4

APPLICATIONS OF DIFFERENTIATION

4.4

Indeterminate Forms and L'Hospital's Rule

In this section, we will learn:

How to evaluate functions whose values cannot be found at certain points.

INDETERMINATE FORMS

Suppose we are trying to analyze the behavior of the function

$$F(x) = \frac{\ln x}{x - 1}$$

Although *F* is not defined when *x* = 1, we need to know how *F* behaves near 1.

In particular, we would like to know the value of the limit

$$\lim_{x \to 1} \frac{\ln x}{x - 1}$$

In computing this limit, we can't apply Law 5 of limits (Section 2.3) because the limit of the denominator is 0.

- In fact, although the limit in Expression 1 exists, its value is not obvious because both numerator and denominator approach 0 and $\frac{0}{0}$ is not defined.

INDETERMINATE FORM —TYPE 0/0

In general, if we have a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, then this limit may or may not exist.

It is called an indeterminate form of type $\frac{0}{0}$.

- We met some limits of this type in Chapter 2.

For rational functions, we can cancel common factors:

$$\lim_{x\to 1}\frac{x^2-x}{x^2-1}=\lim_{x\to 1}\frac{x(x-1)}{(x+1)(x-1)}$$

$$=\lim_{x\to 1}\frac{x}{x+1}=\frac{1}{2}$$

We used a geometric argument to show that:

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

However, these methods do not work for limits such as Expression 1.

- Hence, in this section, we introduce a systematic method, known as l'Hospital's Rule, for the evaluation of indeterminate forms.

Another situation in which a limit is not obvious occurs when we look for a horizontal asymptote of *F* and need to evaluate the limit

$$\lim_{x \to \infty} \frac{\ln x}{x - 1}$$

INDETERMINATE FORMS

It isn't obvious how to evaluate this limit because both numerator and denominator become large as $x \to \infty$.

There is a struggle between the two.

- If the numerator wins, the limit will be ∞.
- If the denominator wins, the answer will be 0.
- Alternatively, there may be some compromise—the answer may be some finite positive number.

INDETERMINATE FORM —TYPE ∞/∞

In general, if we have a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to \infty$ (or $-\infty$) and $g(x) \to \infty$ (or $-\infty$), then the limit may or may not exist.

It is called an indeterminate form of type ∞/∞.

INDETERMINATE FORMS

We saw in Section 2.6 that this type of limit can be evaluated for certain functions—including rational functions—by dividing the numerator and denominator by the highest power of *x* that occurs in the denominator.

- For instance,

$$\lim_{x\to\infty}\frac{x^2-1}{2x^2+1}=\lim_{x\to\infty}\frac{1-\dfrac{1}{x^2}}{2+\dfrac{1}{x^2}}=\frac{1-0}{2+0}=\frac{1}{2}$$

INDETERMINATE FORMS

This method, though, does not work for limits such as Expression 2.

- However, L'Hospital's Rule also applies to this type of indeterminate form.

L'HOSPITAL'S RULE

Suppose *f* and *g* are differentiable and $g'(x) \neq 0$ on an open interval I that contains *a* (except possibly at *a*).

Suppose $\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0$

or that $\lim_{x\to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x\to a} g(x) = \pm\infty$

- In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.

Then,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right exists

(or is ∞ or - ∞).

NOTE 1

L'Hospital's Rule says that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives—provided that the given conditions are satisfied.

- It is especially important to verify the conditions regarding the limits of f and g before using the rule.

NOTE 2

The rule is also valid for one-sided limits and for limits at infinity or negative infinity.

- That is, '$x \to a$' can be replaced by any of the symbols $x \to a^+$, $x \to a^-$, $x \to \infty$, or $x \to -\infty$.

NOTE 3

For the special case in which

$f(a) = g(a) = 0$, f' and g' are continuous,

and $g'(a) \neq 0$, it is easy to see why

the rule is true.

NOTE 3

In fact, using the alternative form of the definition of a derivative, we have:

$$\lim_{x\to a}\frac{f'(x)}{g'(x)}=\frac{f'(a)}{g'(a)}=\frac{\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}}{\lim\limits_{x\to a}\dfrac{g(x)-g(a)}{x-a}}=\lim_{x\to a}\frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}}$$

$$=\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}$$

$$=\lim_{x\to a}\frac{f(x)}{g(x)}$$

NOTE 3

It is more difficult to prove the general version of l'Hospital's Rule.

