6

APPLICATIONS OF INTEGRATION

6.4
Work

In this section, we will learn about:
Applying integration to calculate the amount of work done in performing a certain physical task.

WORK

The term ‘work’ is used in everyday language to mean the total amount of effort required to perform a task.

WORK

In physics, it has a technical meaning that depends on the idea of a ‘force.’

- Intuitively, you can think of a force as describing a push or pull on an object.
- Some examples are: the horizontal push of a book across a table or the downward pull of the earth’s gravity on a ball.

Defn. 1 / Eqn. 1

In general, if an object moves along a straight line with position function $s(t)$, then:

- The force F on the object (in the same direction) is defined by Newton's Second Law of Motion as the product of its mass m and its acceleration.

$$F = m\frac{d^2 s}{dt^2}$$

FORCE

In the SI metric system, the mass is measured in kilograms (kg), the displacement in meters (m), the time in seconds (s), and the force in newtons ($N = kg \cdot m/s^2$).

- Thus, a force of 1 N acting on a mass of 1 kg produces an acceleration of 1 m/s^2.

FORCE

In the US Customary system,
the fundamental unit is chosen to be
the unit of force, which is the pound.

Defn. 2 / Eqn. 2

In the case of constant acceleration:

- The force F is also constant and the work done is defined to be the product of the force F and the distance that the object moves:

$$W = Fd \qquad \text{(work = force x distance)}$$

WORK

If F is measured in newtons and d in meters, then the unit for W is a newton-meter called joule (J).

If F is measured in pounds and d in feet, then the unit for W is a foot-pound (ft-lb), which is about 1.36 J.

Example 1

a. How much work is done in lifting a 1.2-kg book off the floor to put it on a desk that is 0.7 m high?

- Use the fact that the acceleration due to gravity is $g = 9.8\ m/s^2$.

b. How much work is done in lifting a 20-lb weight 6 ft off the ground?

Example 1 a

The force exerted is equal and opposite to that exerted by gravity.

- Therefore, Equation 1 gives:

$$F = mg = (1.2)(9.8) = 11.76 \text{ N}$$

- Thus, Equation 2 gives the work done as:

$$W = Fd = (11.76)(0.7) \approx 8.2 \text{ J}$$

Example 1 b

Here, the force is given as F = 20 lb.

- Therefore, the work done is:

$$W = Fd = 20 \cdot 6 = 120 \text{ ft-lb}$$

- Notice that in (b), unlike (a), we did not have to multiply by g as we were given the ‘weight’ (which is a force) and not the mass of the object.

Equation 2 defines work as long as the force is constant.

However, what happens if the force is variable?

Let's suppose that the object moves along the x-axis in the positive direction, from $x = a$ to $x = b$.

- At each point x between a and b, a force $f(x)$ acts on the object—where f is a continuous function.

We divide the interval [a, b] into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and equal width Δx.

- We choose a sample point x_i^* in the i th subinterval $[x_{i-1}, x_i]$.
- Then, the force at that point is $f(x_i^*)$.

If n is large, then Δx is small, and since f is continuous, the values of f don't change very much over the interval $[x_{i-1}, x_i]$.

- In other words, f is almost constant on the interval.
- So, the work W_i that is done in moving the particle from x_{i-1} to x_i is approximately given by Equation 2: $W_i \approx f(x_i^*)\,\Delta x$

Thus, we can approximate the total work by:

$$W \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$$

WORK

It seems that this approximation becomes better as we make *n* larger.

$$W \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Definition 4

So, we define the work done in moving the object from *a* to *b* as the limit of the quantity as $n \to \infty$.

- As the right side of Equation 3 is a Riemann sum, we recognize its limit as being a definite integral.
- Thus,

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx$$

Example 2

When a particle is located a distance x feet from the origin, a force of $x^2 + 2x$ pounds acts on it.

How much work is done in moving it from $x = 1$ to $x = 3$?

Example 2

$$W = \int_1^3 (x^2 + 2x)\,dx = \left.\frac{x^3}{3} + x^2\right]_1^3 = \frac{50}{3}$$

- The work done is $16\frac{2}{3}$ ft-lb.

