4

APPLICATIONS OF DIFFERENTIATION

4.8
Newton’s Method

In this section, we will learn:

How to solve high degree equations using Newton’s method.

INTRODUCTION

Suppose that a car dealer offers to sell you a car for $18,000 or for payments of $375 per month for five years.

- You would like to know what monthly interest rate the dealer is, in effect, charging you.

Equation 1

To find the answer, you have to solve the equation

$$48x(1+x)^{60} - (1+x)^{60} + 1 = 0$$

- How would you solve such an equation?

HIGH-DEGREE POLYNOMIALS

For a quadratic equation $ax^2 + bx + c = 0$, there is a well-known formula for the roots.

For third- and fourth-degree equations, there are also formulas for the roots.

- However, they are extremely complicated.

HIGH-DEGREE POLYNOMIALS

If *f* is a polynomial of degree 5 or higher, there is no such formula.

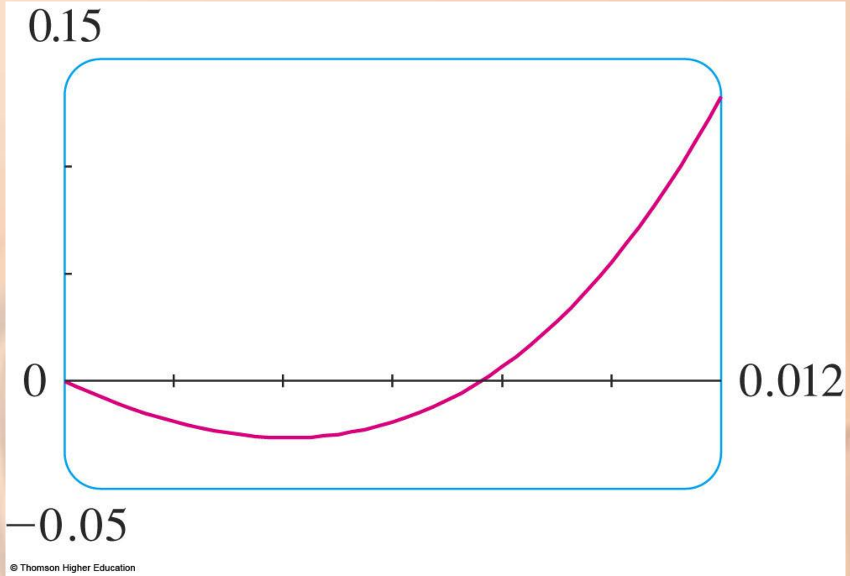
TRANSCENDENTAL EQUATIONS

Likewise, there is no formula that will enable us to find the exact roots of a transcendental equation such as $\cos x = x$.

APPROXIMATE SOLUTION

We can find an approximate solution by plotting the left side of the equation.

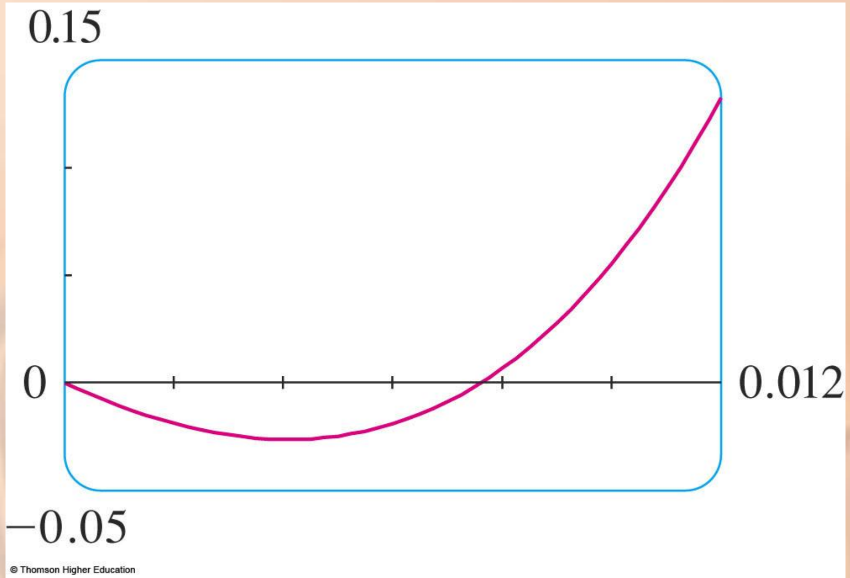
- Using a graphing device, and after experimenting with viewing rectangles, we produce the graph below.



ZOOMING IN

We see that, in addition to the solution $x = 0$, which doesn't interest us, there is a solution between 0.007 and 0.008

- Zooming in shows that the root is approximately 0.0076



ZOOMING IN

If we need more accuracy, we could zoom in repeatedly.

