5

INTEGRALS

In Section 5.3, we saw that the second part of the Fundamental Theorem of Calculus (FTC) provides a very powerful method for evaluating the definite integral of a function.

- This is assuming that we can find an antiderivative of the function.

5.4
Indefinite Integrals and the Net Change Theorem

In this section, we will learn about:

Indefinite integrals and their applications.

In this section, we:

- Introduce a notation for antiderivatives.
- Review the formulas for antiderivatives.
- Use the formulas to evaluate definite integrals.
- Reformulate the second part of the FTC (FTC2) in a way that makes it easier to apply to science and engineering problems.

Both parts of the FTC establish connections between antiderivatives and definite integrals.

- Part 1 says that if, *f* is continuous, then $\int_a^x f(t)\,dt$ is an antiderivative of *f*.
- Part 2 says that $\int_a^b f(x)\,dx$ can be found by evaluating *F*(*b*) – *F*(*a*), where *F* is an antiderivative of *f*.

INDEFINITE INTEGRALS

We need a convenient notation for antiderivatives that makes them easy to work with.

INDEFINITE INTEGRAL

Due to the relation given by the FTC between antiderivatives and integrals, the notation $\int f(x)\,dx$ is traditionally used for an antiderivative of *f* and is called an indefinite integral.

Thus, $\int f(x)\,dx = F(x)$ means $F'(x) = f(x)$

For example, we can write

$$\int x^2\,dx = \frac{x^3}{3} + C \quad \text{because} \quad \frac{d}{dx}\left(\frac{x^3}{3} + C\right) = x^2$$

- Thus, we can regard an indefinite integral as representing an entire family of functions (one antiderivative for each value of the constant *C*).

INDEFINITE VS. DEFINITE INTEGRALS

You should distinguish carefully between definite and indefinite integrals.

- A definite integral $\int_a^b f(x)\,dx$ is a number.
- An indefinite integral $\int f(x)\,dx$ is a function (or family of functions).

INDEFINITE VS. DEFINITE INTEGRALS

The connection between them is given by the FTC2.

If *f* is continuous on [*a*, *b*], then

$$\int_a^b f(x)\,dx = \int f(x)\,dx \Big]_a^b$$

INDEFINITE INTEGRALS

The effectiveness of the FTC depends on having a supply of antiderivatives of functions.

- Therefore, we restate the Table of Antidifferentiation Formulas from Section 4.9, together with a few others, in the notation of indefinite integrals.

Any formula can be verified by differentiating the function on the right side and obtaining the integrand.

For instance, $\int \sec^2 x\, dx = \tan x + C$

because

$$\frac{d}{dx}(\tan x + C) = \sec^2 x$$

TABLE OF INDEFINITE INTEGRALS

Table 1

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C$$

TABLE OF INDEFINITE INTEGRALS

Table 1

$\int \sin x\, dx = -\cos x + C$	$\int \cos x\, dx = \sin x + C$
$\int \sec^2 x\, dx = \tan x + C$	$\int \csc^2 x\, dx = -\cot x + C$
$\int \sec x \tan x\, dx = \sec x + C$	$\int \csc x \cot x\, dx = -\csc x + C$
$\int \frac{1}{x^2+1}\, dx = \tan^{-1} x + C$	$\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + C$
$\int \sinh x\, dx = \cosh x + C$	$\int \cosh x\, dx = \sinh x + C$

INDEFINITE INTEGRALS

Recall from Theorem 1 in Section 4.9 that the most general antiderivative on a given interval is obtained by adding a constant to a particular antiderivative.

- We adopt the convention that, when a formula for a general indefinite integral is given, it is valid only on an interval.

Thus, we write

$$\int \frac{1}{x^2} dx = -\frac{1}{x} + C$$

with the understanding that it is valid on the interval (0, ∞) or on the interval (-∞, 0).