Example 1

Find $\lim_{x \to 1} \dfrac{\ln x}{x-1}$

- $\lim_{x \to 1} \ln x = \ln 1 = 0$ and $\lim_{x \to 1} (x-1) = 0$
- Thus, we can apply l'Hospital's Rule:

$$\lim_{x \to 1} \frac{\ln x}{x-1} = \lim_{x \to 1} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x-1)} = \lim_{x \to 1} \frac{1/x}{1} = \lim_{x \to 1} \frac{1}{x} = 1$$

Example 2

Calculate $\lim_{x \to \infty} \frac{e^x}{x^2}$

- We have $\lim_{x \to \infty} e^x = \infty$ and $\lim_{x \to \infty} x^2 = \infty$
- So, l'Hospital's Rule gives:

$$\lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(x^2)} = \lim_{x \to \infty} \frac{e^x}{2x}$$

Example 2

As $e^x \to \infty$ and $2x \to \infty$ as $x \to \infty$, the limit on the right side is also indeterminate.

However, a second application of l'Hospital's Rule gives:

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

Example 3

Calculate $\lim_{x \to \infty} \frac{\ln x}{\sqrt[3]{x}}$

- As $\ln x \to \infty$ and $\sqrt[3]{x} \to \infty$ as $x \to \infty$, l'Hospital's Rule applies:

$$\lim_{x \to \infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x \to \infty} \frac{1/x}{\frac{1}{3} x^{-2/3}}$$

- Notice that the limit on the right side is now indeterminate of type $\frac{0}{0}$.

Example 3

- However, instead of applying the rule a second time as we did in Example 2, we simplify the expression and see that a second application is unnecessary:

$$\lim_{x\to\infty}\frac{\ln x}{\sqrt[3]{x}}=\lim_{x\to\infty}\frac{1/x}{\frac{1}{3}x^{-2/3}}=\lim_{x\to\infty}\frac{3}{\sqrt[3]{x}}=0$$

Example 4

Find $\lim_{x \to 0} \frac{\tan x - x}{x^3}$

- Noting that both tan $x - x \to 0$ and $x^3 \to 0$ as $x \to 0$, we use l'Hospital's Rule:

$$\lim_{x \to 0} \frac{\tan x - x}{x^3} = \lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2}$$

Example 4

- As the limit on the right side is still indeterminate of type $\frac{0}{0}$, we apply the rule again:

$$\lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2} = \lim_{x\to 0}\frac{2\sec^2 x \tan x}{6x}$$

- Since $\lim_{x\to 0} \sec^2 x = 1$, we simplify the calculation by writing:

$$\lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x} = \frac{1}{3} \lim_{x\to 0} \sec^2 x \lim_{x\to 0} \frac{\tan x}{x}$$

$$= \frac{1}{3} \lim_{x\to 0} \frac{\tan x}{x}$$

Example 4

- We can evaluate this last limit either by using l'Hospital's Rule a third time or by writing tan x as (sin x)/(cos x) and making use of our knowledge of trigonometric limits.

Putting together all the steps, we get:

$$\lim_{x\to 0}\frac{\tan x - x}{x^3} = \lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2}$$

$$= \lim_{x\to 0}\frac{2\sec^2 x \tan x}{6x}$$

$$= \frac{1}{3}\lim_{x\to 0}\frac{\tan x}{x} = \frac{1}{3}\lim_{x\to 0}\frac{\sec^2 x}{1} = \frac{1}{3}$$

Example 5

Find $\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x}$

- If we blindly attempted to use l-Hospital's rule, we would get:

$$\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x} = \lim_{x \to \pi^-} \frac{\cos x}{\sin x} = -\infty$$

This is wrong.

- Although the numerator sin $x \to 0$ as $x \to \pi^-$, notice that the denominator (1 - cos x) does not approach 0.
- So, the rule can't be applied here.

The required limit is, in fact, easy to find because the function is continuous at π and the denominator is nonzero there:

$$\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x} = \frac{\sin \pi}{1 - \cos \pi} = \frac{0}{1 - (-1)} = 0$$

The example shows what can go wrong if you use the rule without thinking.

- Other limits can be found using the rule, but are more easily found by other methods.
- See Examples 3 and 5 in Section 2.3, Example 3 in Section 2.6, and the discussion at the beginning of the section.

So, when evaluating any limit,
you should consider other methods
before using l'Hospital's Rule.

INDETERMINATE PRODUCTS

If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$ (or -∞), then it isn't clear what the value of $\lim_{x \to a} f(x)g(x)$, if any, will be.

INDETERMINATE PRODUCTS

There is a struggle between *f* and *g*.

- If *f* wins, the answer will be 0.
- If *g* wins, the answer will be ∞ (or -∞).
- Alternatively, there may be a compromise where the answer is a finite nonzero number.

INDETERMINATE FORM—TYPE $0 \cdot \infty$

This kind of limit is called an indeterminate form of type $0 \cdot \infty$.