WORK

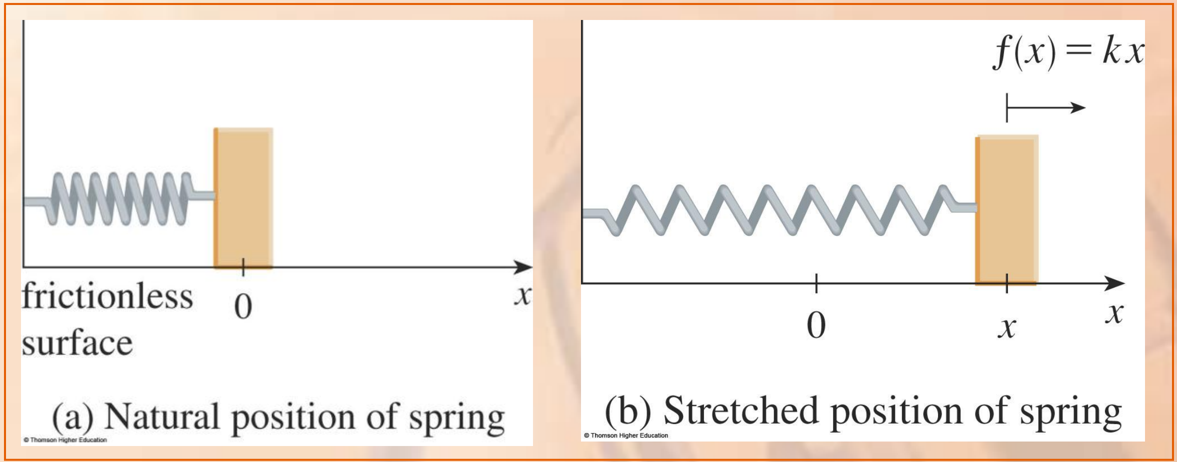
In the next example, we use a law from physics: Hooke’s Law.

HOOKE'S LAW

The force required to maintain a spring stretched x units beyond its natural length is proportional to x

$$f(x) = kx$$

where k is a positive constant (called the spring constant).

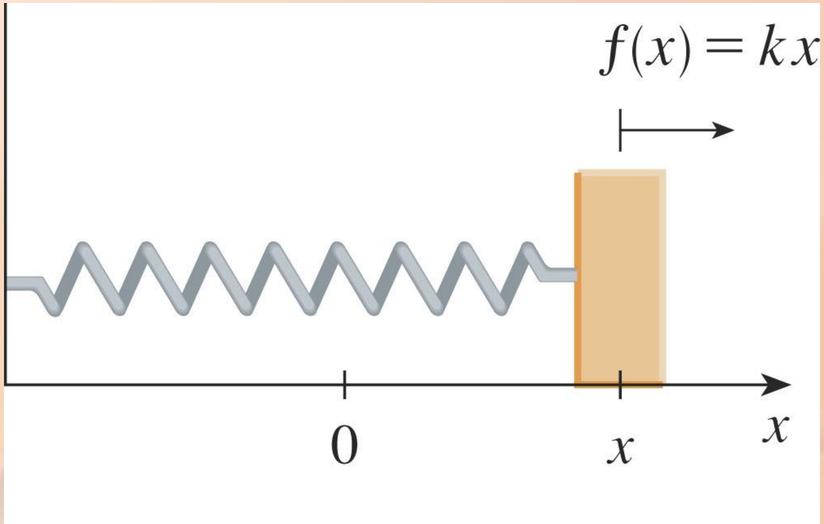


(a) Natural position of spring

(b) Stretched position of spring

HOOKE'S LAW

The law holds provided that *x* is not too large.



(b) Stretched position of spring

Example 3

A force of 40 N is required to hold a spring that has been stretched from its natural length of 10 cm to a length of 15 cm.

How much work is done in stretching the spring from 15 cm to 18 cm?

Example 3

According to Hooke's Law, the force required to hold the spring stretched x meters beyond its natural length is $f(x) = kx$.

- When the spring is stretched from 10 cm to 15 cm, the amount stretched is 5 cm = 0.05 m.
- This means that $f(0.05) = 40$, so $0.05k = 40$.
- Therefore, $k = \frac{40}{0.05} = 800$

Thus, $f(x) = 800x$ and the work done in stretching the spring from 15 cm to 18 cm is:

$$W = \int_{0.05}^{0.08} 800x\,dx = 800 \frac{x^2}{2} \bigg]_{0.05}^{0.08}$$

$$= 400\left[(0.08)^2 - (0.05)^2\right] = 1.56\,\text{J}$$

Example 4

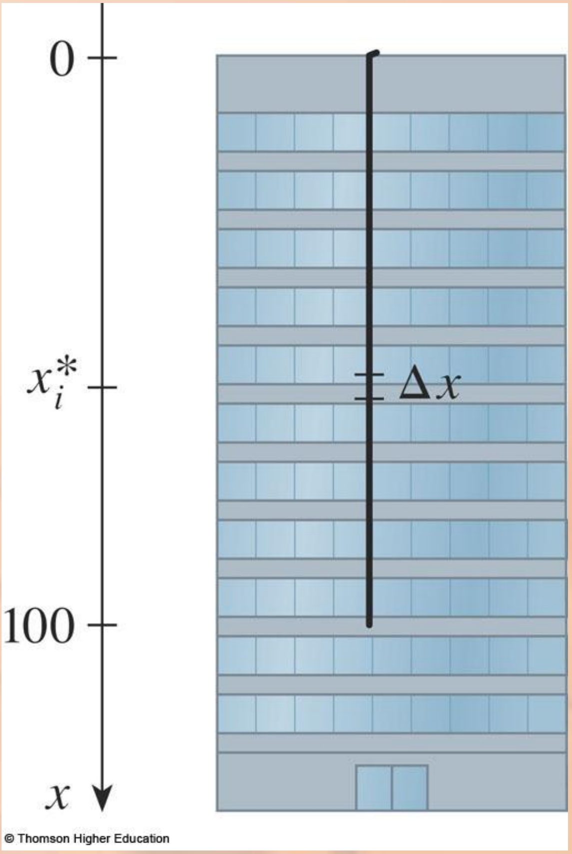
A 200-lb cable is 100 ft long and hangs vertically from the top of a tall building. How much work is required to lift the cable to the top of the building?

- Here, we don’t have a formula for the force function.
- However, we can use an argument similar to the one that led to Definition 4.

Example 4

Let's place the origin at the top of the building and the *x*-axis pointing downward.

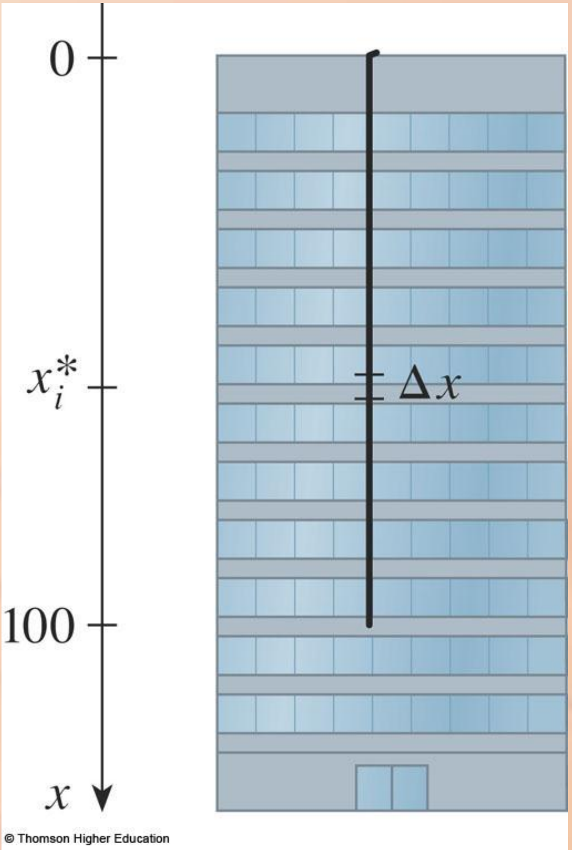
- We divide the cable into small parts with length Δ*x*.



Example 4

If x_i^* is a point in the *i* th such interval, then all points in the interval are lifted by roughly the same amount, namely x_i^*.

- The cable weighs 2 lb/foot.
- So, the weight of the *i* th part is $2\Delta x$.



Example 4

Thus, the work done on the *i* th part, in foot-pounds, is:

$$(2\Delta x) \quad x_i^* = 2x_i^* \Delta x$$

force distance

We get the total work done by adding all these approximations and letting the number of parts become large (so $\Delta x \to 0$):