This is true despite the fact that the general antiderivative of the function $f(x) = 1/x^2$, $x \neq 0$, is:

$$F(x) = \begin{cases} -\dfrac{1}{x} + C_1 & \text{if } x < 0 \\ -\dfrac{1}{x} + C_2 & \text{if } x > 0 \end{cases}$$

Example 1

Find the general indefinite integral

$$\int (10x^4 - 2\sec^2 x)\, dx$$

- Using our convention and Table 1, we have:

$$\begin{aligned}\int (10x^4 - 2\sec^2 x)\, dx &= 10\int x^4\, dx - 2\int \sec^2 x\, dx \\ &= 10(x^5/5) - 2\tan x + C \\ &= 2x^5 - 2\tan x + C\end{aligned}$$

- You should check this answer by differentiating it.

Evaluate $\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$

- This indefinite integral isn't immediately apparent in Table 1.
- So, we use trigonometric identities to rewrite the function before integrating:

$$\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \int \left(\frac{1}{\sin\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right)d\theta$$

$$= \int \csc\theta\cot\theta\,d\theta = -\csc\theta + C$$

Example 3

Evaluate $\int_0^3 (x^3 - 6x)\,dx$

- Using FTC2 and Table 1, we have:

$$\int_0^3 (x^3 - 6x)\,dx = \frac{x^4}{4} - 6\frac{x^2}{2}\Bigg]_0^3$$

$$= \left(\tfrac{1}{4}\cdot 3^4 - 3\cdot 3^2\right) - \left(\tfrac{1}{4}\cdot 0^4 - 3\cdot 0^2\right)$$

$$= \tfrac{81}{4} - 27 - 0 + 0 = -6.75$$

- Compare this with Example 2 b in Section 5.2

Find

$$\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1} \right) dx$$

and interpret the result in terms of areas.

The FTC gives:

$$\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1} \right) dx = 2\frac{x^4}{4} - 6\frac{x^2}{2} + 3\tan^{-1} x \Big]_0^2$$

$$= \tfrac{1}{2} x^4 - 3x^2 + 3\tan^{-1} x \Big]_0^2$$

$$= \tfrac{1}{2}(2^4) - 3(2^2) + 3\tan^{-1} 2 - 0$$

$$= -4 + 3\tan^{-1} 2$$

- This is the exact value of the integral.

Example 4

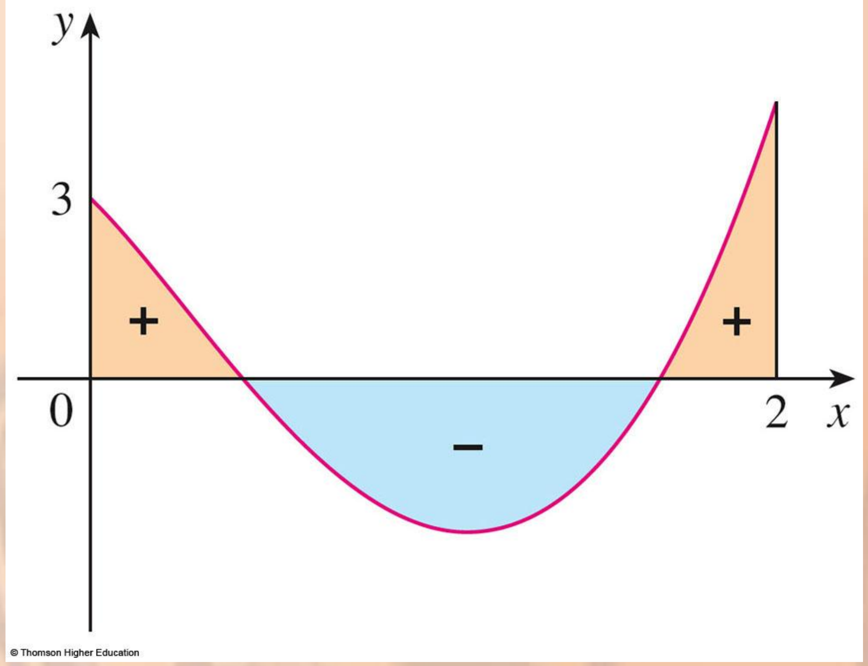
If a decimal approximation is desired, we can use a calculator to approximate $\tan^{-1} 2$.

