5

INTEGRALS

5.5

The Substitution Rule

In this section, we will learn:

To substitute a new variable in place of an existing expression in a function, making integration easier.

INTRODUCTION

Due to the Fundamental Theorem of Calculus (FTC), it’s important to be able to find antiderivatives.

INTRODUCTION

Equation 1

However, our antidifferentiation formulas don’t tell us how to evaluate integrals such as

$$\int 2x\sqrt{1+x^2}\,dx$$

INTRODUCTION

To find this integral, we use the problem-solving strategy of introducing something extra.

- The ‘something extra’ is a new variable.
- We change from the variable x to a new variable u.

INTRODUCTION

Suppose we let u be the quantity under the root sign in Equation 1, $u = 1 + x^2$.

- Then, the differential of u is $du = 2x\,dx$.

INTRODUCTION

Notice that, if the *dx* in the notation for an integral were to be interpreted as a differential, then the differential $2x\,dx$ would occur in Equation 1.

Equation 2

So, formally, without justifying our calculation, we could write:

$$\int 2x\sqrt{1+x^2}\,dx = \int \sqrt{1+x^2}\,2x\,dx$$

$$= \int \sqrt{u}\,du$$

$$= \tfrac{2}{3}u^{3/2} + C$$

$$= \tfrac{2}{3}(x^2+1)^{3/2} + C$$

INTRODUCTION

However, now we can check that we have the correct answer by using the Chain Rule to differentiate the final function of Equation 2:

$$\frac{d}{dx}\left[\tfrac{2}{3}(x^2+1)^{3/2}+C\right]=\tfrac{2}{3}\cdot\tfrac{3}{2}(x^2+1)^{1/2}\cdot 2x$$

$$=2x\sqrt{x^2+1}$$

INTRODUCTION

In general, this method works whenever we have an integral that we can write in the form

$$\int f(g(x))g'(x)\,dx$$

Observe that, if *F'* = *f*, then

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C$$

because, by the Chain Rule,

$$\frac{d}{dx}\left[F(g(x))\right] = F'(g(x))g'(x)$$

INTRODUCTION

If we make the 'change of variable' or 'substitution' $u = g(x)$, from Equation 3, we have:

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C$$

$$= F(u) + C$$

$$= \int F'(u)\,du$$

Writing *F'* = *f*, we get:

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

- Thus, we have proved the following rule.

SUBSTITUTION RULE

Equation 4

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

SUBSTITUTION RULE

Notice that the Substitution Rule for integration was proved using the Chain Rule for differentiation.

Notice also that, if $u = g(x)$, then $du = g'(x)\,dx$.

- So, a way to remember the Substitution Rule is to think of *dx* and *du* in Equation 4 as differentials.

SUBSTITUTION RULE

Thus, the Substitution Rule says:

It is permissible to operate with *dx* and *du* after integral signs as if they were differentials.

Example 1

Find $\int x^3 \cos(x^4 + 2)\, dx$

- We make the substitution $u = x^4 + 2$.
- This is because its differential is $du = 4x^3\, dx$, which, apart from the constant factor 4, occurs in the integral.

Example 1

Thus, using $x^3\,dx = du/4$ and the Substitution Rule, we have:

$$\int x^3 \cos(x^4+2)\,dx = \int \cos u \cdot \tfrac{1}{4}\,du = \tfrac{1}{4}\int \cos u \cdot du$$

$$= \tfrac{1}{4}\sin u + C$$

$$= \tfrac{1}{4}\sin(x^4+2) + C$$

- Notice that, at the final stage, we had to return to the original variable *x*.

SUBSTITUTION RULE

The idea behind the Substitution Rule is to replace a relatively complicated integral by a simpler integral.

- This is accomplished by changing from the original variable *x* to a new variable *u* that is a function of *x*.

- Thus, in Example 1, we replaced the integral $\int x^3\cos(x^4 + 2)\,dx$ by the simpler integral $\frac{1}{4}\int \cos u\,du$.

The main challenge in using the rule is to think of an appropriate substitution.

- You should try to choose *u* to be some function in the integrand whose differential also occurs—except for a constant factor.
- This was the case in Example 1.