That becomes tiresome, though.

NUMERICAL ROOTFINDERS

A faster alternative is to use a numerical rootfinder on a calculator or computer algebra system.

- If we do so, we find that the root, correct to nine decimal places, is 0.007628603

How do those numerical rootfinders work?

- They use a variety of methods.
- Most, though, make some use of Newton’s method, also called the Newton-Raphson method.

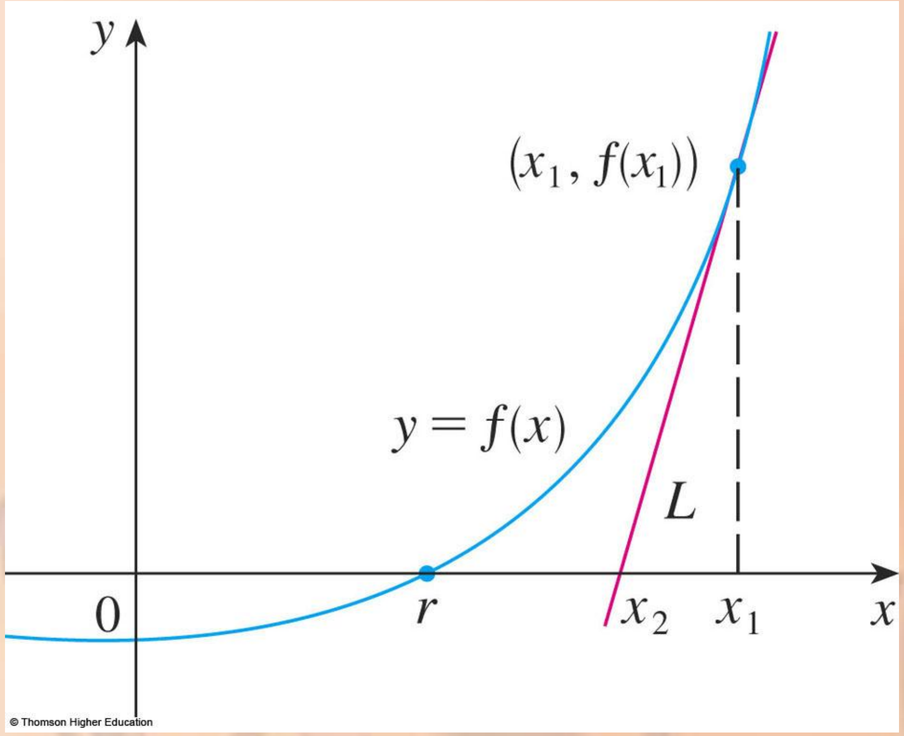
We will explain how the method works, for two reasons:

- To show what happens inside a calculator or computer
- As an application of the idea of linear approximation

NEWTON'S METHOD

The geometry behind Newton's method is shown here.

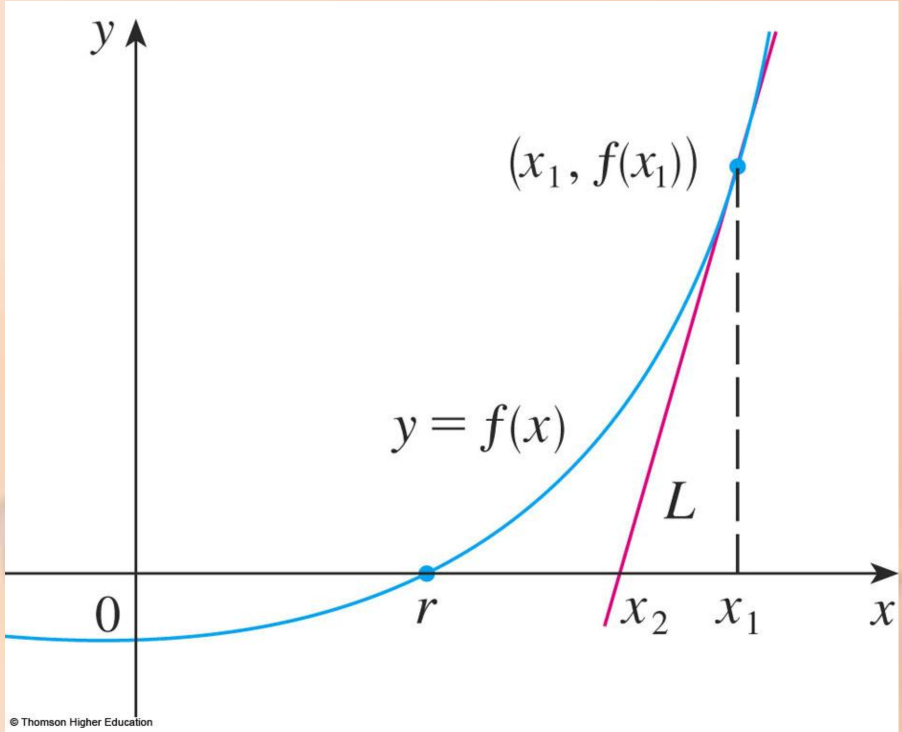
- The root that we are trying to find is labeled *r*.



