4

APPLICATIONS OF DIFFERENTIATION

4.9
Antiderivatives

In this section, we will learn about:

Antiderivatives and how they are useful in solving certain scientific problems.

INTRODUCTION

A physicist who knows the velocity of a particle might wish to know its position at a given time.

INTRODUCTION

An engineer who can measure the variable rate at which water is leaking from a tank wants to know the amount leaked over a certain time period.

INTRODUCTION

A biologist who knows the rate at which a bacteria population is increasing might want to deduce what the size of the population will be at some future time.

ANTIDERIVATIVES

In each case, the problem is to find a function F whose derivative is a known function f.

If such a function F exists, it is called an antiderivative of f.

DEFINITION

A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

For instance, let $f(x) = x^2$.

- It isn't difficult to discover an antiderivative of *f* if we keep the Power Rule in mind.
- In fact, if $F(x) = \frac{1}{3} x^3$, then $F'(x) = x^2 = f(x)$.

However, the function $G(x) = \frac{1}{3} x^3 + 100$ also satisfies $G'(x) = x^2$.

- Therefore, both *F* and *G* are antiderivatives of *f*.

ANTIDERIVATIVES

Indeed, any function of the form

$H(x) = \frac{1}{3} x^3 + C$, where C is a constant,

is an antiderivative of f.

- The question arises: Are there any others?

To answer the question, recall that, in Section 4.2, we used the Mean Value Theorem.

- It was to prove that, if two functions have identical derivatives on an interval, then they must differ by a constant (Corollary 7).

ANTIDERIVATIVES

Thus, if F and G are any two antiderivatives of f, then $F'(x) = f(x) = G(x)$

So, $G(x) – F(x) = C$, where C is a constant.

- We can write this as $G(x) = F(x) + C$.
- Hence, we have the following theorem.

Theorem 1

If F is an antiderivative of f on an interval I, the most general antiderivative of f on I is

$$F(x) + C$$

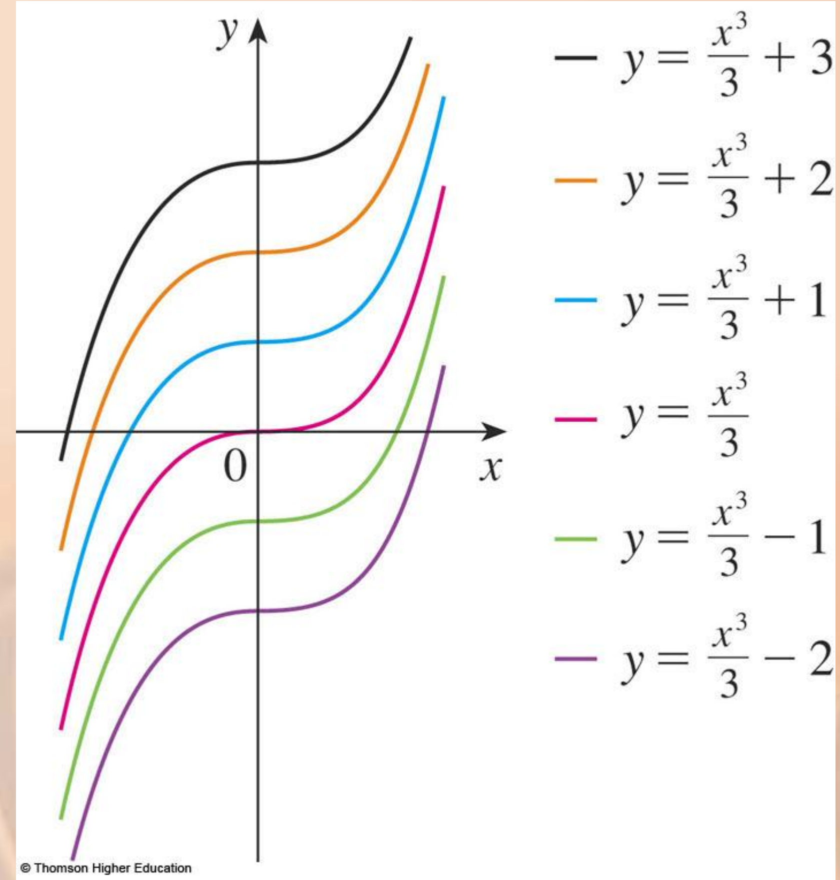
where C is an arbitrary constant.

Going back to the function $f(x) = x^2$, we see that the general antiderivative of f is $\frac{1}{3} x^3 + C$.

FAMILY OF FUNCTIONS

By assigning specific values to *C*, we obtain a family of functions.