If that is not possible, try choosing *u* to be some complicated part of the integrand—perhaps the inner function in a composite function.

Finding the right substitution is a bit of an art.

- It’s not unusual to guess wrong.
- If your first guess doesn’t work, try another substitution.

Evaluate $\int \sqrt{2x+1}\, dx$

- Let $u = 2x + 1$.
- Then, $du = 2\, dx$.
- So, $dx = du/2$.

E. g. 2—Solution 1

- Thus, the rule gives:

$$\int \sqrt{2x+1}\,dx = \int \sqrt{u}\,\frac{du}{2}$$

$$= \tfrac{1}{2}\int u^{1/2}du$$

$$= \tfrac{1}{2}\cdot\frac{u^{3/2}}{3/2} + C$$

$$= \tfrac{1}{3}u^{3/2} + C$$

$$= \tfrac{1}{3}(2x+1)^{3/2} + C$$

Another possible substitution is $u = \sqrt{2x+1}$

Then, $du = \dfrac{dx}{\sqrt{2x+1}}$

So, $dx = \sqrt{2x+1}$

- Alternatively, observe that $u^2 = 2x + 1$.
- So, $2u\,du = 2\,dx$.

Thus, $\int \sqrt{2x+1}\,dx = \int u \cdot u\,du$

$$= \int u^2\,du$$

$$= \frac{u^3}{3} + C$$

$$= \tfrac{1}{3}(2x+1)^{3/2} + C$$

Example 3

Find $\int \frac{x}{\sqrt{1-4x^2}}\,dx$

- Let $u = 1 - 4x^2$.
- Then, $du = -8x\,dx$.
- So, $x\,dx = -1/8\,du$ and

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx = -\tfrac{1}{8}\int \frac{1}{\sqrt{u}}\,du = -\tfrac{1}{8}\int u^{-1/2}\,du$$

$$= -\tfrac{1}{8}(2\sqrt{u}) + C = -\tfrac{1}{4}\sqrt{1-4x^2} + C$$

SUBSTITUTION RULE

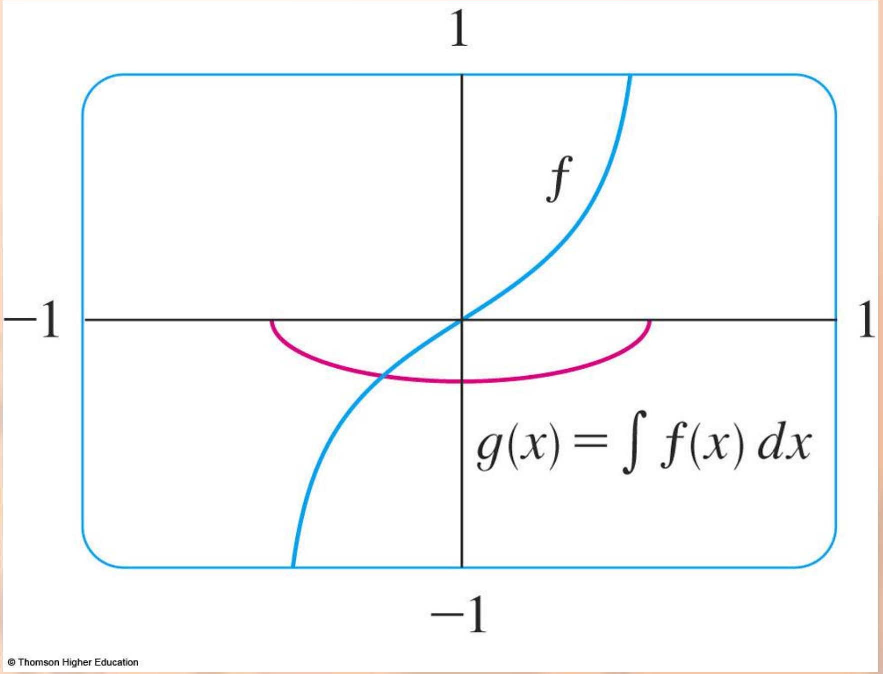
The answer to the example could be checked by differentiation.

Instead, let’s check it with a graph.