NEWTON'S METHOD

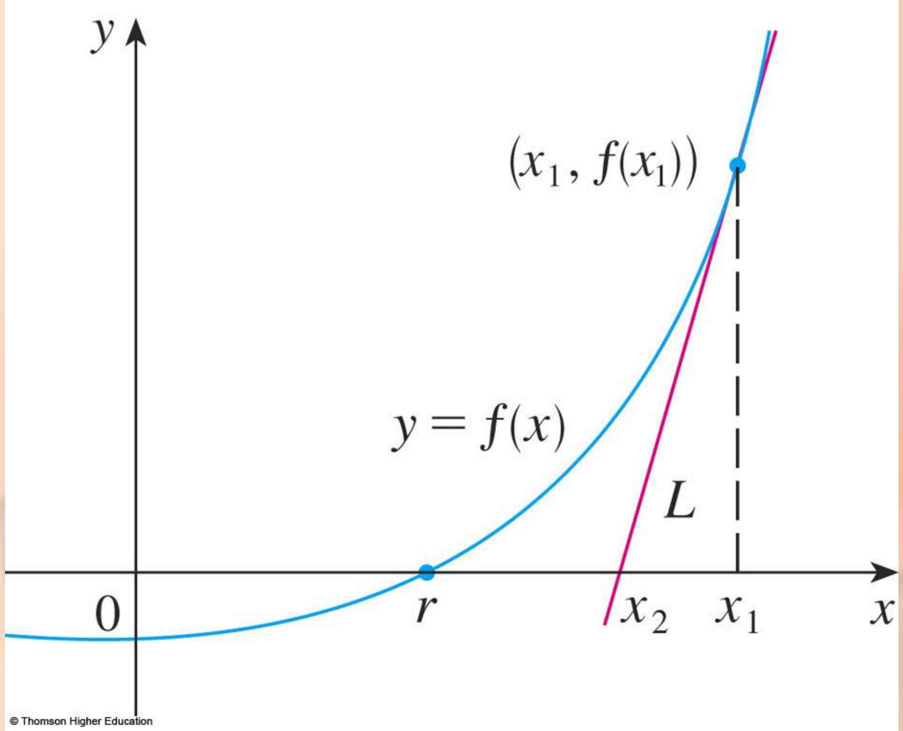
We start with a first approximation x_1, which is obtained by one of the following methods:

- Guessing
- A rough sketch of the graph of *f*
- A computer-generated graph of *f*