- Their graphs are vertical translates of one another.
- This makes sense, as each curve must have the same slope at any given value of *x*.



Find the most general antiderivative of each function.

a. $f(x) = \sin x$

b. $f(x) = 1/x$

c. $f(x) = x^n,\ n \neq -1$

If $f(x) = -\cos x$, then $F'(x) = \sin x$.

- So, an antiderivative of $\sin x$ is $-\cos x$.
- By Theorem 1, the most general antiderivative is: $G(x) = -\cos x + C$

Recall from Section 3.6 that

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

- So, on the interval $(0, \infty)$, the general antiderivative of $1/x$ is $\ln x + C$.

We also learned that

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}$$

for all $x \neq 0$.

- Theorem 1 then tells us that the general antiderivative of $f(x) = 1/x$ is $\ln|x| + C$ on any interval that doesn't contain 0.

In particular, this is true on each of the intervals (-∞, 0) and (0, ∞).

- So, the general antiderivative of *f* is:

$$F(x) = \begin{cases} \ln x + C_1 & \text{if } x > 0 \\ \ln(-x) + C_2 & \text{if } x < 0 \end{cases}$$

We use the Power Rule to discover an antiderivative of x^n.

In fact, if $n \neq -1$, then

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{(n+1)x^n}{n+1} = x^n$$

Therefore, the general antiderivative of $f(x) = x^n$ is:

$$F(x) = \frac{x^{n+1}}{n+1} + C$$

- This is valid for $n \geq 0$ since then $f(x) = x^n$ is defined on an interval.

- If n is negative (but $n \neq -1$), it is valid on any interval that doesn't contain 0.

ANTIDERIVATIVE FORMULA

As in the example, every differentiation formula, when read from right to left, gives rise to an antidifferentiation formula.

ANTIDERIVATIVE FORMULA

Table 2

Here, we list some particular antiderivatives.

Function	Particular antiderivative	Function	Particular antiderivative
$cf(x)$	$cF(x)$	$\sin x$	$-\cos x$
$f(x) + g(x)$	$F(x) + G(x)$	$\sec^2 x$	$\tan x$
$x^n \quad (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\sec x \tan x$	$\sec x$
$1/x$	$\ln \lvert x \rvert$	$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
e^x	e^x	$\dfrac{1}{1+x^2}$	$\tan^{-1} x$
$\cos x$	$\sin x$		

ANTIDERIVATIVE FORMULA

Each formula is true because the derivative of the function in the right column appears in the left column.

Function	Particular antiderivative	Function	Particular antiderivative
$cf(x)$	$cF(x)$	$\sin x$	$-\cos x$
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$1/x$	$\ln \lvert x \rvert$	$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
e^x	e^x	$\dfrac{1}{1+x^2}$	$\tan^{-1} x$
$\cos x$	$\sin x$		

ANTIDERIVATIVE FORMULA

In particular, the first formula says that the antiderivative of a constant times a function is the constant times the antiderivative of the function.

Function	Particular antiderivative	Function	Particular antiderivative
$cf(x)$	$cF(x)$	$\sin x$	$-\cos x$
$f(x) + g(x)$	$F(x) + G(x)$	$\sec^2 x$	$\tan x$
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$1/x$	$\ln \lvert x \rvert$	$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
e^x	e^x	$\dfrac{1}{1+x^2}$	$\tan^{-1} x$
$\cos x$	$\sin x$		

ANTIDERIVATIVE FORMULA

The second formula says that the antiderivative of a sum is the sum of the antiderivatives.

- We use the notation *F'* = *f*, *G'* = *g*.

Function	Particular antiderivative	Function	Particular antiderivative
$cf(x)$	$cF(x)$	$\sin x$	$-\cos x$
$f(x) + g(x)$	$F(x) + G(x)$	$\sec^2 x$	$\tan x$
$x^n \quad (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\sec x \tan x$	$\sec x$
$1/x$	$\ln \lvert x \rvert$	$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
e^x	e^x		
$\cos x$	$\sin x$	$\dfrac{1}{1+x^2}$	$\tan^{-1} x$

Find all functions *g* such that

$$g'(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x}$$

First, we rewrite the given function:

$$g'(x) = 4\sin x + \frac{2x^5}{x} - \frac{\sqrt{x}}{x} = 4\sin x + 2x^4 - \frac{1}{\sqrt{x}}$$

- Thus, we want to find an antiderivative of:

$$g'(x) = 4\sin x + 2x^4 - x^{-1/2}$$

Using the formulas in Table 2 together with Theorem 1, we obtain:

$$g(x) = 4(-\cos x) + 2\frac{x^5}{5} - \frac{x^{1/2}}{\frac{1}{2}} + C$$

$$= -4\cos x + \tfrac{2}{5}x^5 - 2\sqrt{x} + C$$

ANTIDERIVATIVES

In applications of calculus, it is very common to have a situation as in the example—where it is required to find a function, given knowledge about its derivatives.

