) = \pm\sqrt{3}$$

So, the equations of the tangents at (3, 0) are:

$$y = \sqrt{3}\left(x - 3\right)$$

and

$$y = -\sqrt{3}\left(x - 3\right)$$

C has a horizontal tangent when $dy/dx = 0$, that is, when $dy/dt = 0$ and $dx/dt \neq 0$.

- Since $dy/dt = 3t^2 - 3$, this happens when $t^2 = 1$, that is, $t = \pm 1$.
- The corresponding points on *C* are (1, -2) and (1, 2).

C has a vertical tangent when $dx/dt = 2t = 0$, that is, $t = 0$.

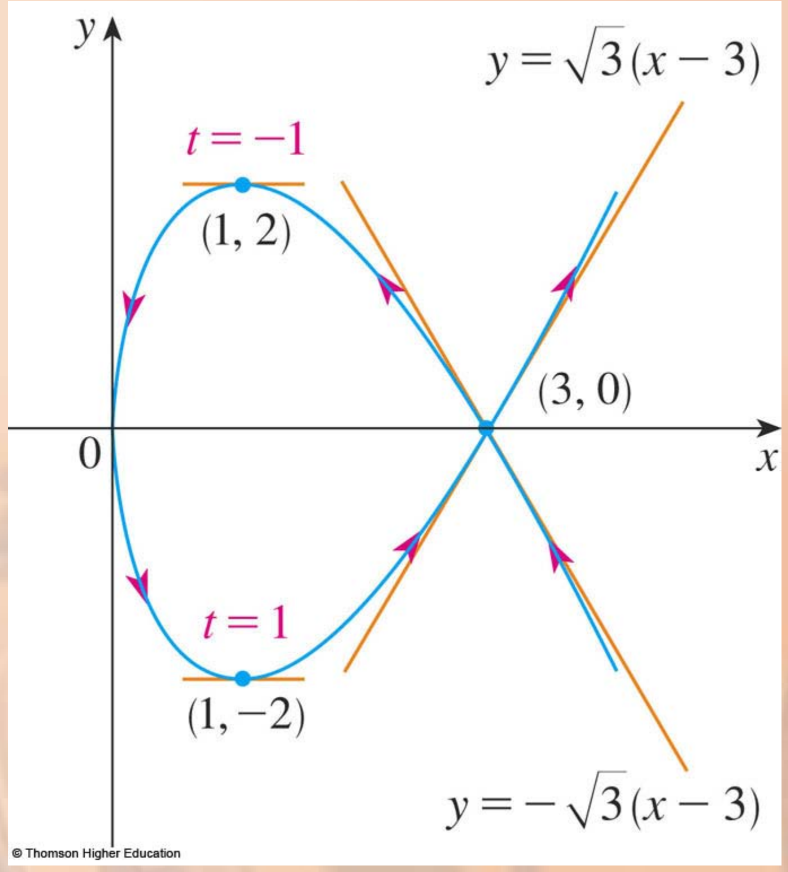
- Note that $dy/dt \neq 0$ there.
- The corresponding point on C is (0, 0).

To determine concavity, we calculate the second derivative:

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}} = \frac{\dfrac{3}{2}\left(1+\dfrac{1}{t^2}\right)}{2t} = \frac{3\left(t^2+1\right)}{4t^3}$$

- The curve is concave upward when $t > 0$.
- It is concave downward when $t < 0$.

Using the information from (b) and (c), we sketch *C*.



a. Find the tangent to the cycloid

$x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$

at the point where $\theta = \pi/3$.

- See Example 7 in Section 10.1

b. At what points is the tangent horizontal? When is it vertical?

The slope of the tangent line is:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\sin\theta}{r(1-\cos\theta)} = \frac{\sin\theta}{1-\cos\theta}$$

When $\theta = \pi/3$, we have

$$x = r\left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right) = r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$$

$$y = r\left(1 - \cos\frac{\pi}{3}\right) = \frac{r}{2}$$

and

$$\frac{dy}{dx} = \frac{\sin(\pi/3)}{1 - \cos(\pi/3)} = \frac{\sqrt{3}/2}{1 - \frac{1}{2}} = \sqrt{3}$$

Hence, the slope of the tangent is $\sqrt{3}$.