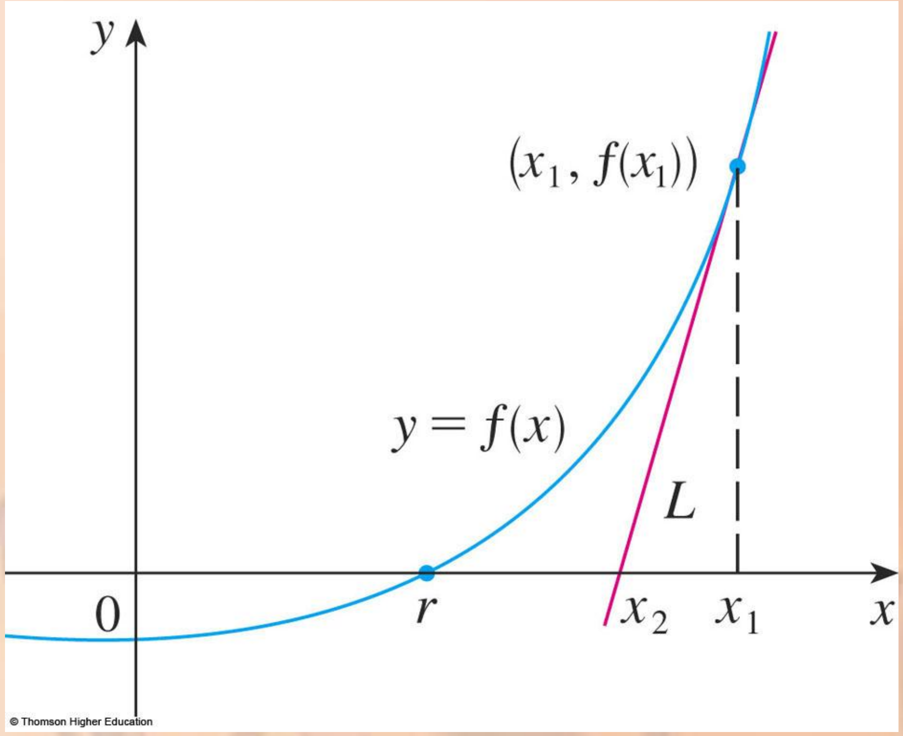
NEWTON'S METHOD

Consider the tangent line L to the curve $y = f(x)$ at the point $(x_1, f(x_1))$ and look at the x-intercept of L, labeled x_2.



Here's the idea behind the method.

- The tangent line is close to the curve.
- So, its *x*-intercept, x_2 , is close to the *x*-intercept of the curve (namely, the root *r* that we are seeking).
- As the tangent is a line, we can easily find its *x*-intercept.



To find a formula for x_2 in terms of x_1, we use the fact that the slope of L is $f'(x_1)$.

So, its equation is:

$$y - f(x_1) = f'(x_1)(x - x_1)$$

SECOND APPROXIMATION

As the x-intercept of L is x_2, we set $y = 0$ and obtain: $0 - f(x_1) = f'(x_1)(x_2 - x_1)$

If $f'(x_1) \neq 0$, we can solve this equation for x_2:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

- We use x_2 as a second approximation to r.

THIRD APPROXIMATION

Next, we repeat this procedure with x_1 replaced by x_2, using the tangent line at $(x_2, f(x_2))$.

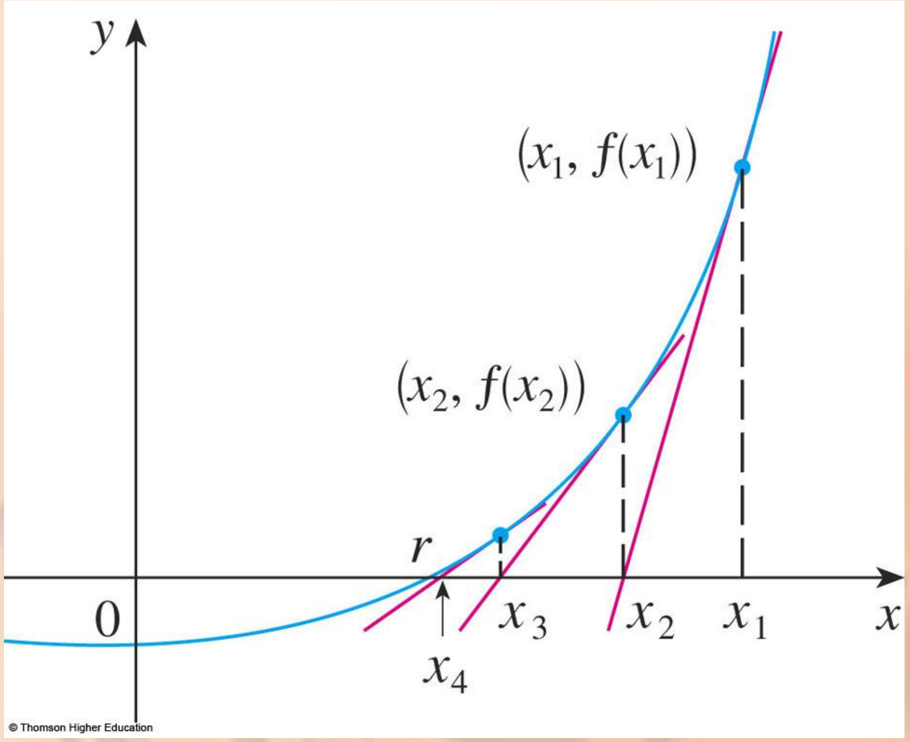
- This gives a third approximation:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

SUCCESSIVE APPROXIMATIONS

If we keep repeating this process, we obtain a sequence of approximations

$x_1, x_2, x_3, x_4, \ldots$



In general, if the *n*th approximation is x_n and $f'(x_n) \neq 0$, then the next approximation is given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

CONVERGENCE

If the numbers x_n become closer and closer to r as n becomes large, then we say that the sequence converges to r and we write:

$$\lim_{n \to \infty} x_n = r$$

CONVERGENCE

The sequence of successive approximations converges to the desired root for functions of the type illustrated in the previous figure.