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} 2x_i * \Delta x$$

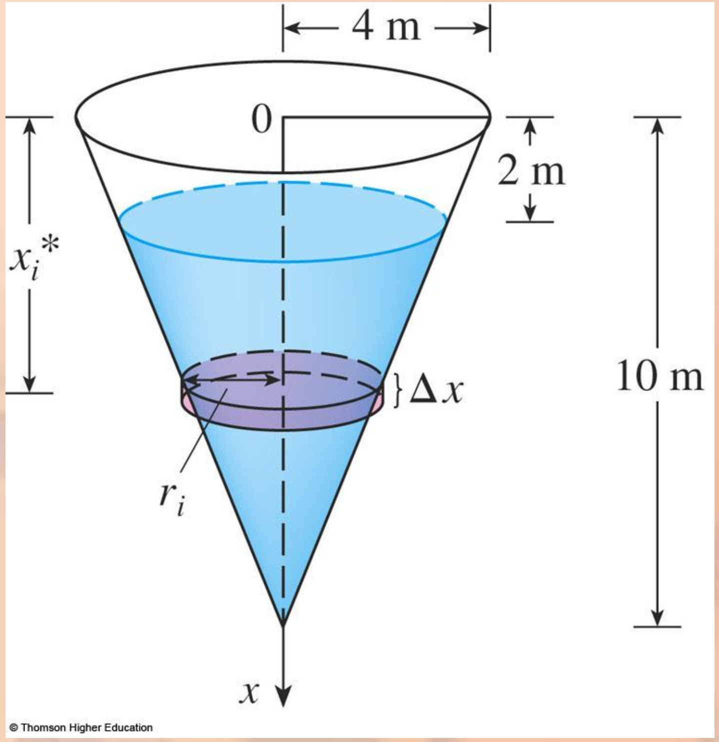
$$= \int_0^{100} 2x\,dx = x^2 \Big]_0^{100}$$

$$= 10{,}000 \text{ ft-lb}$$

Example 5

A tank has the shape of an inverted circular cone with height 10 m and base radius 4 m. It is filled with water to a height of 8 m.

- Find the work required to empty the tank by pumping all the water to the top of the tank.
- The density of water is 1,000 kg/m^3.

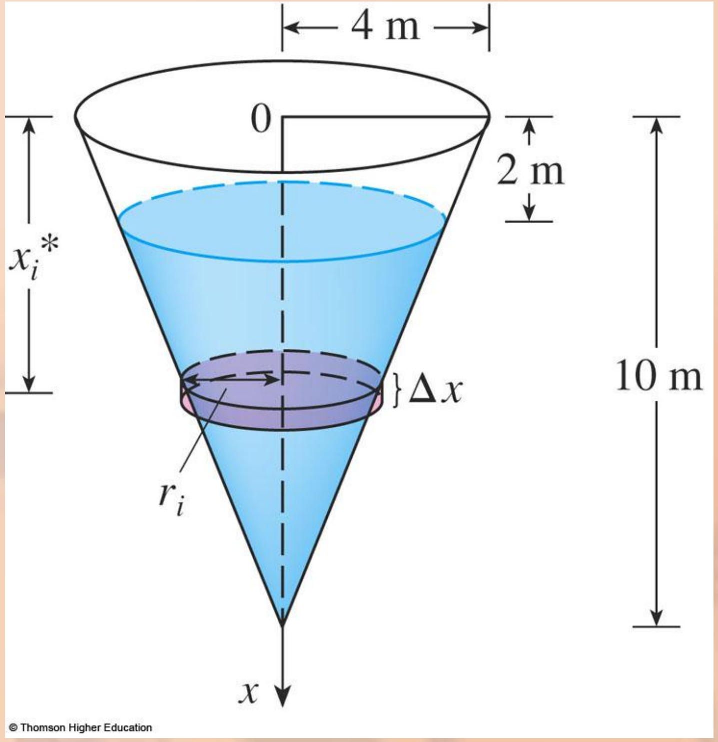


Example 5

Let's measure depths from the top of the tank by introducing a vertical coordinate line.

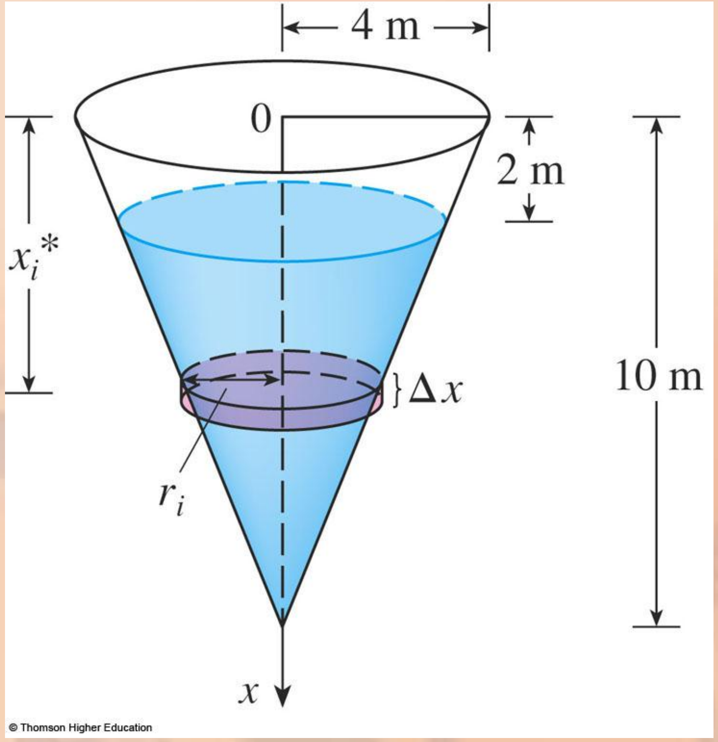
- The water extends from a depth of 2 m to a depth of 10 m.