Doing so, we get:

$$\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1} \right) dx \approx -0.67855$$

The figure shows the graph of the integrand in the example.

- We know from Section 5.2 that the value of the integral can be interpreted as the sum of the areas labeled with a plus sign minus the area labeled with a minus sign.



Evaluate $\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\, dt$

- First, we need to write the integrand in a simpler form by carrying out the division:

$$\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\, dt = \int_1^9 (2 + t^{1/2} - t^{-2})\, dt$$

Example 5

- Then, $\int_1^9 (2 + t^{1/2} - t^{-2})\,dt$

$$= 2t + \frac{t^{3/2}}{\frac{3}{2}} - \frac{t^{-1}}{-1}\Bigg]_1^9$$

$$= 2t + \frac{2}{3}t^{3/2} + \frac{1}{t}\Bigg]_1^9$$

$$= (2\cdot 9 + \tfrac{2}{3}\cdot 9^{3/2} + \tfrac{1}{9}) - (2\cdot 1 + \tfrac{2}{3}\cdot 1^{3/2} + \tfrac{1}{1})$$

$$= 18 + 18 + \tfrac{1}{9} - 2 - \tfrac{2}{3} - 1 = 32\tfrac{4}{9}$$

APPLICATIONS

The FTC2 says that, if *f* is continuous on [*a*, *b*], then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where *F* is any antiderivative of *f*.

- This means that *F'* = *f*.
- So, the equation can be rewritten as:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

We know $F'(x)$ represents the rate of change of $y = F(x)$ with respect to x and $F(b) – F(a)$ is the change in y when x changes from a to b.

- Note that y could, for instance, increase, then decrease, then increase again.
- Although y might change in both directions, $F(b) – F(a)$ represents the net change in y.

NET CHANGE THEOREM

So, we can reformulate the FTC2 in words, as follows.

The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

NET CHANGE THEOREM

This principle can be applied to all the rates of change in the natural and social sciences that we discussed in Section 3.7

The following are a few instances of the idea.

NET CHANGE THEOREM

If $V(t)$ is the volume of water in a reservoir at time t, its derivative $V'(t)$ is the rate at which water flows into the reservoir at time t.

- So, $\int_{t_1}^{t_2} V'(t)\,dt = V(t_2) - V(t_1)$

  is the change in the amount of water in the reservoir between time t_1 and time t_2.

NET CHANGE THEOREM

If $[C](t)$ is the concentration of the product of a chemical reaction at time t, then the rate of reaction is the derivative $d[C]/dt$.

- So, $\int_{t_1}^{t_2} \frac{d[C]}{dt}\,dt = [C](t_2) - [C](t_1)$

  is the change in the concentration of C from time t_1 to time t_2.

NET CHANGE THEOREM

If the mass of a rod measured from the left end to a point *x* is *m*(*x*), then the linear density is $\rho(x) = m'(x)$.

- So, $\int_a^b \rho(x)\,dx = m(b) - m(a)$

  is the mass of the segment of the rod that lies between $x = a$ and $x = b$.

NET CHANGE THEOREM

If the rate of growth of a population is *dn*/*dt*,

then $$\int_{t_1}^{t_2} \frac{dn}{dt}\,dt = n(t_2) - n(t_1)$$

is the net change in population during the time period from t_1 to t_2.

- The population increases when births happen and decreases when deaths occur.
- The net change takes into account both births and deaths.

NET CHANGE THEOREM

If $C(x)$ is the cost of producing x units of a commodity, then the marginal cost is the derivative $C'(x)$.

- So, $\int_{x_1}^{x_2} C'(x)\,dx = C(x_2) - C(x_1)$

  is the increase in cost when production is increased from x_1 units to x_2 units.

NET CHANGE THEOREM

Equation 2

If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$.

- So, $\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$

  is the net change of position, or displacement, of the particle during the time period from t_1 to t_2.