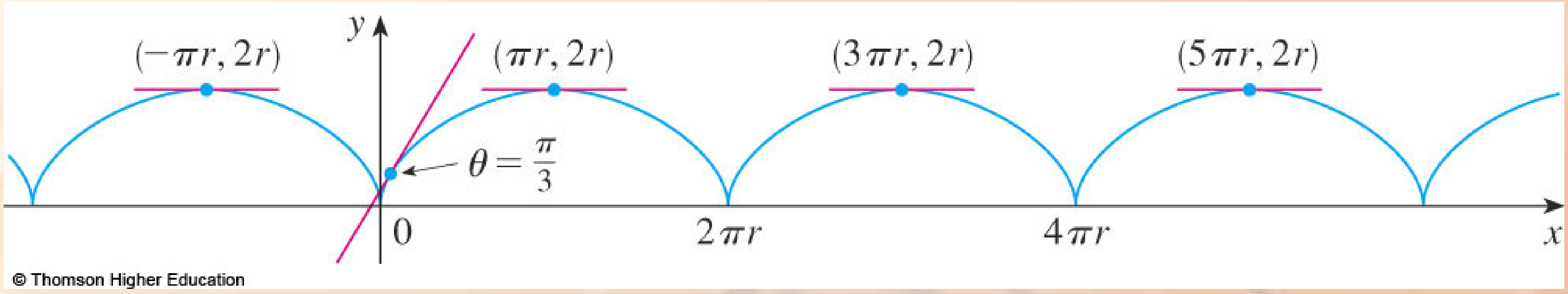
Its equation is:

$$y - \frac{r}{2} = \sqrt{3}\left(x - \frac{r\pi}{3} + \frac{r\sqrt{3}}{2} \right)$$

or

$$\sqrt{3}x - y = r\left(\frac{\pi}{\sqrt{3}} - 2 \right)$$

The tangent is sketched here.



The tangent is horizontal when

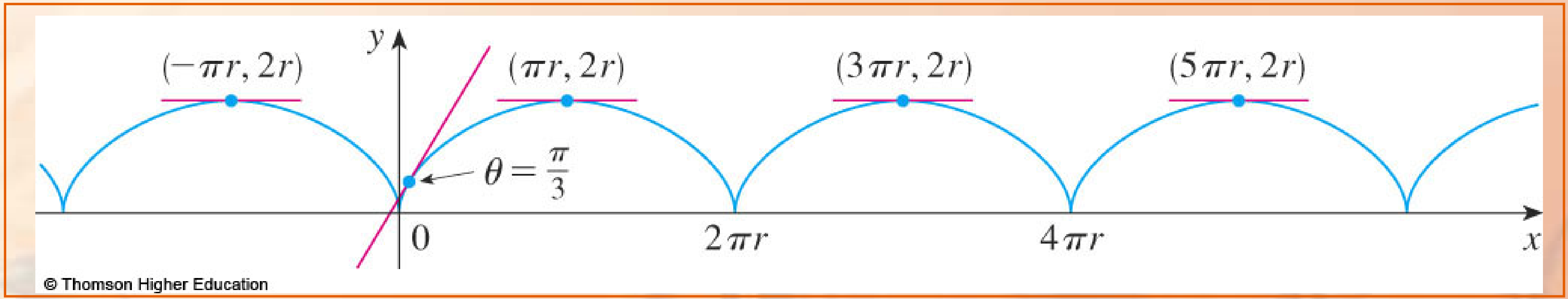
$dy/dx = 0$.

- This occurs when $\sin \theta = 0$ and $1 – \cos \theta \neq 0$, that is, $\theta = (2n – 1)\pi$, n an integer.
- The corresponding point on the cycloid is $((2n – 1)\pi r, 2r)$.

Example 2 b

When $\theta = 2n\pi$, both $dx/d\theta$ and $dy/d\theta$ are 0.

It appears from the graph that there are vertical tangents at those points.