DIFFERENTIAL EQUATIONS

An equation that involves the derivatives of a function is called a differential equation.

- These will be studied in some detail in Chapter 9.
- For the present, we can solve some elementary differential equations.

DIFFERENTIAL EQUATIONS

The general solution of a differential equation involves an arbitrary constant (or constants), as in Example 2.

- However, there may be some extra conditions given that will determine the constants and, therefore, uniquely specify the solution.

Find *f* if

$f'(x) = e^x + 20(1 + x^2)^{-1}$ and $f(0) = -2$

- The general antiderivative of

$$f'(x) = e^x + \frac{20}{1+x^2}$$

is

$$f'(x) = e^x + 20\tan^{-1} x + C$$

To determine C, we use the fact that $f(0) = -2$:

$$f(0) = e^0 + 20 \tan^{-1} 0 + C = -2$$

- Thus, we have: $C = -2 - 1 = -3$

- So, the particular solution is:

$$f(x) = e^x + 20 \tan^{-1} x - 3$$

Find f if $f''(x) = 12x^2 + 6x - 4$,

$f(0) = 4$, and $f(1) = 1$.

The general antiderivative of

$f''(x) = 12x^2 + 6x - 4$ is:

$$f'(x) = 12\frac{x^3}{3} + 6\frac{x^2}{2} - 4x + C$$

$$= 4x^3 + 3x^2 - 4x + C$$

Using the antidifferentiation rules once more, we find that:

$$f(x) = 4\frac{x^4}{4} + 3\frac{x^3}{3} - 4\frac{x^2}{2} + Cx + D$$

$$= x^4 + x^3 - 2x^2 + Cx + D$$

To determine C and D, we use the given conditions that $f(0) = 4$ and $f(1) = 1$.

- As $f(0) = 0 + D = 4$, we have: $D = 4$
- As $f(1) = 1 + 1 – 2 + C + 4 = 1$, we have: $C = –3$

Therefore, the required function is:

$$f(x) = x^4 + x^3 - 2x^2 - 3x + 4$$

GRAPH

If we are given the graph of a function *f*, it seems reasonable that we should be able to sketch the graph of an antiderivative *F*.

Suppose we are given that $F(0) = 1$.

- We have a place to start—the point (0, 1).
- The direction in which we move our pencil is given at each stage by the derivative $F'(x) = f(x)$.

GRAPH

In the next example, we use the principles of this chapter to show how to graph F even when we don’t have a formula for f.

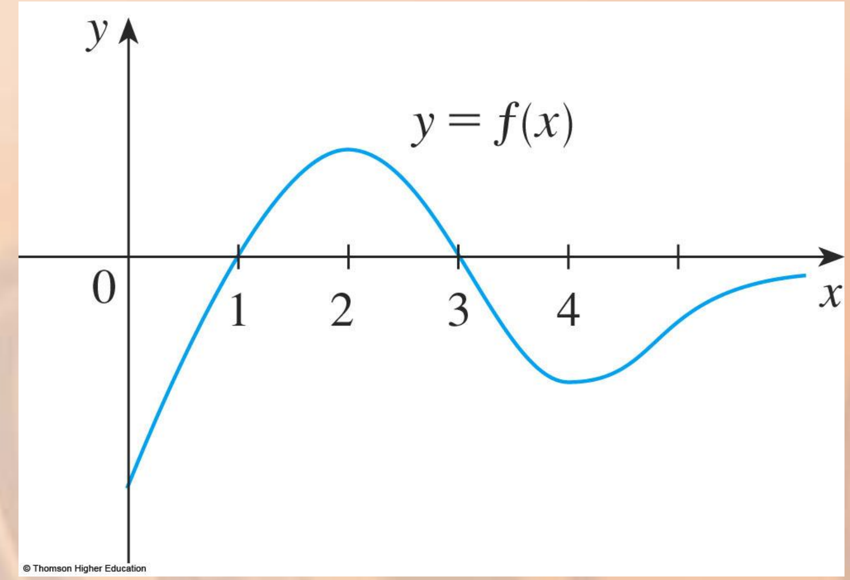
- This would be the case, for instance, when $f(x)$ is determined by experimental data.