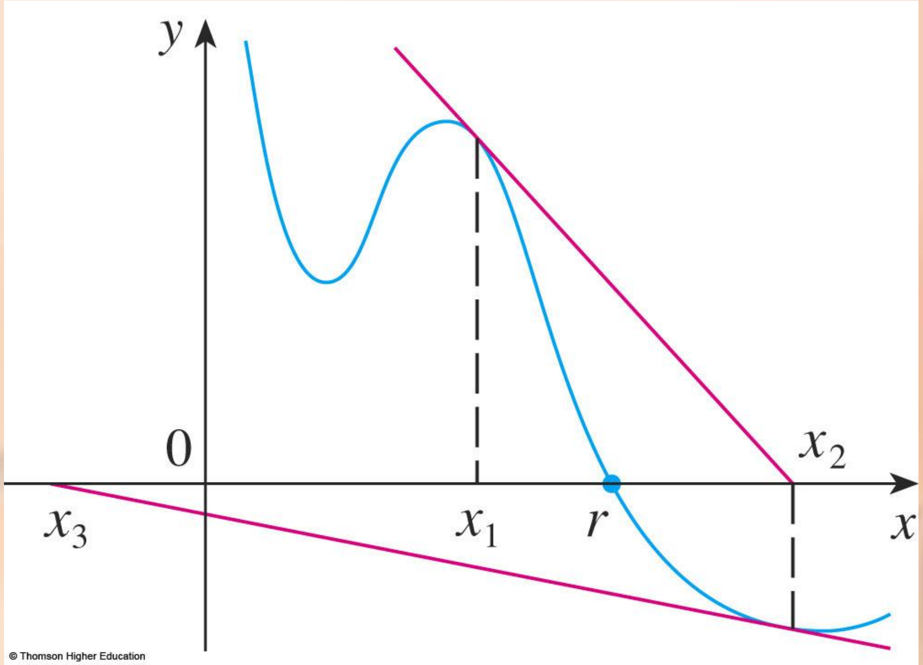
However, in certain circumstances, it may not converge.

NON-CONVERGENCE

Consider the situation shown here.

You can see that x_2 is a worse approximation than x_1.

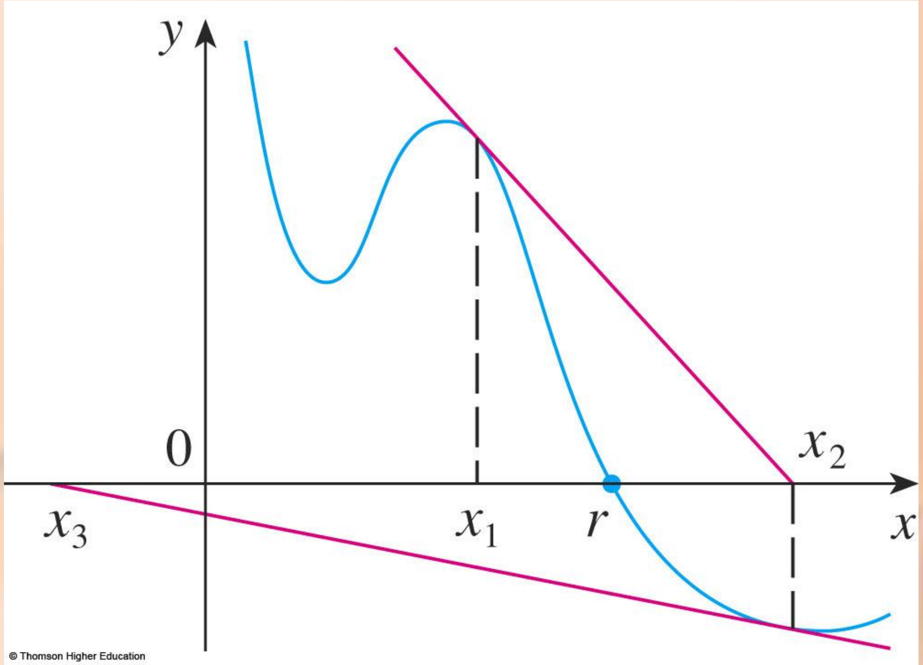
- This is likely to be the case when $f'(x_1)$ is close to 0.



NON-CONVERGENCE

It might even happen that an approximation falls outside the domain of f, such as x_3.

- Then, Newton's method fails.
- In that case, a better initial approximation x_1 should be chosen.



NON-CONVERGENCE

See Exercises 31–34 for specific examples in which Newton's method works very slowly or does not work at all.