- So, we divide the interval [2, 10] into *n* subintervals with endpoints $x_0, x_1, \ldots, x_n$ and choose x_i^* in the *i* th subinterval.



This divides the water into *n* layers.

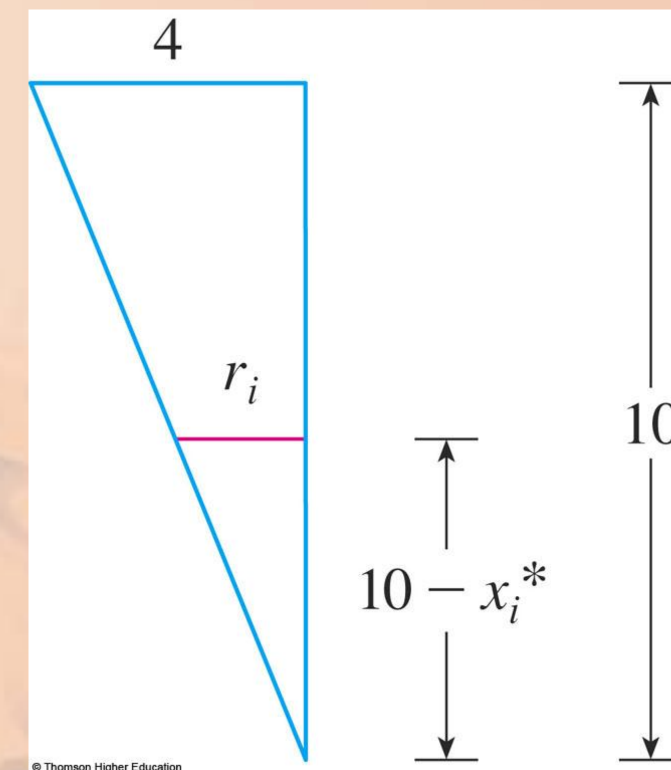
- The *i* th layer is approximated by a circular cylinder with radius r_i and height Δx.



Example 5

We can compute r_i from similar triangles:

$$\frac{r_i}{10 - x_i^*} = \frac{4}{10} \qquad r_i = \tfrac{2}{5}\left(10 - x_i^*\right)$$



Example 5

Thus, an approximation to the volume of the *i* th layer of water is:

$$V_i \approx \pi r_i^{\,2} \Delta x = \frac{4\pi}{25}\left(10 - x_i^{\,*}\right)^2 \Delta x$$

- So, its mass is: m_i = density x volume

$$\approx 1000 \cdot \frac{4\pi}{25}\left(10 - x_i^{\,*}\right)^2 \Delta x$$

$$= 160\pi\left(10 - x_i^{\,*}\right)^2 \Delta x$$

The force required to raise this layer must overcome the force of gravity and so:

$$
\begin{aligned}
F_i &= m_i g \\
&\approx (9.8)160\pi(10 - x_i^*)^2 \Delta x \\
&\approx 1570\pi(10 - x_i^*)^2 \Delta x
\end{aligned}
$$

Example 5

Each particle in the layer must travel a distance of approximately x_i*.

- The work W_i done to raise this layer to the top is approximately the product of the force F_i and the distance x_i*:

$$W_i \approx F_i x_i^*$$

$$\approx 1570\pi x_i^*(10 - x_i^*)^2 \Delta x$$

Example 5

To find the total work done in emptying the entire tank, we add the contributions of each of the layers and then take the limit as $n \to \infty$.

Example 5

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} 1570\pi x_i {}^*(10 - x_i {}^*)^2 \Delta x$$

$$= \int_2^{10} 1570\pi x_i {}^* (10 - x)^2 \, dx$$

$$= 1570\pi \int_2^{10} \left(100x - 20x + x^2\right) dx$$

$$= 1570\pi \left[50x^2 - \frac{20x^3}{3} + \frac{x^4}{4} \right]_2^{10}$$

$$= 1570\pi \left(\tfrac{2048}{3} \right) \approx 3.4 \times 10^6 \text{ J}$$