We can verify this by using l'Hospital's Rule as follows:

$$\lim_{\theta \to 2n\pi^+} \frac{dy}{dx} = \lim_{\theta \to 2n\pi^+} \frac{\sin\theta}{1-\cos\theta}$$

$$= \lim_{\theta \to 2n\pi^+} \frac{\cos\theta}{\sin\theta} = \infty$$

A similar computation shows that $dy/dx \to -\infty$ as $\theta \to 2n\pi^{-}$.

- So, indeed, there are vertical tangents when $\theta = 2n\pi$, that is, when $x = 2n\pi r$.

AREAS

We know that the area under a curve $y = F(x)$ from a to b is

$$A = \int_a^b F(x)\,dx$$

where $F(x) \geq 0$.

Suppose the curve is traced out once by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \le t \le \beta$.

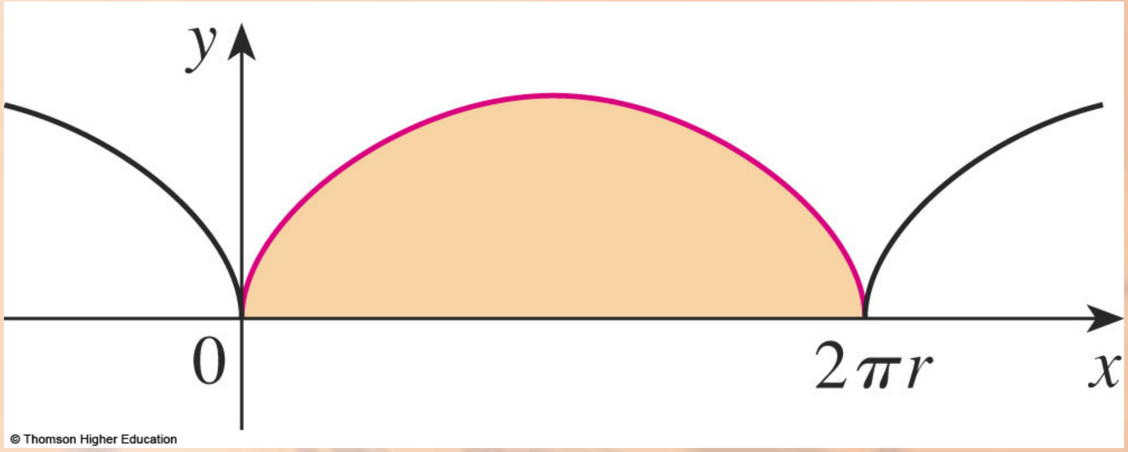
- Then, we can calculate an area formula by using the Substitution Rule for Definite Integrals.

$$A = \int_{\alpha}^{\beta} y\,dx = \int_{\alpha}^{\beta} g(t)\,f'(t)\,dt$$

$$\left[\text{or } \int_{\beta}^{\alpha} g(t)\,f'(t)\,dt\right]$$

Find the area under one arch of the cycloid

$$x = r(\theta - \sin\theta) \qquad y = r(1 - \cos\theta)$$



One arch of the cycloid is given by $0 \le \theta \le 2\pi$.

Using the Substitution Rule with

$y = r(1 - \cos\theta)$ and $dx = r(1 - \cos\theta)\, d\theta$,

we have the following result.

Example 3

$$A = \int_0^{2\pi r} y\, dx$$

$$= \int_0^{2\pi} r\left(1 - \cos\theta\right) r\left(1 - \cos\theta\right) d\theta$$

$$= r^2 \int_0^{2\pi} \left(1 - \cos\theta\right)^2 d\theta$$

$$= r^2 \int_0^{2\pi} \left(1 - 2\cos\theta + \cos^2\theta\right) d\theta$$

$$= r^2 \int_0^{2\pi} \left[1 - 2\cos\theta + \tfrac{1}{2}\left(1 + \cos 2\theta\right)\right] d\theta$$

$$= r^2 \left[\tfrac{3}{2}\theta - 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} = r^2\left(\tfrac{3}{2}.2\pi\right) = 3\pi r^2$$

The result of Example 3 says that the area under one arch of the cycloid is three times the area of the rolling circle that generates the cycloid (Example 7 in Section 10.1).

- Galileo guessed this result.
- However, it was first proved by the French mathematician Roberval and the Italian mathematician Torricelli.