GRAPH

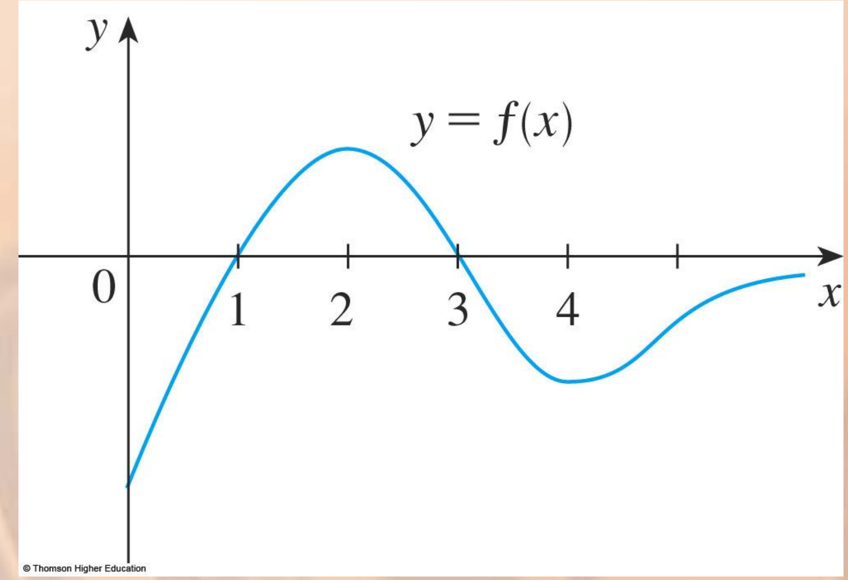
Example 5

The graph of a function *f* is given.

Make a rough sketch of an antiderivative *F*, given that *F*(0) = 2.



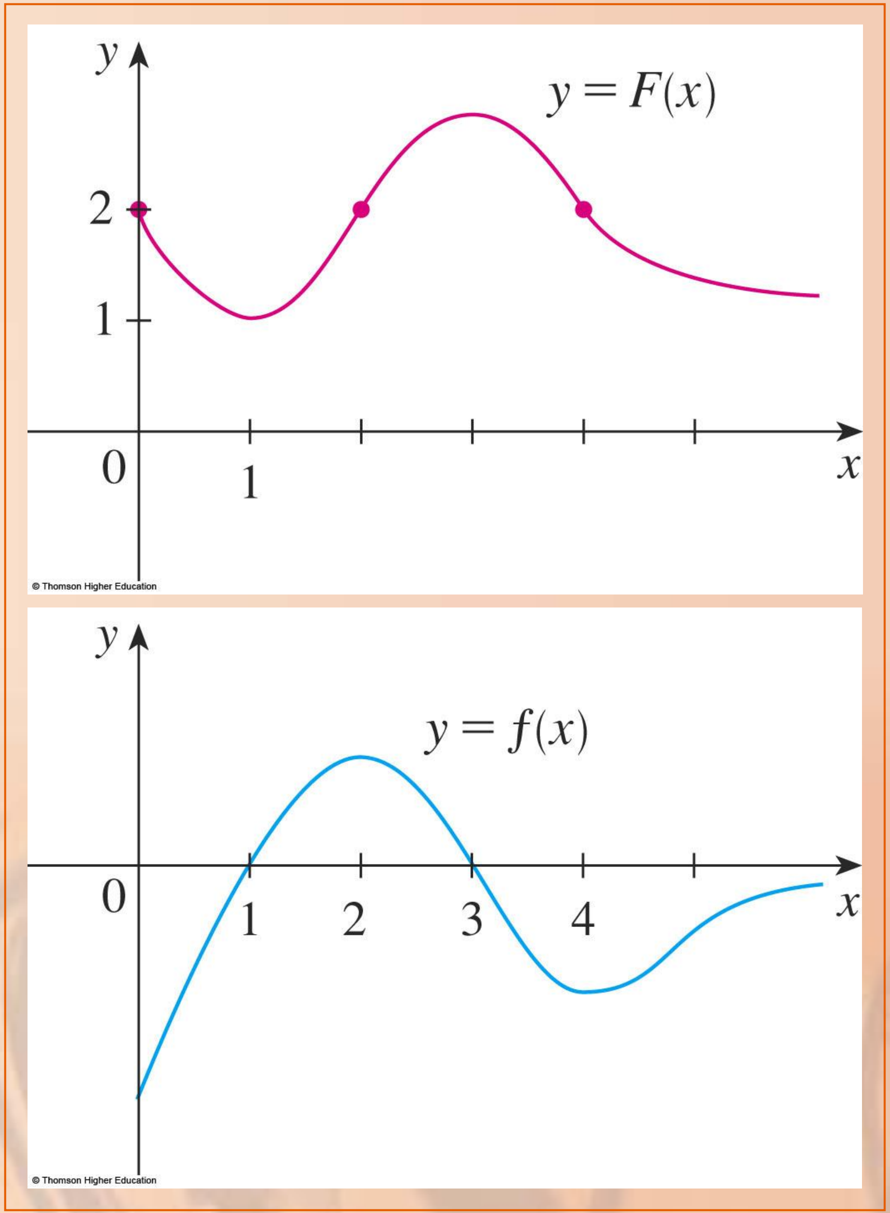
We are guided by the fact that the slope of $y = F(x)$ is $f(x)$.



