The Inverse Hyperbolic Function and Their Derivatives

1. The Inverse Hyperbolic Sine Function

a) **Definition**

The **inverse hyperbolic sine function** is defined as follows:

$y = \sinh^{-1} x$ iff $\sinh y = x$ with y *in* $(-\infty, +\infty)$ and x *in* $(-\infty, +\infty)$

$f(x) = \sinh^{-1} x : (-\infty, \infty) \to (-\infty, \infty)$

Domain: $(-\infty, \infty) = R$

Range: $(-\infty, \infty) = R$

b) **Expression**: Show that $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$

Proof

Start with $y = \sinh x = \dfrac{e^x - e^{-x}}{2}$

To find the inverse solve for x and then interchange x and y.

$y = \dfrac{e^x - e^{-x}}{2} \Rightarrow 2y = e^x - \dfrac{1}{e^x}$; Let $e^x = z > 0 \Rightarrow 2y = z - \dfrac{1}{z} \Rightarrow z^2 - 2zy - 1 = 0$

The quadratic equation in z gives: $z = y \pm \sqrt{y^2 + 1}$

Because $z = e^x > 0 \Rightarrow$ choose $z = y + \sqrt{y^2 + 1}$ $\therefore e^x = y + \sqrt{y^2 + 1} \Rightarrow x = \ln(y + \sqrt{y^2 + 1})$

Now, interchange x and y to obtain:

$y = \sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$

$$\boxed{\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})}$$

c) **Derivative**

$$\frac{d}{dx}\sinh^{-1} x = \frac{d}{dx}[\ln(x + \sqrt{x^2 + 1})] = \frac{(x + \sqrt{x^2 + 1})'}{x + \sqrt{x^2 + 1}} = \frac{1 + \dfrac{2x}{2\sqrt{x^2 + 1}}}{x + \sqrt{x^2 + 1}} = \frac{\dfrac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}}}{x + \sqrt{x^2 + 1}} = \frac{1}{\sqrt{x^2 + 1}}$$

$$\boxed{\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{x^2 + 1}}}$$

2. **The Inverse Hyperbolic Cosine Function**

$\cosh hx = y \; : \; R \to (0,1)$ is **not** invertible

The restricted $\cosh hx = y \; : \; [0,\infty) \to (1,\infty)$ **is** invertible

a) **Definition**

The **inverse hyperbolic cosine function** is defined as follows:

$y = \cosh^{-1} x$ iff $\cosh hy = x$ with y *in* $(1,+\infty)$ and x *in* $(0,+\infty)$

$f(x) = \sinh^{-1} x : (1,\infty) \to (0,\infty)$ Domain: $(1,\infty)$ Range: $(0,\infty)$

b) **Expression**: Show that $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$

Proof

Start with $y = \cosh x = \dfrac{e^x + e^{-x}}{2}$

To find the inverse solve for x and then interchange x and y.

$$y = \frac{e^x + e^{-x}}{2} \Rightarrow 2y = e^x + \frac{1}{e^x} \;;$$

Let $e^x = z \geq 1, \quad x \geq 0 \Rightarrow 2y = z + \dfrac{1}{z} \Rightarrow z^2 - 2zy + 1 = 0$

The quadratic equation in z gives: $z = y \pm \sqrt{y^2 - 1}$

Because $z = e^x \geq 1 \Rightarrow$ choose $z = y + \sqrt{y^2-1} \quad \therefore e^x = y + \sqrt{y^2-1} \Rightarrow x = \ln(y + \sqrt{y^2-1})$

Now, interchange x and y to obtain:

$y = \cosh^{-1} x = \ln(x + \sqrt{x^2-1})$

$$\boxed{\cosh^{-1} x = \ln(x + \sqrt{x^2-1})}$$

c) **Derivative**

$$\frac{d}{dx}\cosh^{-1} x = \frac{d}{dx}[\ln(x+\sqrt{x^2-1})] = \frac{(x+\sqrt{x^2-1})'}{x+\sqrt{x^2-1})} = \frac{1+\dfrac{2x}{2\sqrt{x^2-1}}}{x+\sqrt{x^2-1})} = \frac{\dfrac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}}{x+\sqrt{x^2-1})} = \frac{1}{\sqrt{x^2+1}}$$

$$\boxed{\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2+1}}}$$

3. **<u>The Inverse Hyperbolic Tangent Function</u>**

$\tanh x = y \ : \ R \to (-1,1)$ is invertible

a) <u>Definition</u>

The **inverse hyperbolic tangent function** is defined as follows:

$y = \tanh^{-1} x$ iff $\tanh y = x$ with y *in* $(-1,+1)$ and x *in* $(-\infty,+\infty)$

$f(x) = \tanh^{-1} x : (1,+1) \to (-\infty,\infty)$

Domain: $(1,+1) \to$ Range: $(-\infty,\infty)$

b) **<u>Expression</u>**: Show that $\tanh^{-1} x = \frac{1}{2}\ln(\frac{x+1}{x-1})$

Proof

Start with $y = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

To find the inverse solve for x and then interchange x and y.