ARC LENGTH

We already know how to find the length L of a curve C given in the form

$$y = F(x),\ a \le x \le b$$

Formula 3 in Section 8.1 says that, if *F'* is continuous, then

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

ARC LENGTH

Suppose that *C* can also be described by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \le t \le \beta$, where $dx/dt = f'(t) > 0$.

- This means that *C* is traversed once, from left to right, as *t* increases from α to β and $f(\alpha) = a$ and $f(\beta) = b$.

Putting Formula 2 into Formula 3 and using the Substitution Rule, we obtain:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, \frac{dx}{dt}\, dt$$

Since *dx*/*dt* > 0, we have:

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Even if C can't be expressed in the form $y = f(x)$, Formula 4 is still valid.

However, we obtain it by polygonal approximations.

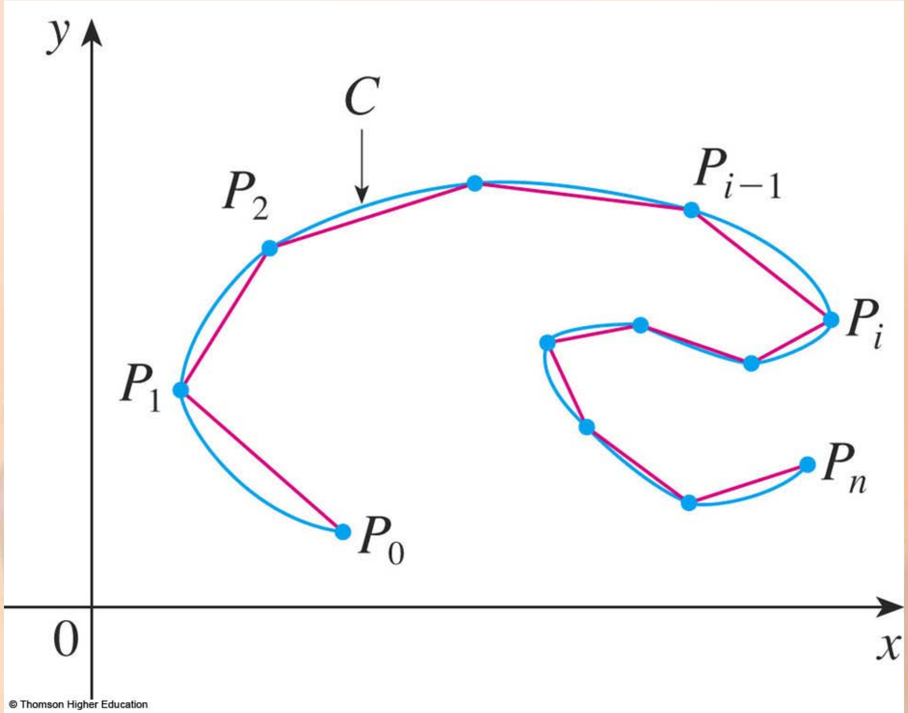
ARC LENGTH

We divide the parameter interval [α, β] into n subintervals of equal width Δt.

If $t_0, t_1, t_2, \ldots, t_n$ are the endpoints of these subintervals, then $x_i = f(t_i)$ and $y_i = g(t_i)$ are the coordinates of points $P_i(x_i, y_i)$ that lie on C.

ARC LENGTH

So, the polygon with vertices P_0, P_1 , ..., P_n approximates *C*.



ARC LENGTH

As in Section 8.1, we define the length *L* of *C* to be the limit of the lengths of these approximating polygons as $n \to \infty$:

$$L = \lim_{n \to \infty} \sum_{i-1}^{n} \left| P_{i-1} P_i \right|$$

The Mean Value Theorem, when applied to f on the interval $[t_{i-1}, t_i]$, gives a number t_i^* in (t_{i-1}, t_i) such that:

$$f(t_i) - f(t_{i-1}) = f'(t_i^*)(t_i - t_{i-1})$$

- If we let $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$, the equation becomes:

$$\Delta x_i = f'(t_i^*)\, \Delta t$$

Similarly, when applied to *g*, the Mean Value Theorem gives a number t_i^{**} in (t_{i-1}, t_i) such that:

$$\Delta y_i = g'(t_i^{**})\,\Delta t$$

ARC LENGTH

Therefore,

$$\left|P_{i-1}P_i\right| = \sqrt{\left(\Delta x_i\right)^2 + \left(\Delta y_i\right)^2}$$

$$= \sqrt{\left[f'\left(t_i^*\right)\Delta t\right]^2 + \left[g'\left(t_i^{**}\right)\Delta t\right]^2}$$

$$= \sqrt{\left[f'\left(t_i^*\right)\right]^2 + \left[g'\left(t_i^{**}\right)\right]^2}\,\Delta t$$

Equation 5

Thus,

$$L = \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{\left[f'\left(t_i^*\right)\right]^2+\left[g'\left(t_i^{**}\right)\right]^2}\,\Delta t$$

The sum in Equation 5 resembles a Riemann sum for the function $\sqrt{\left[f'(t)\right]^2+\left[g'(t)\right]^2}$.

However, is not exactly a Riemann sum because $t_i^* \neq t_i^{**}$ in general.

Nevertheless, if *f'* and *g'* are continuous, it can be shown that the limit in Equation 5 is the same as if t_i^* and t_i^{**} were equal, namely,

$$L = \int_{\alpha}^{\beta} \sqrt{\left[f'(t) \right]^2 + \left[g'(t) \right]^2} \, dt$$

Thus, using Leibniz notation, we have the following result—which has the same form as Formula 4.

THEOREM **Theorem 6**

Let a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$, where:

- f' and g' are continuous on $[\alpha, \beta]$.
- C is traversed exactly once as t increases from α to β.

Then, the length of C is:

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Notice that the formula in Theorem 6 is consistent with these general formulas from Section 8.1:

$$L = \int ds \qquad \text{and} \qquad (ds)^2 = (dx)^2 + (dy)^2$$

Suppose we use the representation of the unit circle given in Example 2 in Section 10.1:

$$x = \cos t \qquad y = \sin t \qquad 0 \le t \le 2\pi$$

- Then,

$$dx/dt = -\sin t \quad \text{and} \quad dy/dt = \cos t$$

- So, as expected, Theorem 6 gives:

$$\int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t}\, dt$$

$$= \int_0^{2\pi} dt 2\pi$$

$$= 2\pi$$

On the other hand, suppose we use the representation given in Example 3 in Section 10.1:

$$x = \sin 2t \qquad y = \cos 2t \qquad 0 \le t \le 2\pi$$

- Then,

$$dx/dt = 2\cos 2t \quad \text{and} \quad dy/dt = -2\sin 2t$$

- Then, the integral in Theorem 6 gives:

$$\int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{4\cos^2 2t + 4\sin^2 2t}\, dt$$

$$= \int_0^{2\pi} 2\, dt$$

$$= 4\pi$$

Notice that the integral gives twice the arc length of the circle.

- This is because, as t increases from 0 to 2π, the point $(\sin 2t, \cos 2t)$ traverses the circle twice.

In general, when finding the length of a curve *C* from a parametric representation, we have to be careful to ensure that *C* is traversed only once as t increases from α to β.

Find the length of one arch of the cycloid

$$x = r(\theta - \sin\theta) \qquad y = r(1 - \cos\theta)$$

- From Example 3, we see that one arch is described by the parameter interval $0 \le \theta \le 2\pi$.

We have:

$$\frac{dx}{d\theta} = r(1 - \cos\theta)$$

and

$$\frac{dy}{d\theta} = r\sin\theta$$

Thus,

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{r^2 (1-\cos\theta)^2 + r^2 \sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{\mathrm{r}^2 \left(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta\right)}\, d\theta$$

$$= r\int_0^{2\pi} \sqrt{2(1-\cos\theta)}\, d\theta$$

To evaluate this integral, we use the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ with $\theta = 2x$.

- This gives $1 - \cos \theta = 2\sin^2(\theta/2)$.

Since $0 \le \theta \le 2\pi$, we have $0 \le \theta/2 \le \pi$, and so $\sin(\theta/2) \ge 0$.