GRAPH

Example 5

We start at (0, 2) and draw *F* as an initially decreasing function since *f*(*x*) is negative when 0 < *x* < 1.



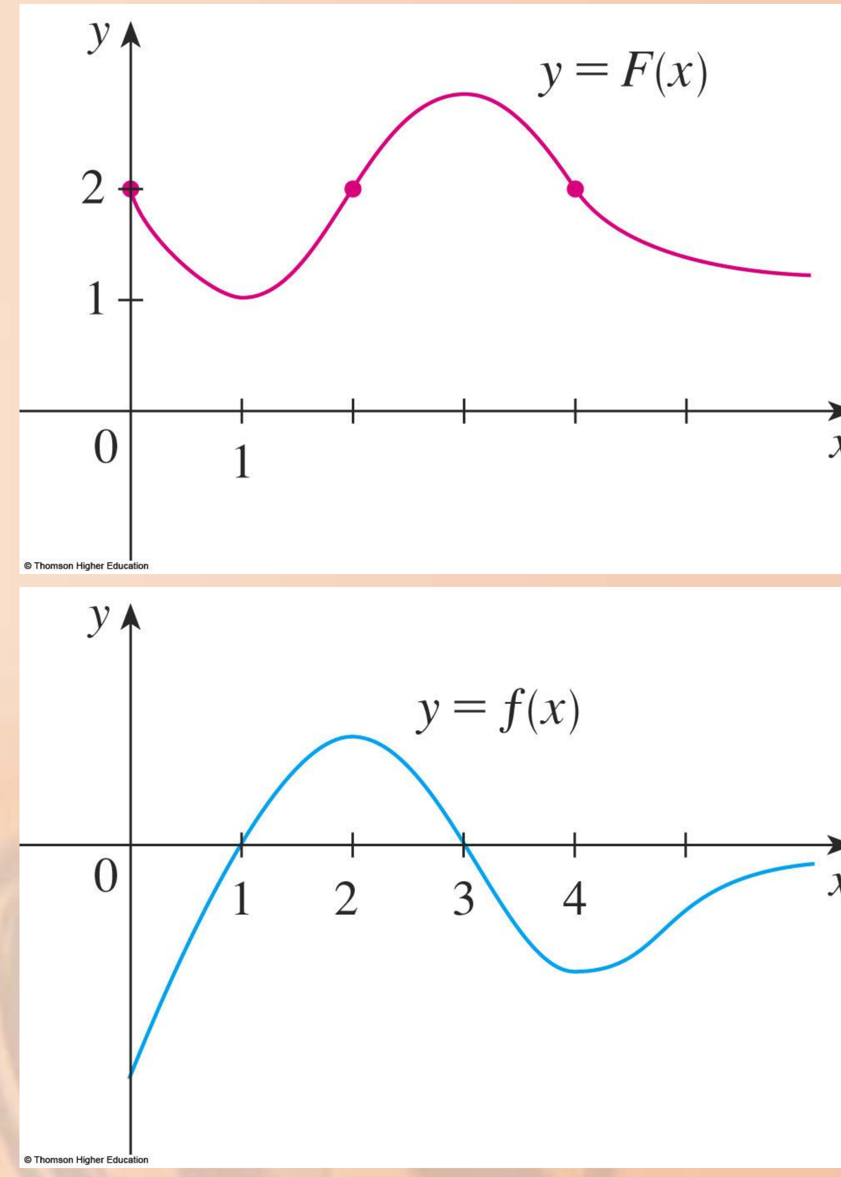
GRAPH

Example 5

Notice $f(1) = f(3) = 0$.

So, F has horizontal tangents when $x = 1$ and $x = 3$.

- For $1 < x < 3$, $f(x)$ is positive.
- Thus, F is increasing.