SUBSTITUTION RULE

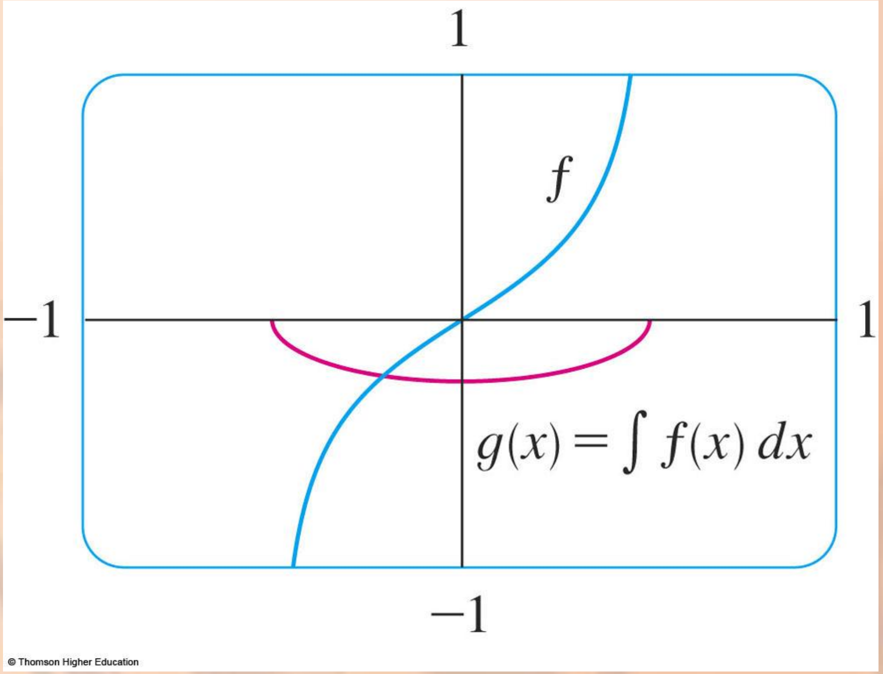
Here, we have used a computer to graph both the integrand $f(x) = x/\sqrt{1-4x^2}$ and its indefinite integral $g(x) = -\frac{1}{4}\sqrt{1-4x^2}$

- We take the case $C = 0$.



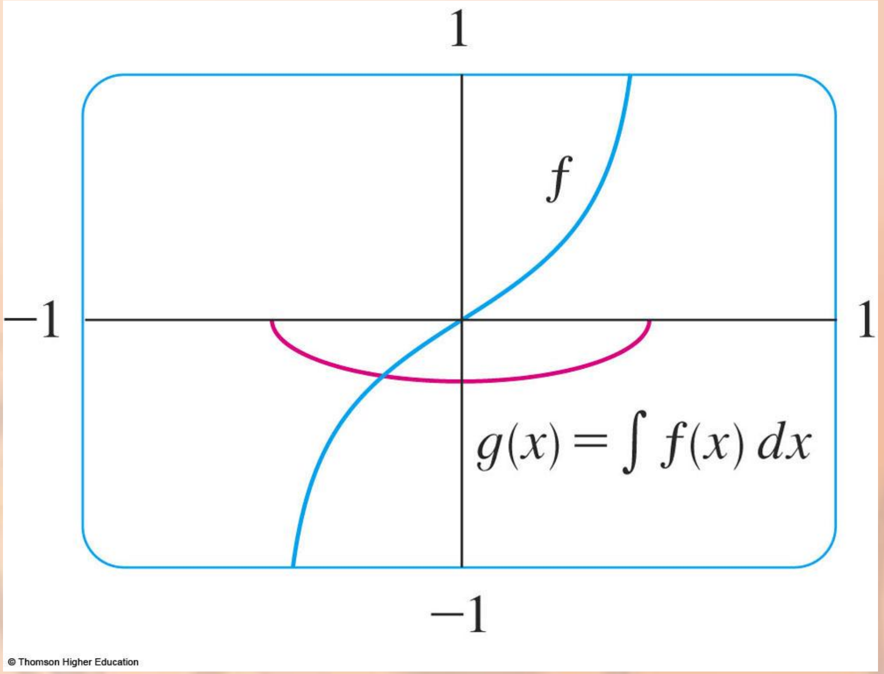
Notice that *g*(*x*):

- Decreases when *f*(*x*) is negative
- Increases when *f*(*x*) is positive
- Has its minimum value when *f*(*x*) = 0



SUBSTITUTION RULE

So, it seems reasonable, from the graphical evidence, that *g* is an antiderivative of *f*.