Therefore,

$$\begin{aligned}\sqrt{2(1-\cos\theta)} &= \sqrt{4\sin^2(\theta/2)} \\ &= 2\left|\sin(\theta/2)\right| \\ &= 2\sin(\theta/2)\end{aligned}$$

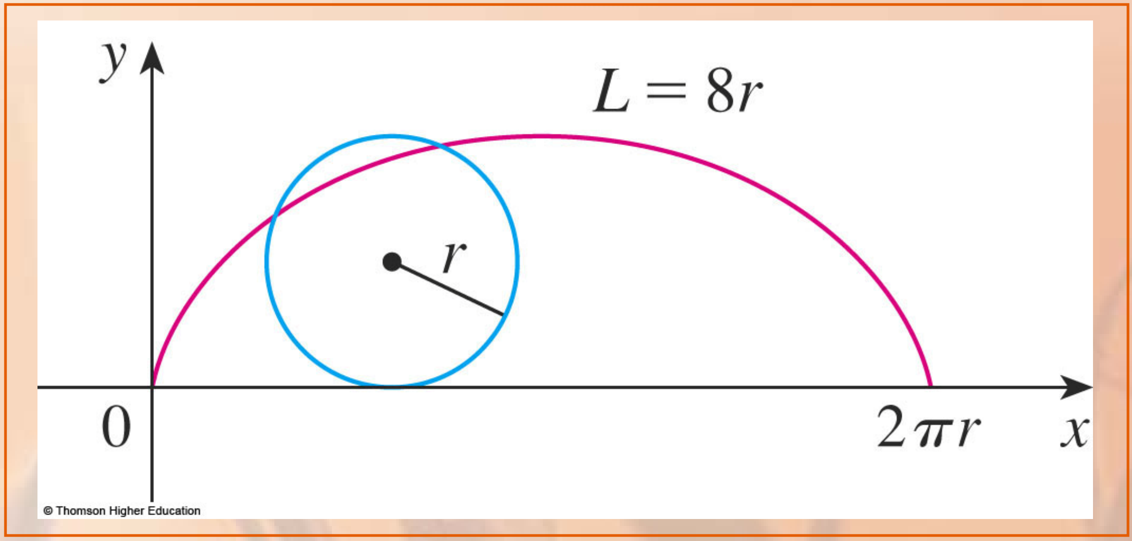
Hence,

$$\begin{aligned} L &= 2r\int_0^{2\pi} \sin(\theta/2)\,d\theta \\ &= 2r\Big[-2\cos(\theta/2)\Big]_0^{2\pi} \\ &= 2r[2+2] \\ &= 8r \end{aligned}$$

ARC LENGTH

The result of Example 5 says that the length of one arch of a cycloid is eight times the radius of the generating circle.

- This was first proved in 1658 by Sir Christopher Wren.



SURFACE AREA

In the same way as for arc length, we can adapt Formula 5 in Section 8.2 to obtain a formula for surface area.

Let the curve given by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$, be rotated about the *x*-axis, where:

- f', g' are continuous.
- $g(t) \ge 0$.

Then, the area of the resulting surface is given by:

$$S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

SURFACE AREA

The general symbolic formulas

$S = \int 2\pi y \, ds$ and $S = \int 2\pi x \, ds$ (Formulas 7 and 8 in Section 8.2) are still valid.

However, for parametric curves, we use:

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

Show that the surface area of a sphere of radius r is $4\pi r^2$.

- The sphere is obtained by rotating the semicircle

$$x = r\cos t \qquad y = r\sin t \qquad 0 \le t \le \pi$$

about the x-axis.

- So, from Formula 7, we get:

$$S = \int_0^{\pi} 2\pi r \sin t \sqrt{(-r \sin t)^2 + (r \cos t)^2}\, dt$$

$$= 2\pi \int_0^{\pi} r \sin t \sqrt{r^2 \left(\sin^2 t + \cos^2 t\right)}\, dt$$

$$= 2\pi \int_0^{\pi} r \sin t * r\, dt$$

$$= 2\pi r^2 \int_0^{\pi} \sin t\, dt$$

$$= 2\pi r^2 (-\cos t)\Big]_0^{\pi} = 4\pi r^2$$