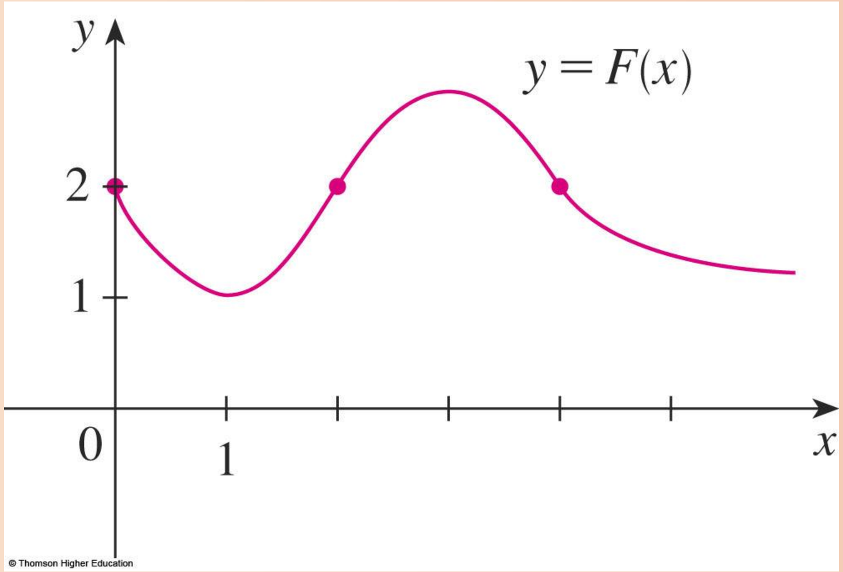


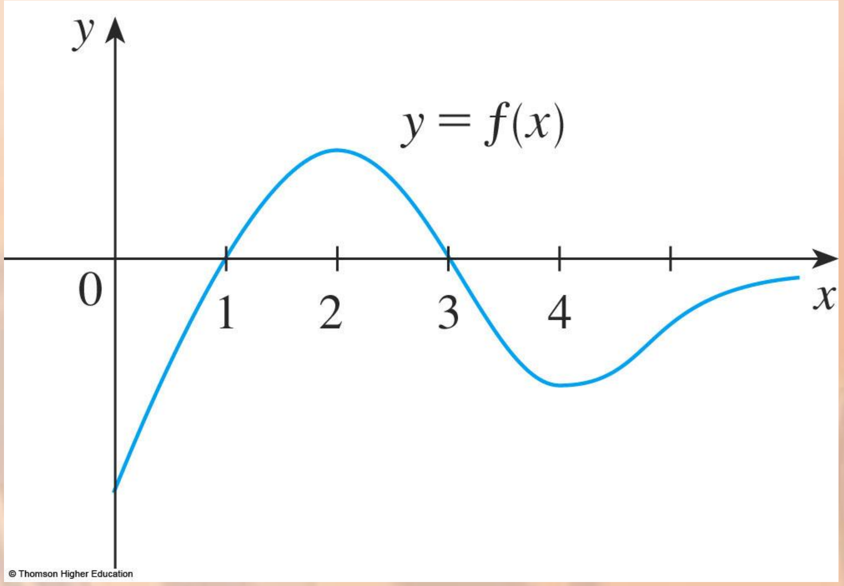
GRAPH

Example 5

We see F has a local minimum when $x = 1$ and a local maximum when $x = 3$.

- For $x > 3$, $f(x)$ is negative.
- Thus, F is decreasing on $(3, \infty)$.