NET CHANGE THEOREM

In Section 5.1, we guessed that this was true for the case where the object moves in the positive direction.

Now, however, we have proved that it is always true.

NET CHANGE THEOREM

If we want to calculate the distance the object travels during that time interval, we have to consider the intervals when:

- $v(t) \geq 0$ (the particle moves to the right)
- $v(t) \leq 0$ (the particle moves to the left)

NET CHANGE THEOREM

Equation 3

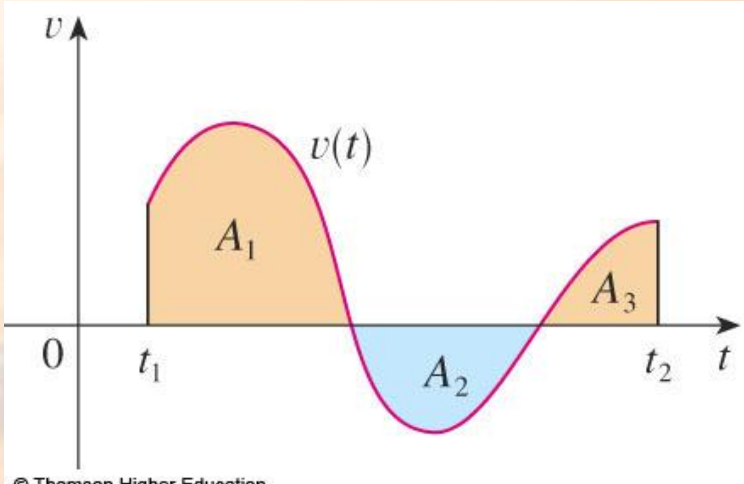
In both cases, the distance is computed by integrating $|v(t)|$, the speed.

Therefore,

$$\int_{t_1}^{t_2} |v(t)|\,dt = \text{total distance traveled}$$

NET CHANGE THEOREM

The figure shows how both displacement and distance traveled can be interpreted in terms of areas under a velocity curve.



$$\text{displacement} = \int_{t_1}^{t_2} v(t)\,dt = A_1 - A_2 + A_3$$

$$\text{distance} = \int_{t_1}^{t_2} |v(t)|\,dt = A_1 + A_2 + A_3$$

NET CHANGE THEOREM

The acceleration of the object is

$a(t) = v'(t)$.

- So, $\int_{t_1}^{t_2} a(t)\,dt = v(t_2) - v(t_1)$

  is the change in velocity from time t_1 to time t_2.

Example 6

A particle moves along a line so that its velocity at time t is:

$$v(t) = t^2 - t - 6 \text{ (in meters per second)}$$

a) Find the displacement of the particle during the time period $1 \leq t \leq 4$.

b) Find the distance traveled during this time period.

Example 6 a

By Equation 2, the displacement is:

$$s(4) - s(1) = \int_1^4 v(t)\,dt = \int_1^4 (t^2 - t - 6)\,dt$$

$$= \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t \right]_1^4 = -\frac{9}{2}$$

- This means that the particle moved 4.5 m toward the left.

Note that

$$v(t) = t^2 - t - 6 = (t - 3)(t + 2)$$

- Thus,
  $v(t) \leq 0$ on the interval [1, 3] and $v(t) \geq 0$ on [3, 4]

Example 6 b

So, from Equation 3, the distance traveled is:

$$\int_1^4 |v(t)|\,dt = \int_1^3 [-v(t)]\,dt + \int_3^4 v(t)\,dt$$

$$= \int_1^3 (-t^2 + t + 6)\,dt + \int_3^4 (t^2 - t - 6)\,dt$$

$$= \left[-\frac{t^3}{3} + \frac{t^2}{2} + 6t \right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t \right]_3^4$$