Example 4

Calculate $\int e^{5x}\,dx$

- If we let $u = 5x$, then $du = 5\,dx$.
- So, $dx = 1/5\,du$.
- Therefore, $\int e^{5x}dx = \frac{1}{5}\int e^{u}du$

$$= \frac{1}{5}e^{u} + C$$

$$= \frac{1}{5}e^{5x} + C$$

Example 5

Find $\int \sqrt{1+x^2}\, x^5 \, dx$

- An appropriate substitution becomes more obvious if we factor x^5 as $x^4 \cdot x$.
- Let $u = 1 + x^2$.
- Then, $du = 2x\, dx$.
- So, $x\, dx = du/2$.

Also, $x^2 = u - 1$; so, $x^4 = (u - 1)^2$:

$$\int \sqrt{1+x^2}\, x^5\, dx = \int \sqrt{1+x^2}\, x^4 \cdot x\, dx = \int \sqrt{u}\,(u-1)^2 \frac{du}{2}$$

$$= \tfrac{1}{2} \int \sqrt{u}\,(u^2 - 2u + 1)\, du$$

$$= \tfrac{1}{2} \int (u^{5/2} - 2u^{3/2} + u^{1/2})\, du$$

$$= \tfrac{1}{2} \left(\tfrac{2}{7} u^{7/2} - 2 \cdot \tfrac{2}{5} u^{5/2} + \tfrac{2}{3} u^{3/2}\right) + C$$

$$= \tfrac{1}{7} (1+x^2)^{7/2} - \tfrac{2}{5} (1+x^2)^{5/2} + \tfrac{1}{3} (1+x^2)^{3/2} + C$$

Calculate $\int \tan x \, dx$

- First, we write tangent in terms of sine and cosine:

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$$

- This suggests that we should substitute $u = \cos x$, since then $du = -\sin x \, dx$, and so $\sin x \, dx = -du$:

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\int \frac{du}{u}$$

$$= -\ln |u| + C$$

$$= -\ln |\cos x| + C$$

SUBSTITUTION RULE

Equation 5

Since $-\ln|\cos x| = \ln(|\cos x|^{-1})$

$= \ln(1/|\cos x|)$

$= \ln|\sec x|$,

the result of the example can also be written

as $$\int \tan x\, dx = \ln|\sec x| + C$$

DEFINITE INTEGRALS

When evaluating a definite integral by substitution, two methods are possible.

One method is to evaluate the indefinite integral first and then use the FTC.

- For instance, using the result of Example 2, we have:

$$\int_0^4 \sqrt{2x+1}\,dx = \int \sqrt{2x+1}\,dx\Big]_0^4$$

$$= \tfrac{1}{3}(2x+1)^{3/2}\Big]_0^4$$

$$= \tfrac{1}{3}(9)^{3/2} - \tfrac{1}{3}(1)^{3/2}$$

$$= \tfrac{1}{3}(27-1) = \tfrac{26}{3}$$

Another method, which is usually preferable, is to change the limits of integration when the variable is changed.

Thus, we have the substitution rule for definite integrals.

SUB. RULE FOR DEF. INTEGRALS **Equation 6**

If *g'* is continuous on [*a*, *b*] and *f* is continuous on the range of *u* = *g*(*x*), then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Let *F* be an antiderivative of *f*.