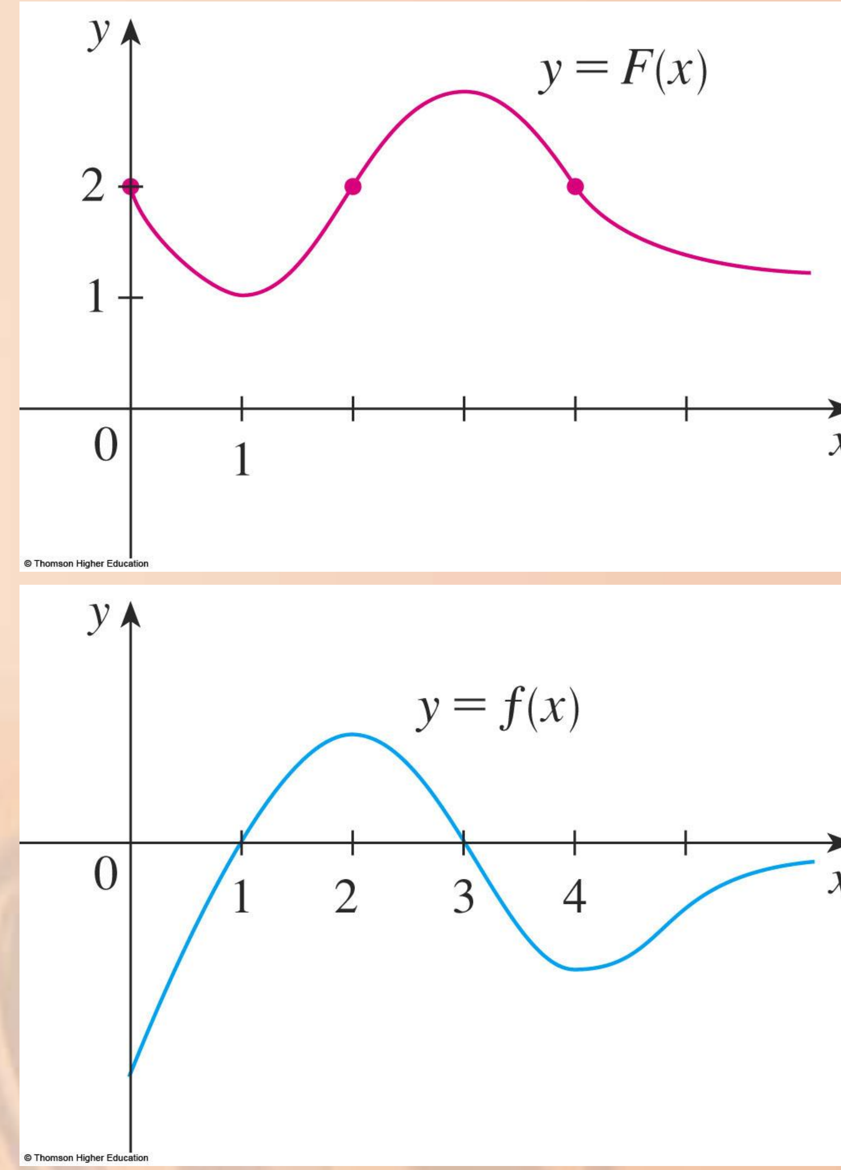




GRAPH

Example 5

Since $f(x) \to 0$ as $x \to \infty$, the graph of F becomes flatter as $x \to \infty$.