$$= \frac{61}{6} \approx 10.17\,\text{m}$$

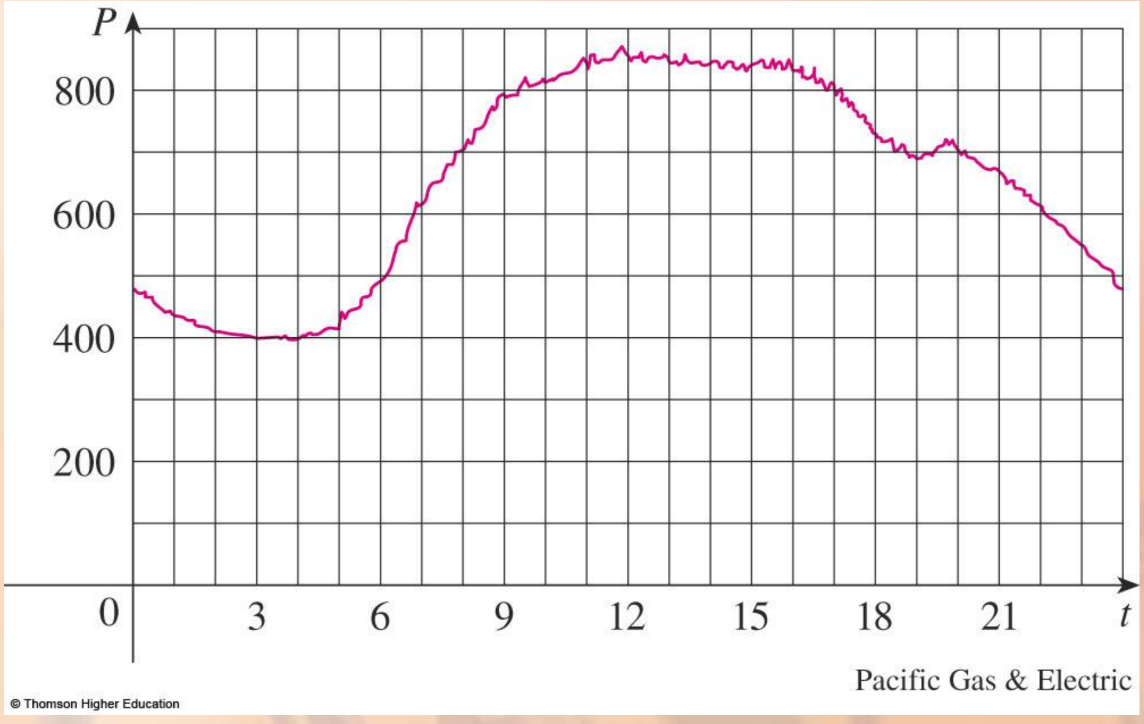
NET CHANGE THEOREM

Example 7

The figure shows the power consumption in San Francisco for a day in September.

- *P* is measured in megawatts.
- *t* is measured in hours starting at midnight.

Estimate the energy used on that day.



Example 7

Power is the rate of change of energy:

$$P(t) = E'(t)$$

- So, by the Net Change Theorem,

$$\int_0^{24} P(t)\,dt = \int_0^{24} E'(t)\,dt = E(24) - E(0)$$

is the total amount of energy used that day.

We approximate the value of the integral using the Midpoint Rule with 12 subintervals and $\Delta t = 2$, as follows.

$$\int_0^{24} P(t)\,dt$$

$$\approx [P(1) + P(3) + P(5) + \ldots + P(21) + P(23)]\Delta t$$

$$\approx (440 + 400 + 420 + 620 + 790 + 840 + 850$$

$$+ 840 + 810 + 690 + 670 + 550)(2)$$

$$= 15,840$$

- The energy used was approximately 15,840 megawatt-hours.

NET CHANGE THEOREM

How did we know what units to use for energy in the example?

NET CHANGE THEOREM

The integral $\int_0^{24} P(t)\,dt$ is defined as the limit of sums of terms of the form $P(t_i^*)\,\Delta t$.

Now, $P(t_i^*)$ is measured in megawatts and Δt is measured in hours.

- So, their product is measured in megawatt-hours.
- The same is true of the limit.

NET CHANGE THEOREM

In general, the unit of measurement for

$$\int_a^b f(x)\,dx$$

is the product of the unit for *f*(*x*) and the unit for *x*.