- Then, by Equation 3, $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$.
- So, by Part 2 of the FTC (FTC2), we have:

$$\int_a^b f(g(x))g'(x)\,dx = F(g(x))\Big]_a^b$$

$$= F(g(b)) - F(g(a))$$

- However, applying the FTC2 a second time, we also have:

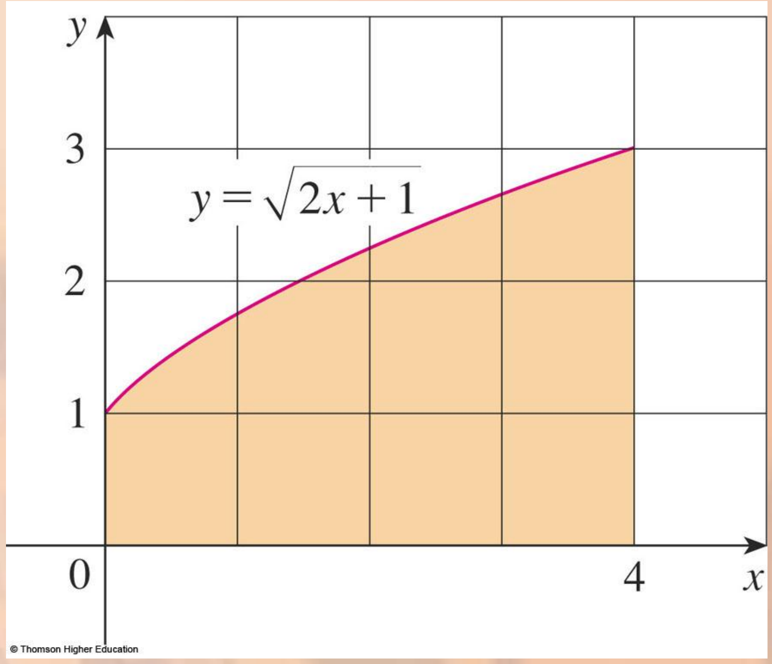
$$\int_{g(a)}^{g(b)} f(u)\,du = F(u)\Big]_{g(a)}^{g(b)}$$

$$= F(g(b)) - F(g(a))$$

 Example 7

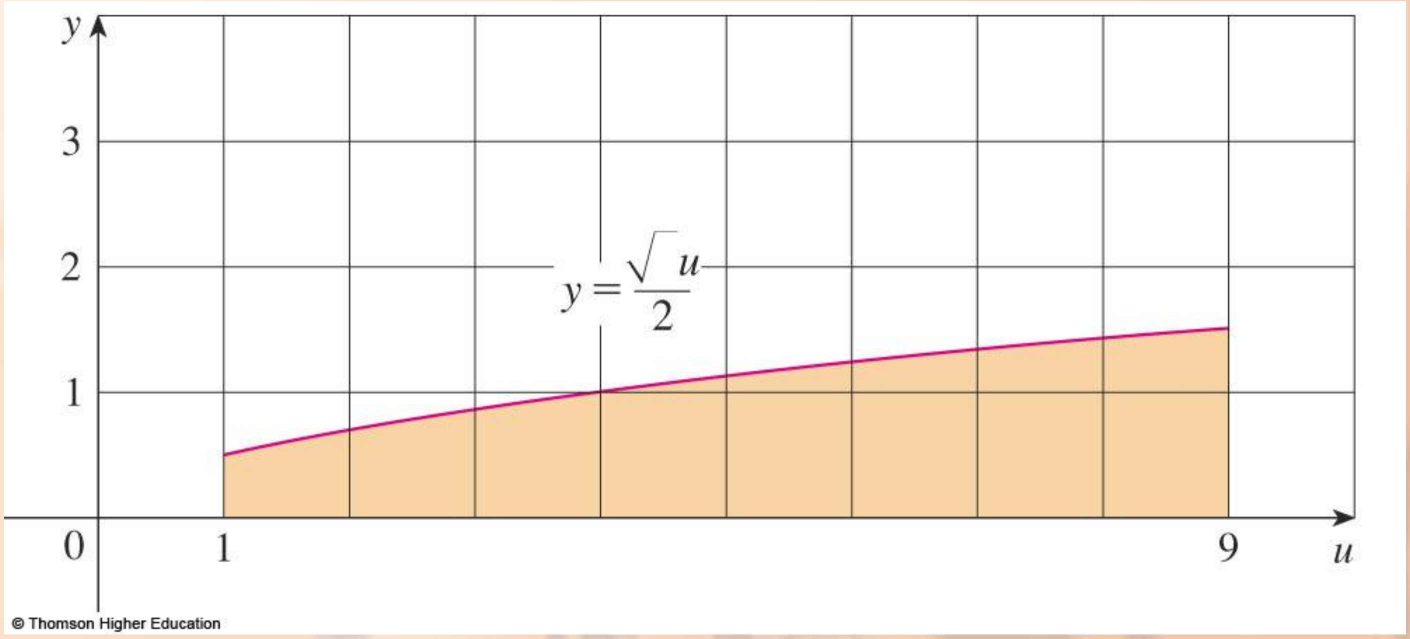
Evaluate $\int_0^4 \sqrt{2x+1}\,dx$ using Equation 6.