Starting with $x_1 = 2$, find the third approximation x_3 to the root of the equation

$$x^3 - 2x - 5 = 0$$

We apply Newton's method with

$$f(x) = x^3 - 2x - 5 \quad \text{and} \quad f'(x) = 3x^2 - 2$$

- Newton himself used this equation to illustrate his method.
- He chose $x_1 = 2$ after some experimentation because $f(1) = -6$, $f(2) = -1$ and $f(3) = 16$

Equation 2 becomes:

$$x_n + 1 = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$$

With n = 1, we have:

$$x_2 = x_1 - \frac{x_1^3 - 2x_1 - 5}{3x_1^2 - 2}$$

$$= 2 - \frac{2^3 - 2(2) - 5}{3(2)^2 - 2}$$

$$= 2.1$$

With n = 2, we obtain:

$$x_3 = x_2 - \frac{x_2^3 - 2x_2 - 5}{3x_2^2 - 2}$$

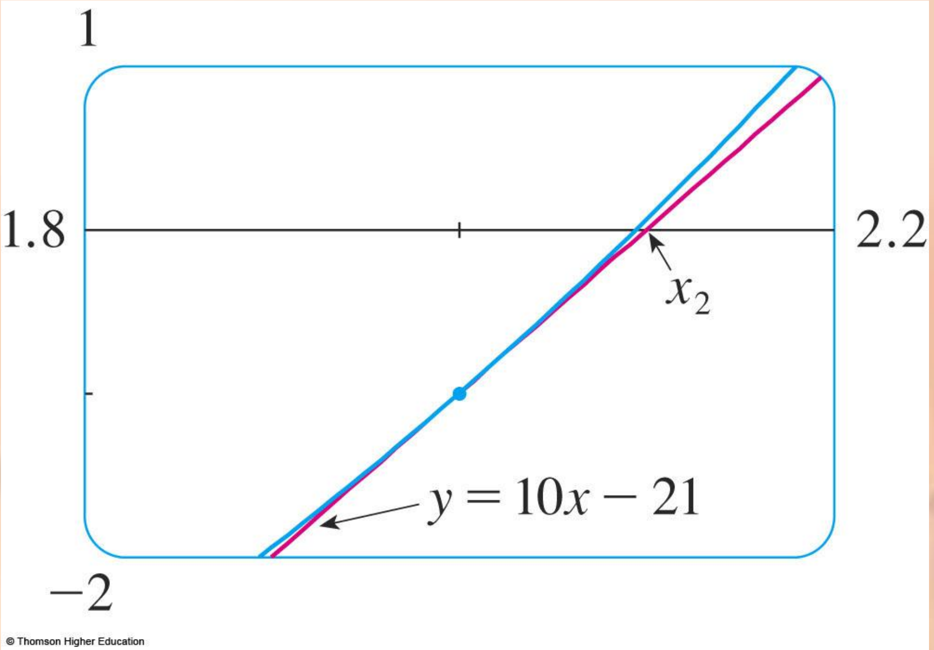
$$= 2.1 - \frac{2.1^3 - 2(2.1) - 5}{3(2.1)^2 - 2}$$

$$\approx 2.0946$$

- It turns out that this third approximation $x_3 \approx 2.0946$ is accurate to four decimal places.

The figure shows the geometry behind the first step in Newton's method in the example.

- As $f'(2) = 10$, the tangent line to $y = x^3 - 2x - 5$ at $(2, -1)$ has equation
  $y = 10x - 21$
- So, its x-intercept is $x_2 = 2.1$



NEWTON'S METHOD

Suppose that we want to achieve a given accuracy—say, to eight decimal places—using Newton's method.

- How do we know when to stop?

NEWTON'S METHOD

The rule of thumb that is generally used is that we can stop when successive approximations x_n and x_{n+1} agree to eight decimal places.

- A precise statement concerning accuracy in the method will be given in Exercise 37 in Section 11.11

ITERATIVE PROCESS

Notice that the procedure in going from n to $n + 1$ is the same for all values of n.

It is called an iterative process.

- This means that the method is particularly convenient for use with a programmable calculator or a computer.

Example 2

Use Newton's method to find $\sqrt[6]{2}$ correct to eight decimal places.

- First, we observe that finding $\sqrt[6]{2}$ is equivalent to finding the positive root of the equation $x^6 - 2 = 0$
- So, we take $f(x) = x^6 - 2$
- Then, $f'(x) = 6x^5$

So, Formula 2 (Newton's method) becomes:

$$x_{n+1} = x_n - \frac{x_n^6 - 2}{6x_n^5}$$

Choosing $x_1 = 1$ as the initial approximation, we obtain:

$$x_2 \approx 1.16666667$$

$$x_3 \approx 1.12644368$$

$$x_4 \approx 1.12249707$$

$$x_5 \approx 1.12246205$$

$$x_6 \approx 1.12246205$$

- As x_5 and x_6 agree to eight decimal places, we conclude that $\sqrt[6]{2} \approx 1.12246205$ to eight decimal places.