- We can deal with it by writing the product *fg* as a quotient:

$$fg = \frac{f}{1/g} \quad \text{or} \quad fg = \frac{g}{1/f}$$

- This converts the given limit into an indeterminate form of type $\frac{0}{0}$ or ∞/∞, so that we can use l'Hospital's Rule.

Evaluate $\lim_{x \to 0^+} x \ln x$

- The given limit is indeterminate because, as $x \to 0^+$, the first factor (x) approaches 0, whereas the second factor ($\ln x$) approaches $-\infty$.

Example 6

- Writing $x = 1/(1/x)$, we have $1/x \to \infty$ as $x \to 0^+$.

- So, l'Hospital's Rule gives:

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2}$$

$$= \lim_{x\to 0^+} (-x) = 0$$

Note

In solving the example, another possible option would have been to write:

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{x}{1/\ln x}$$

- This gives an indeterminate form of the type 0/0.
- However, if we apply l'Hospital's Rule, we get a more complicated expression than the one we started with.

INDETERMINATE PRODUCTS

Note

In general, when we rewrite an indeterminate product, we try to choose the option that leads to the simpler limit.

INDETERMINATE FORM—TYPE ∞ - ∞

If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$, then the limit

$$\lim_{x \to a} [f(x) - g(x)]$$

is called an indeterminate form of type ∞ - ∞.

INDETERMINATE DIFFERENCES

Again, there is a contest between *f* and *g*.

- Will the answer be ∞ (*f* wins)?
- Will it be - ∞ (*g* wins)?
- Will they compromise on a finite number?

INDETERMINATE DIFFERENCES

To find out, we try to convert the difference into a quotient (for instance, by using a common denominator, rationalization, or factoring out a common factor) so that we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.

Compute $\lim_{x \to (\pi/2)^{-}} (\sec x - \tan x)$

- First, notice that sec $x \to \infty$ and tan $x \to \infty$ as $x \to (\pi/2)^{-}$.
- So, the limit is indeterminate.

Here, we use a common denominator:

$$\lim_{x \to (\pi/2)^-} (\sec x - \tan x) = \lim_{x \to (\pi/2)^-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right)$$

$$= \lim_{x \to (\pi/2)^-} \frac{1 - \sin x}{\cos x}$$

$$= \lim_{x \to (\pi/2)^-} \frac{-\cos x}{-\sin x} = 0$$

- Note that the use of l'Hospital's Rule is justified because 1 – sin $x \to 0$ and cos $x \to 0$ as $x \to (\pi/2)^-$.

INDETERMINATE POWERS

Several indeterminate forms arise from the limit $\lim_{x \to a} [f(x)]^{g(x)}$

1. $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ type 0^0
2. $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = 0$ type ∞^0
3. $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \pm\infty$ type 1^∞

INDETERMINATE POWERS

Each of these three cases can be treated in either of two ways.

- Taking the natural logarithm:

  Let $y = [f(x)]^{g(x)}$, then $\ln y = g(x) \ln f(x)$

- Writing the function as an exponential:

  $[f(x)]^{g(x)} = e^{g(x) \ln f(x)}$

INDETERMINATE POWERS

Recall that both these methods were used in differentiating such functions.

- In either method, we are led to the indeterminate product $g(x) \ln f(x)$, which is of type $0 \cdot \infty$.

Example 8

Calculate $\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x}$

- First, notice that, as $x \to 0^+$, we have $1 + \sin 4x \to 1$ and $\cot x \to \infty$.
- So, the given limit is indeterminate.

Example 8

Let $y = (1 + \sin 4x)^{\cot x}$

Then, $\ln y = \ln[(1 + \sin 4x)^{\cot x}]$

$= \cot x \ln(1 + \sin 4x)$

So, l'Hospital's Rule gives:

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\tan x}$$

$$= \lim_{x\to 0^+} \frac{\dfrac{4\cos 4x}{1+\sin 4x}}{\sec^2 x} = 4$$

So far, we have computed the limit of $\ln y$.

However, what we want is the limit of *y*.

- To find this, we use the fact that $y = e^{\ln y}$:

$$\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x} = \lim_{x \to 0^+} y$$

$$= \lim_{x \to 0^+} e^{\ln y} = e^4$$

Example 9

Find $\lim_{x \to 0^+} x^x$

- Notice that this limit is indeterminate since $0^x = 0$ for any $x > 0$ but $x^0 = 1$ for any $x \neq 0$.

Example 9

We could proceed as in Example 8 or by writing the function as an exponential:

$$x^x = (e^{\ln x})^x = e^{x \ln x}$$

- In Example 6, we used l'Hospital's Rule to show that $\lim_{x \to 0^+} x \ln x = 0$
- Therefore, $\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{x \ln x} = e^0 = 1$