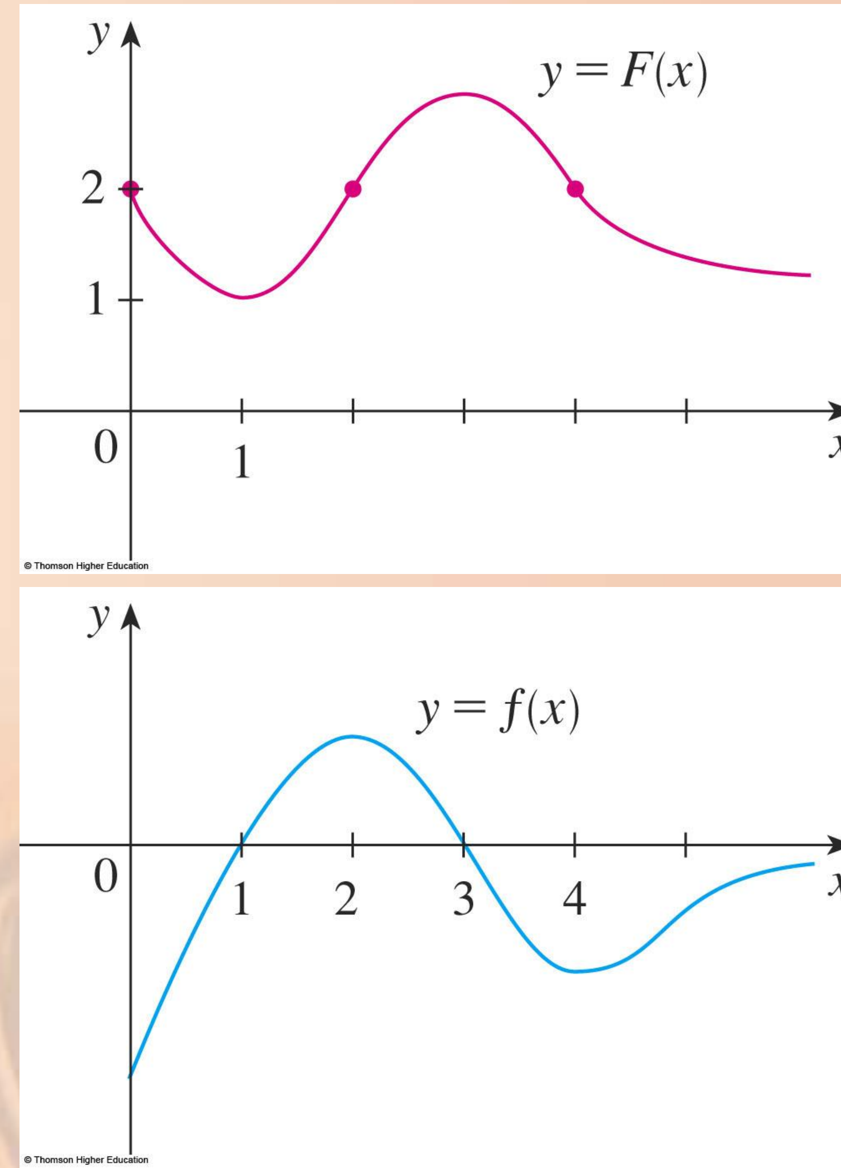


GRAPH

Example 5

Also, $F''(x) = f'(x)$ changes from positive to negative at $x = 2$ and from negative to positive at $x = 4$.

- So, F has inflection points when $x = 2$ and $x = 4$.



RECTILINEAR MOTION

Antidifferentiation is particularly useful in analyzing the motion of an object moving in a straight line.

RECTILINEAR MOTION

Recall that, if the object has position function $s = f(t)$, then the velocity function is $v(t) = s'(t)$.

- This means that the position function is an antiderivative of the velocity function.

RECTILINEAR MOTION

Likewise, the acceleration function is $a(t) = v'(t)$.

- So, the velocity function is an antiderivative of the acceleration.

RECTILINEAR MOTION

If the acceleration and the initial values $s(0)$ and $v(0)$ are known, then the position function can be found by antidifferentiating twice.

Example 6

A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$.

Its initial velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm.

- Find its position function $s(t)$.

As $v'(t) = a(t) = 6t + 4$, antidifferentiation gives:

$$v(t) = 6\frac{t^2}{2} + 4t + C$$

$$= 3t^2 + 4t + C$$

Note that $v(0) = C$.

However, we are given that $v(0) = -6$, so $C = -6$.

- Therefore, we have:

$$v(t) = 3t^2 + 4t - 6$$

As $v(t) = s'(t)$, s is the antiderivative of v:

$$s(t) = 3\frac{t^3}{3} + 4\frac{t^2}{2} - 6t + D$$

$$= t^3 + 2t^2 - 6t + D$$

- This gives $s(0) = D$.
- We are given that $s(0) = 9$, so $D = 9$.

The required position function is:

$$s(t) = t^3 + 2t^2 - 6t + 9$$

An object near the surface of the earth is subject to a gravitational force that produces a downward acceleration denoted by *g*.

For motion close to the ground, we may assume that *g* is constant.

- Its value is about 9.8 m/s^2 (or 32 ft/s^2).

Example 7

A ball is thrown upward with a speed of 48 ft/s from the edge of a cliff 432 ft above the ground.

- Find its height above the ground *t* seconds later.
- When does it reach its maximum height?
- When does it hit the ground?

The motion is vertical, and we choose the positive direction to be upward.

- At time *t*, the distance above the ground is *s*(*t*) and the velocity *v*(*t*) is decreasing.
- So, the acceleration must be negative and we have:

$$a(t) = \frac{dv}{dt} = -32$$

Taking antiderivatives, we have

$$v(t) = -32t + C$$

To determine C, we use the information that $v(0) = 48$.

- This gives: $48 = 0 + C$
- So, $v(t) = -32t + 48$

The maximum height is reached when

$$v(t) = 0\text{, that is, after 1.5 s}$$

As $s'(t) = v(t)$, we antidifferentiate again and obtain:

$$s(t) = -16t^2 + 48t + D$$

Using the fact that $s(0) = 432$, we have $432 = 0 + D$. So,

$$s(t) = -16t^2 + 48t + 432$$

The expression for $s(t)$ is valid until the ball hits the ground.

This happens when $s(t) = 0$, that is, when

$$-16t^2 + 48t + 432 = 0$$

- Equivalently: $t^2 - 3t - 27 = 0$

Using the quadratic formula to solve this equation, we get:

$$t = \frac{3 \pm 3\sqrt{13}}{2}$$

- We reject the solution with the minus sign—as it gives a negative value for *t*.

Therefore, the ball hits the ground after

$$3(1+\sqrt{13})/2 \approx 6.9 \text{ s}$$