Example 3

Find, correct to six decimal places, the root of the equation cos $x = x$.

- We rewrite the equation in standard form: cos $x - x = 0$
- Therefore, we let $f(x) = \cos x - x$
- Then, $f'(x) = -\sin x - 1$

So, Formula 2 becomes:

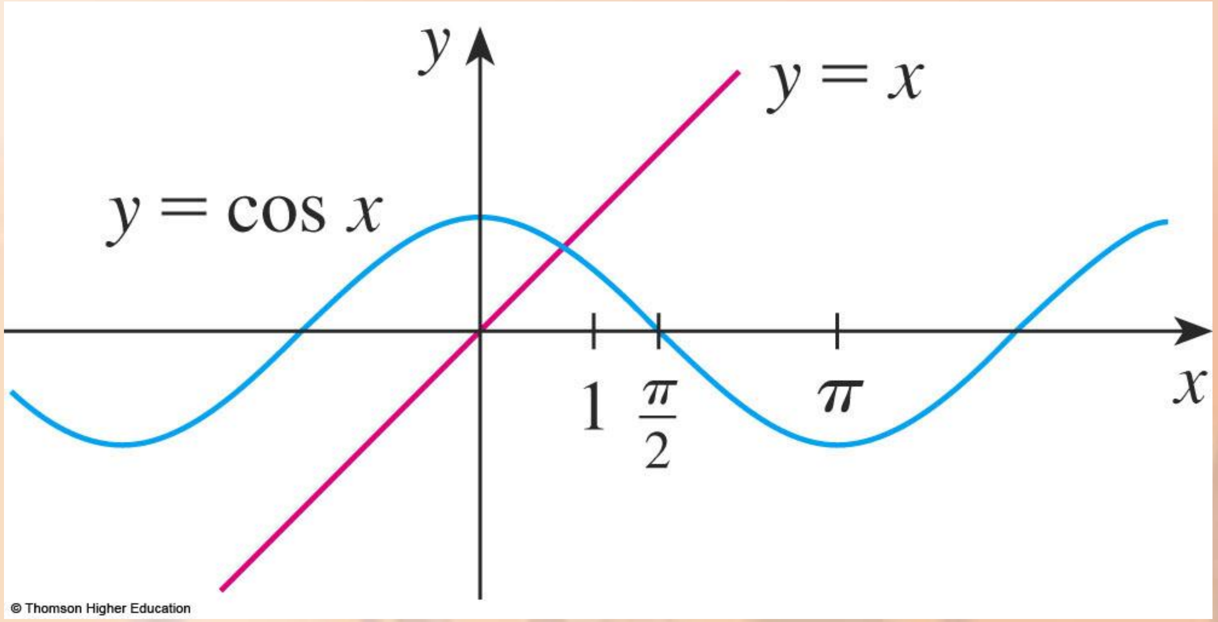
$$x_{n+1} = x_n - \frac{\cos x_n - x_n}{-\sin x_n - 1}$$

$$= x_n + \frac{\cos x_n - x_n}{\sin x_n + 1}$$

Example 3

To guess a suitable value for x_1, we sketch the graphs of $y = \cos x$ and $y = x$.

- It appears they intersect at a point whose x-coordinate is somewhat less than 1.



So, let's take $x_1 = 1$ as a convenient first approximation.