- Using the substitution from Solution 1 of Example 2, we have: $u = 2x + 1$ and $dx = du/2$



To find the new limits of integration, we note that:

- When $x = 0$, $u = 2(0) + 1 = 1$
- When $x = 4$, $u = 2(4) + 1 = 9$



Thus, $\int_0^4 \sqrt{2x+1}\,dx = \int_1^9 \tfrac{1}{2}\sqrt{u}\,du$

$$= \tfrac{1}{2}\cdot\tfrac{2}{3}u^{3/2}\Big]_1^9$$

$$= \tfrac{1}{3}(9^{3/2} - 1^{3/2})$$

$$= \tfrac{26}{3}$$

Observe that, when using Equation 6, we do not return to the variable *x* after integrating.

- We simply evaluate the expression in *u* between the appropriate values of *u*.

 Example 8

Evaluate $\int_1^2 \frac{dx}{(3-5x)^2}$

- Let $u = 3 - 5x$.
- Then, $du = -5\ dx$, so $dx = -du/5$.
- When $x = 1$, $u = -2$, and when $x = 2$, $u = -7$.

 Example 8

Thus, $\displaystyle\int_1^2 \frac{dx}{(3-5x)^2} = -\frac{1}{5}\int_{-2}^{-7} \frac{du}{u^2}$

$$= -\frac{1}{5}\left[-\frac{1}{u}\right]_{-5}^{-7}$$

$$= \frac{1}{5u}\bigg]_{-2}^{-7}$$

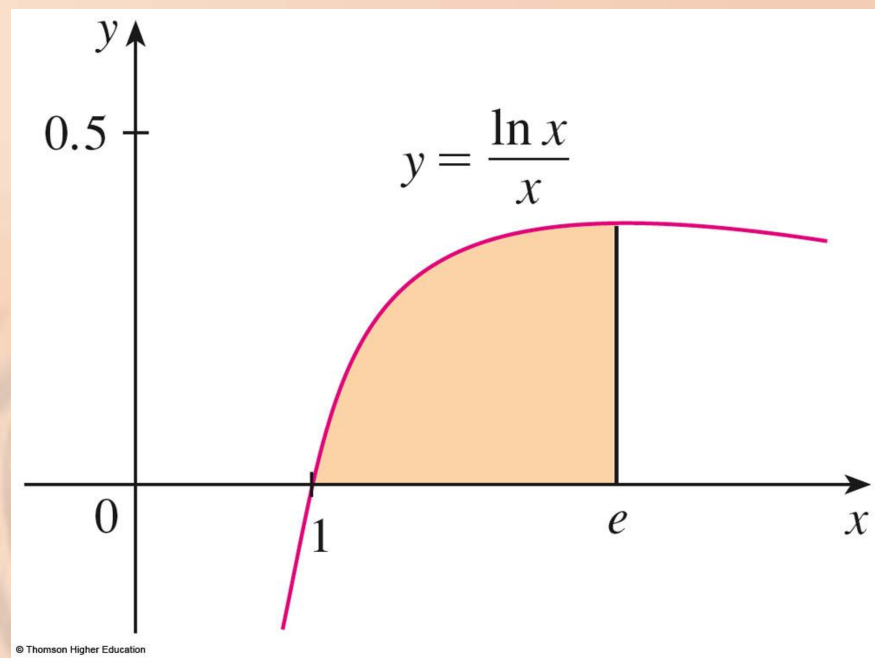
$$= \frac{1}{5}\left(-\frac{1}{7}+\frac{1}{2}\right) = \frac{1}{14}$$

Calculate $\int_1^e \frac{\ln x}{x}\,dx$

- We let $u = \ln x$ because its differential $du = dx/x$ occurs in the integral.
- When $x = 1$, $u = \ln 1$, and when $x = e$, $u = \ln e = 1$.
- Thus, $\int_1^e \frac{\ln x}{x}\,dx = \int_0^1 u\,du = \frac{u^2}{2}\Big]_0^1 = \frac{1}{2}$

Example 9

As the function $f(x) = (\ln x)/x$ in the example is positive for $x > 1$, the integral represents the area of the shaded region in this figure.