$$y = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^x - \frac{1}{e^x}}{e^x + \frac{1}{e^x}} \ ;$$

Let $e^x = z > 0 \quad \Rightarrow y = \frac{z - \frac{1}{z}}{z + \frac{1}{z}} \Rightarrow y = \frac{z^2 - 1}{z^2 + 1} \Rightarrow (z^2+1)y = z^2 - 1 \Rightarrow (y-1)z^2 + y + 1 = 0$

The quadratic equation in z gives: $z = \pm\sqrt{\frac{y+1}{1-y}}$

Because $z = e^x > 0 \Rightarrow$ choose $z = +\sqrt{\frac{y+1}{1-y}} \quad \therefore e^x = \sqrt{\frac{y+1}{1-y}} \Rightarrow x = \ln(\sqrt{\frac{y+1}{1-y}})$

Now, interchange x and y to obtain:

$$y = \tanh^{-1} x = \ln(\sqrt{\frac{x+1}{1-x}}) = \frac{1}{2}\ln(\frac{x+1}{1-x})$$

$$\boxed{\tanh^{-1} x = \frac{1}{2}\ln(\frac{x+1}{1-x})}$$

c) <u>Derivative</u>

$$\frac{d}{dx}\tanh^{-1} x = \frac{d}{dx}[\frac{1}{2}\ln(\frac{x+1}{1-x})] = \frac{1}{2}\frac{d}{dx}[\ln(\frac{x+1}{1-x})] = \frac{1}{2}\frac{d}{dx}[\ln(x+1) - \ln(1-x)] =$$

$$= \frac{1}{2}[\frac{1}{1+x} - \frac{-1}{1-x}] = \frac{1}{2}[\frac{1}{1+x} + \frac{1}{1-x}] = \frac{1}{2}[\frac{1+x+1-x}{(1+x)(1-x)}] = \frac{1}{1-x^2}$$

$$\boxed{\frac{d}{dx}\tanh^{-1} x = \frac{1}{1-x^2}}$$

4. **The Inverse Hyperbolic Cotangent Function**

$\coth x = y \ : \ (-\infty,0)\cup(0,\infty)\to(-\infty,-1)\cup(1,\infty)$ is invertible

a) **Definition**

The **inverse hyperbolic cotangent function** is defined as follows:

$y=\coth^{-1}x$ **iff** $\coth y = x$ with $y \ in\,(-\infty,0)\cup(0,\infty)$ and $x \ in\ (-\infty,-1)\cup(1,\infty)$

$f(x)=\tanh^{-1}x:(-\infty,-1)\cup(1,\infty)\to(-\infty,0)\cup(0,\infty)$

Domain: $(-\infty,-1)\cup(1,\infty)\to$ Range: $(-\infty,0)\cup(0,\infty)$

b) **Expression**: Show that $\coth^{-1}x=\frac{1}{2}\ln(\frac{x+1}{x-1})$

Proof

Start with $y=\coth x=\frac{e^x+e^{-x}}{e^x-e^{-x}}$

To find the inverse solve for x and then interchange x and y.

$$y=\frac{e^x+e^{-x}}{e^x-e^{-x}}=\frac{e^x+\frac{1}{e^x}}{e^x-\frac{1}{e^x}}\ ;$$

Let $e^x=z>0 \quad z\neq 1, x\neq 0 \Rightarrow y=\frac{z+\frac{1}{z}}{z-\frac{1}{z}}\Rightarrow y=\frac{z^2+1}{z^2-1}\Rightarrow z^2(y-1)=1+y$

The quadratic equation in z gives: $z=\pm\sqrt{\frac{1+y}{y-1}}$

Because $z=e^x>0\Rightarrow$ choose $z=+\sqrt{\frac{1+y}{y-1}} \quad \therefore e^x=\sqrt{\frac{1+y}{y-1}}\Rightarrow x=\ln(\sqrt{\frac{1+y}{y-1}})$

Now, interchange x and y to obtain:

$$y=\coth^{-1}x=\ln(\sqrt{\frac{1+x}{x-1}})=\frac{1}{2}\ln(\frac{1+x}{x-1})$$

$$\boxed{\coth^{-1}x=\frac{1}{2}\ln(\frac{1+x}{x-1})}$$

c) **Derivative**

$$\frac{d}{dx}\coth^{-1}x=\frac{d}{dx}[\frac{1}{2}\ln(\frac{1+x}{x-1})]=\frac{1}{2}\frac{d}{dx}[\ln(\frac{1+x}{x-1})]=\frac{1}{2}\frac{d}{dx}[\ln(1+x)-\ln(x-1)]=$$

$$=\frac{1}{2}[\frac{1}{1+x}-\frac{1}{1-x}]=\frac{1}{2}[\frac{1}{1+x}-\frac{1}{1-x}]=\frac{1}{2}[\frac{1-x-1+x}{(1+x)(1-x)}]=\frac{1}{1-x^2}$$

$$\boxed{\frac{d}{dx}\tanh^{-1}x=\frac{1}{1-x^2}}$$

5. **The Inverse Hyperbolic Secant Function**

$\sec hx = y\ :\ (-\infty,\infty) \to (0,1])$ is **NOT** invertible

Restrict $\sec hx = y : (0,\infty) \to (0,1]$

a) **Definition**

The **inverse hyperbolic secant function** is defined as follows:

$y = \sec h^{-1}x$ **iff** $\sec hy = x$ with $y\ in\,(0,\infty)$ and $x\ in\ (0,1]$

$f(x) = \sec h^{-1}x : (0,1] \to (0,\infty)$

Domain: $(0,1] \to$ Range: $(0,\infty)$

b) **Expression**: Show that $\sec h^{-1}x = \ln(\frac{1+\sqrt{1-x^2}}{x})$

Proof

Start with $y = \sec hx = \dfrac{2}{e^x + e^{-x}}$

To find the inverse solve for x and then interchange x and y.

$$y = \frac{2}{e^x + e^{-x}} = \frac{2}{e^x + \frac{1}{e^x}}\ ;$$

Let $e^x = z > 1 \quad, x > 0 \Rightarrow y = \dfrac{2}{z + \frac{1}{z}} \Rightarrow y = \dfrac{2z}{z^2+1} \Rightarrow z^2 y - 2z + y = 0$

The quadratic equation in z gives: $z = \dfrac{1 \pm \sqrt{1-y^2}}{y}$

Because $z = e^x > 1 \Rightarrow$ choose $z = \dfrac{1+\sqrt{1-y^2}}{y} > 1$

$$\therefore e^x = \frac{1+\sqrt{1-y^2}}{y} \Rightarrow x = \ln(\frac{1+\sqrt{1-y^2}}{y})$$

Now, interchange x and y to obtain:

$$y = \sec h^{-1}x = \ln(\frac{1+\sqrt{1-x^2}}{x})$$

$$\boxed{\sec h^{-1}x = \ln(\frac{1+\sqrt{1-x^2}}{x})}$$

c) **Derivative**

$$\frac{d}{dx}\sec h^{-1}x = \frac{d}{dx}[\ln(\frac{1+\sqrt{1-x^2}}{x})] = \frac{d}{dx}[\ln(1+\sqrt{1-x^2}) - \ln x] = \cdots = -\frac{1}{x\sqrt{1-x^2}}$$

$$\boxed{\frac{d}{dx}\sec h^{-1}x = -\frac{1}{x\sqrt{1-x^2}}}$$

6. <u>The Inverse Hyperbolic Cosecant Function</u>

$\operatorname{csc} hx = y \ : \ (-\infty,0) \cup (0,\infty) \to (-\infty,0) \cup (0,\infty)$ is invertible

d) <u>Definition</u>

The **inverse hyperbolic cosecant function** is defined as follows:

$y = \operatorname{csc} h^{-1} x$ **iff** $\operatorname{csc} hy = x$ with $y \ in (0,\infty)$ and $x \ in \ (0,1]$

$f(x) = \sec h^{-1} x : (-\infty,0) \cup (0,\infty) \to (-\infty,0) \cup (0,\infty)$

<u>Expression</u> Start with $y = \operatorname{csc} h\, x = \dfrac{2}{e^x - e^{-x}}$

To find the inverse solve for x and then interchange x and y.

$$y = \frac{2}{e^x - e^{-x}} = \frac{2}{e^x - \dfrac{1}{e^x}}$$

Let $e^x = z > 0 \quad , z \neq 1, x \neq 0 \Rightarrow y = \dfrac{2}{z - \dfrac{1}{z}} \Rightarrow y = \dfrac{2z}{z^2 - 1} \Rightarrow z^2 y - 2z - y = 0$

The quadratic equation in z gives: $z = \dfrac{1 \pm \sqrt{1+y^2}}{y}$

$$z = \frac{1+\sqrt{1+y^2}}{y} = \frac{1}{y} + \frac{\sqrt{1+y^2}}{y} > 0 \ \ if \ \ y > 0$$

$$z = \frac{1-\sqrt{1+y^2}}{y} = \frac{1}{y} - \frac{\sqrt{1+y^2}}{y} = \frac{1}{y} + \frac{\sqrt{1+y^2}}{-y} > 0 \ \ if \ \ y < 0 \qquad \Rightarrow z = \frac{1}{y} + \frac{\sqrt{1+y^2}}{|y|}$$

$$\therefore e^x = \frac{1}{y} + \frac{\sqrt{1+y^2}}{|y|} \Rightarrow x = \ln\left(\frac{1}{y} + \frac{\sqrt{1+y^2}}{|y|}\right)$$

Now, interchange x and y to obtain: $y = \operatorname{csc} h^{-1} x = \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{1+x^2}}{|x|}\right)$

$$\boxed{\sec h^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)}$$

e) <u>Derivative</u>

$$\frac{d}{dx} \operatorname{csc} h^{-1} x = \frac{d}{dx}\left[\ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)\right] = \cdots = -\frac{1}{|x|\sqrt{1+x^2}}$$

$$\boxed{\frac{d}{dx} \sec h^{-1} x = -\frac{1}{|x|\sqrt{1+x^2}}}$$