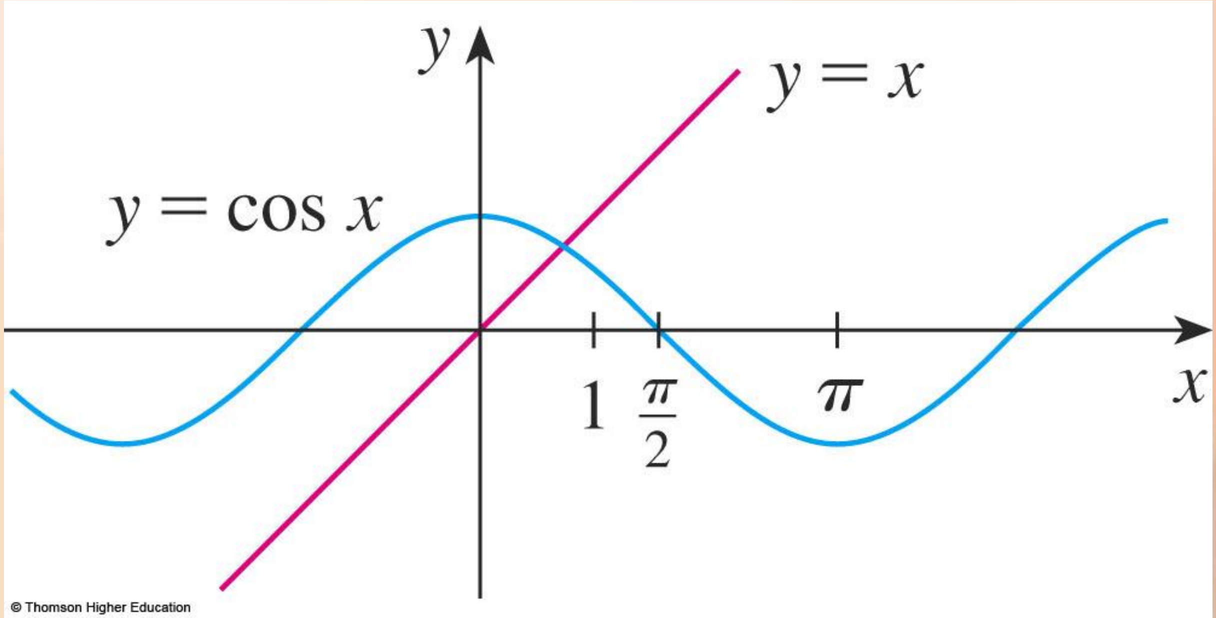
- Then, remembering to put our calculator in radian mode, we get:

$$x_2 \approx 0.75036387$$
$$x_3 \approx 0.73911289$$
$$x_4 \approx 0.73908513$$
$$x_5 \approx 0.73908513$$

- As x_4 and x_5 agree to six decimal places (eight, in fact), we conclude that the root of the equation, correct to six decimal places, is 0.739085

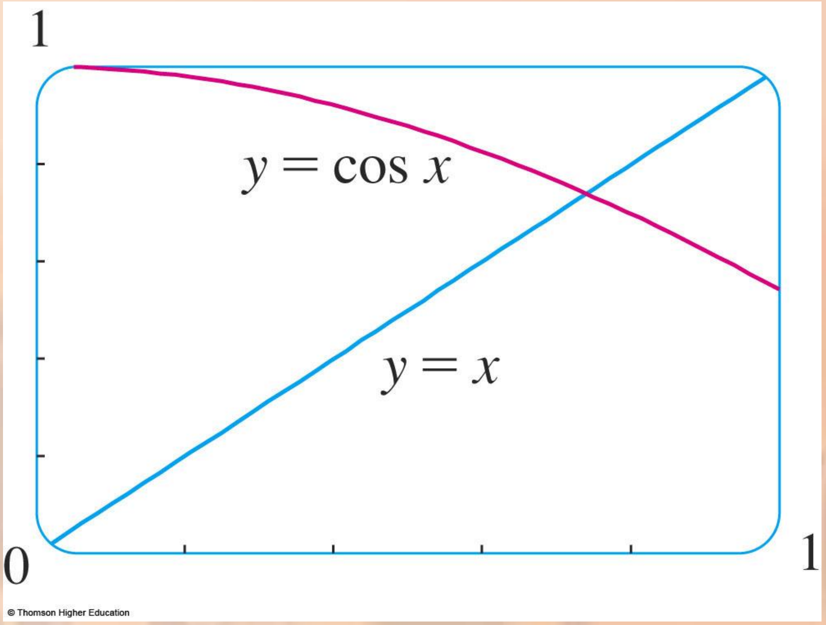
NEWTON'S METHOD

Instead of using this rough sketch to get a starting approximation for the method in the example, we could have used the more accurate graph that a calculator or computer provides.



NEWTON'S METHOD

This figure suggests that we use $x_1 = 0.75$ as the initial approximation.



Then, Newton's method gives:

$$x_2 \approx 0.73911114$$

$$x_3 \approx 0.73908513$$

$$x_4 \approx 0.73908513$$

- So we obtain the same answer as before—but with one fewer step.

NEWTON'S METHOD VS. GRAPHING DEVICES

You might wonder why we bother at all with Newton's method if a graphing device is available.

- Isn't it easier to zoom in repeatedly and find the roots as we did in Section 1.4?

NEWTON'S METHOD VS. GRAPHING DEVICES

If only one or two decimal places of accuracy are required, then indeed the method is inappropriate and a graphing device suffices.

However, if six or eight decimal places are required, then repeated zooming becomes tiresome.

NEWTON'S METHOD VS. GRAPHING DEVICES

It is usually faster and more efficient to use a computer and the method in tandem.

- You start with the graphing device and finish with the method.