SYMMETRY

The next theorem uses the Substitution Rule for Definite Integrals to simplify the calculation of integrals of functions that possess symmetry properties.

Theorem 7

Suppose *f* is continuous on [–*a* , *a*].

a. If *f* is even, [*f*(–*x*) = *f*(*x*)], then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

b. If *f* is odd, [*f*(-*x*) = -*f*(*x*)], then

$$\int_{-a}^{a} f(x)\,dx = 0$$

Proof—Equation 8

We split the integral in two:

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

Proof

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

In the first integral in the second part, we make the substitution $u = -x$.

- Then, $du = -dx$, and when $x = -a$, $u = a$.

Therefore,

$$-\int_0^{-a} f(x)\,dx = -\int_0^{a} f(-u)(-du)$$

$$= \int_0^{a} f(-u)\,du$$

Proof—Equation 9

So, Equation 8 becomes:

$$\int_{-a}^{a} f(x)\,dx$$

$$= \int_{0}^{a} f(-u)\,du + \int_{0}^{a} f(x)\,dx$$

If f is even, then $f(-u) = f(u)$.

So, Equation 9 gives:

$$\int_{-a}^{a} f(x)\,dx$$

$$= \int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx$$

$$= 2\int_{0}^{a} f(x)\,dx$$

If f is odd, then $f(-u) = -f(u)$.

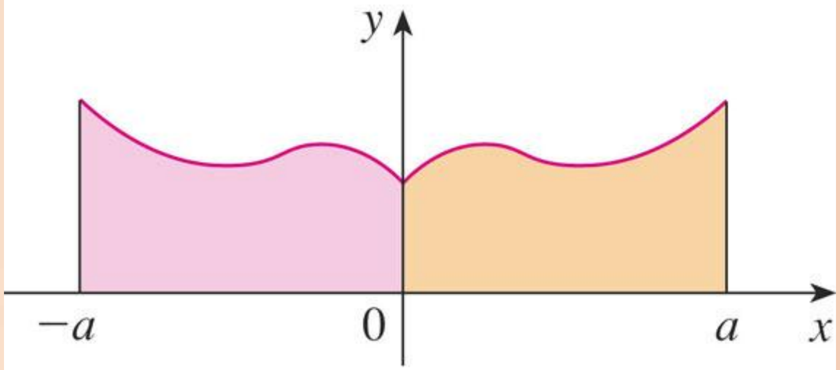
So, Equation 9 gives: $\int_{-a}^{a} f(x)\,dx$

$$= -\int_0^a f(u)\,du + \int_0^a f(x)\,dx$$

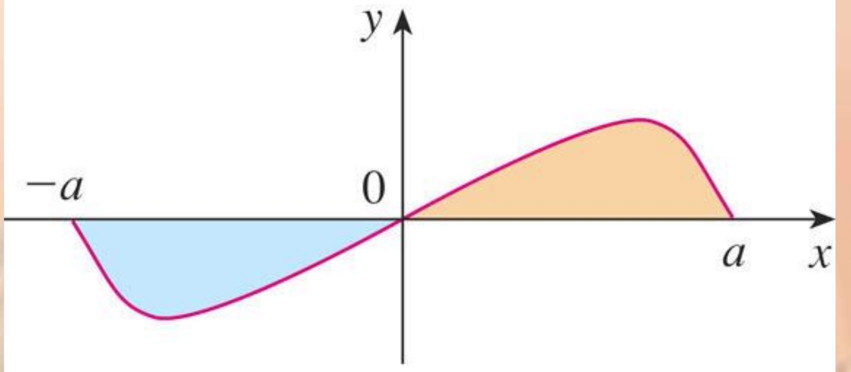
$$= 0$$

INTEGS. OF SYMM. FUNCTIONS

Theorem 7 is illustrated here.



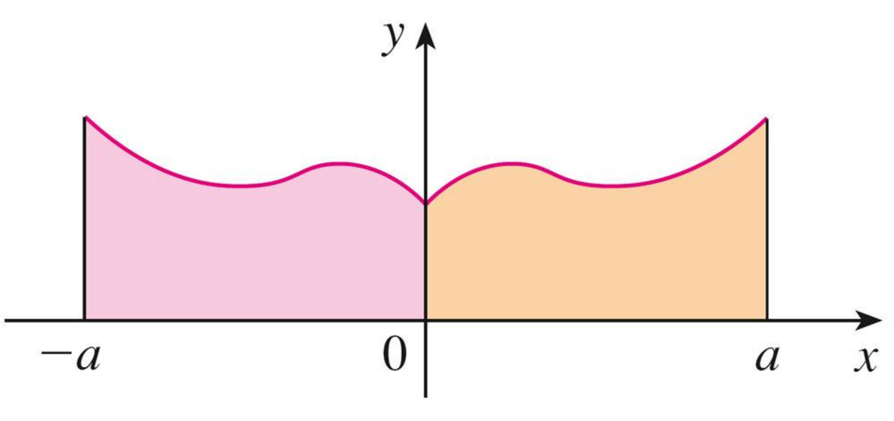
(a) f even, $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$



(b) f odd, $\int_{-a}^{a} f(x)\,dx = 0$

INTEGS. OF SYMM. FUNCTIONS

For the case where *f* is positive and even, part (a) says that the area under $y = f(x)$ from $-a$ to a is twice the area from 0 to a because of symmetry.



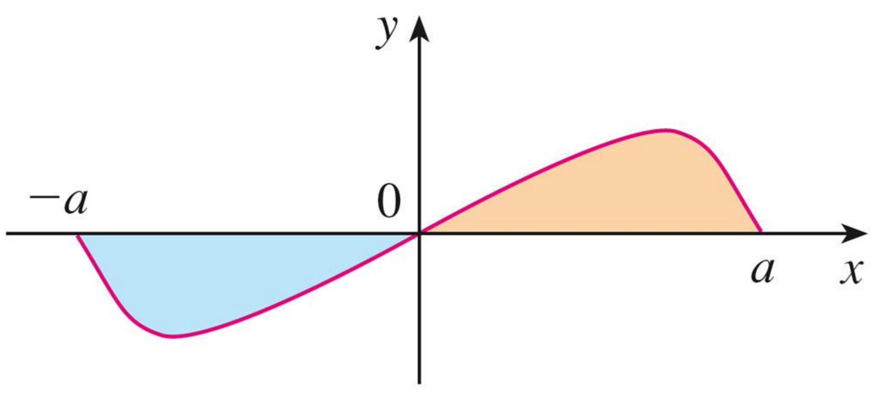
(a) f even, $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

INTEGS. OF SYMM. FUNCTIONS

Recall that an integral $\int_a^b f(x)\,dx$ can be expressed as the area above the *x*–axis and below $y = f(x)$ minus the area below the axis and above the curve.

INTEGS. OF SYMM. FUNCTIONS

Therefore, part (b) says the integral is 0 because the areas cancel.



(b) f odd, $\int_{-a}^{a} f(x)\,dx = 0$

Example 10

As $f(x) = x^6 + 1$ satisfies $f(-x) = f(x)$, it is even.

So, $\int_{-2}^{2} (x^6 + 1)\, dx = 2\int_{0}^{2} (x^6 + 1)\, dx$

$$= 2\left[\tfrac{1}{7}x^7 + x\right]_0^2$$

$$= 2\left(\tfrac{128}{7} + 2\right)$$

$$= \tfrac{284}{7}$$

Example 11

As $f(x) = (\tan x)/ (1 + x^2 + x^4)$ satisfies $f(-x) = -f(x)$, it is odd.

So, $$\int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^4}\, dx = 0$$