3

DIFFERENTIATION RULES

We know that, if $y = f(x)$, then the derivative dy/dx can be interpreted as the rate of change of y with respect to x.

3.7

Rates of Change in the Natural and Social Sciences

In this section, we will examine:

Some applications of the rate of change to physics, chemistry, biology, economics, and other sciences.

RATES OF CHANGE

Let's recall from Section 2.7 the basic idea behind rates of change.

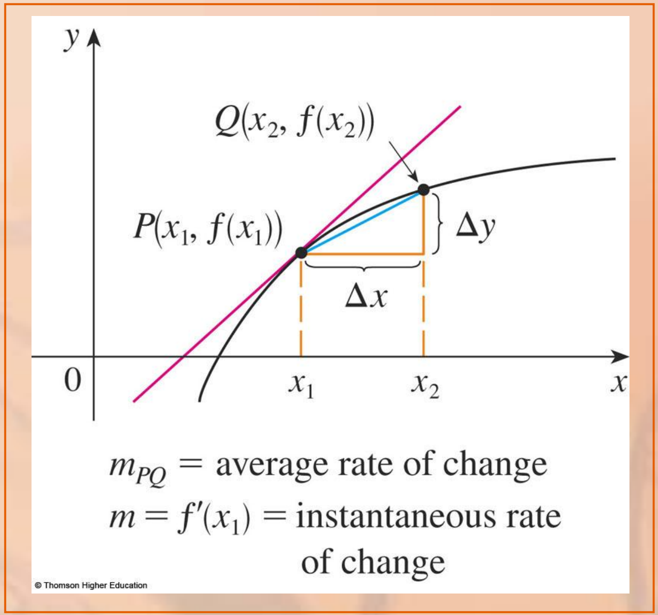
- If x changes from x_1 to x_2, then the change in x is $\Delta x = x_2 - x_1$
- The corresponding change in y is $\Delta y = f(x_2) - f(x_1)$

AVERAGE RATE

The difference quotient $$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is the average rate of change of *y* with respect to *x* over the interval $[x_1, x_2]$.

- It can be interpreted as the slope of the secant line *PQ*.

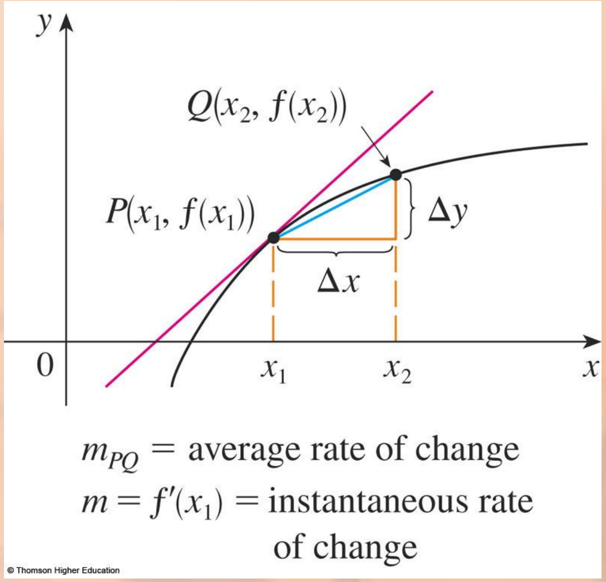


INSTANTANEOUS RATE

Its limit as $\Delta x \to 0$ is the derivative

$f'(x_1)$.

- This can therefore be interpreted as the instantaneous rate of change of *y* with respect to *x* or the slope of the tangent line at $P(x_1,f(x_1))$.



RATES OF CHANGE

Using Leibniz notation, we write the process in the form $\dfrac{dy}{dx} = \lim_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$

RATES OF CHANGE

Whenever the function $y = f(x)$ has a specific interpretation in one of the sciences, its derivative will have a specific interpretation as a rate of change.

- As we discussed in Section 2.7, the units for dy/dx are the units for y divided by the units for x.

NATURAL AND SOCIAL SCIENCES

We now look at some of these interpretations in the natural and social sciences.

Let $s = f(t)$ be the position function of a particle moving in a straight line.

Then,

- $\Delta s/\Delta t$ represents the average velocity over a time period Δt
- $v = ds/dt$ represents the instantaneous velocity (velocity is the rate of change of displacement with respect to time)
- The instantaneous rate of change of velocity with respect to time is acceleration: $a(t) = v'(t) = s''(t)$

These were discussed in Sections 2.7 and 2.8

- However, now that we know the differentiation formulas, we are able to solve problems involving the motion of objects more easily.

The position of a particle is given by the equation $s = f(t) = t^3 - 6t^2 + 9t$ where t is measured in seconds and s in meters.

a) Find the velocity at time t.

b) What is the velocity after 2 s? After 4 s?

c) When is the particle at rest?

Example 1

a) When is the particle moving forward (that is, in the positive direction)?

b) Draw a diagram to represent the motion of the particle.

c) Find the total distance traveled by the particle during the first five seconds.

a) Find the acceleration at time *t* and after 4 s.

b) Graph the position, velocity, and acceleration functions for $0 \leq t \leq 5$.

c) When is the particle speeding up? When is it slowing down?

The velocity function is the derivative of the position function.

$$s = f(t) = t^3 - 6t^2 + 9t$$

$$v(t) = ds/dt = 3t^2 - 12t + 9$$

The velocity after 2 s means the instantaneous velocity when t = 2, that is,

$$v(2) = \left.\frac{ds}{dt}\right|_{t=2} = 3(2)^2 - 12(2) + 9 = -3 \;\; m/s$$

The velocity after 4 s is:

$$v(4) = 3(4)^2 - 12(4) + 9 = 9 \;\; m/s$$

Example 1 c

The particle is at rest when $v(t) = 0$,

that is,

$3t^2 - 12t + 9 = 3(t^2 - 4t + 3) = 3(t - 1)(t - 3) = 0$

- This is true when $t = 1$ or $t = 3$.
- Thus, the particle is at rest after 1 s and after 3 s.

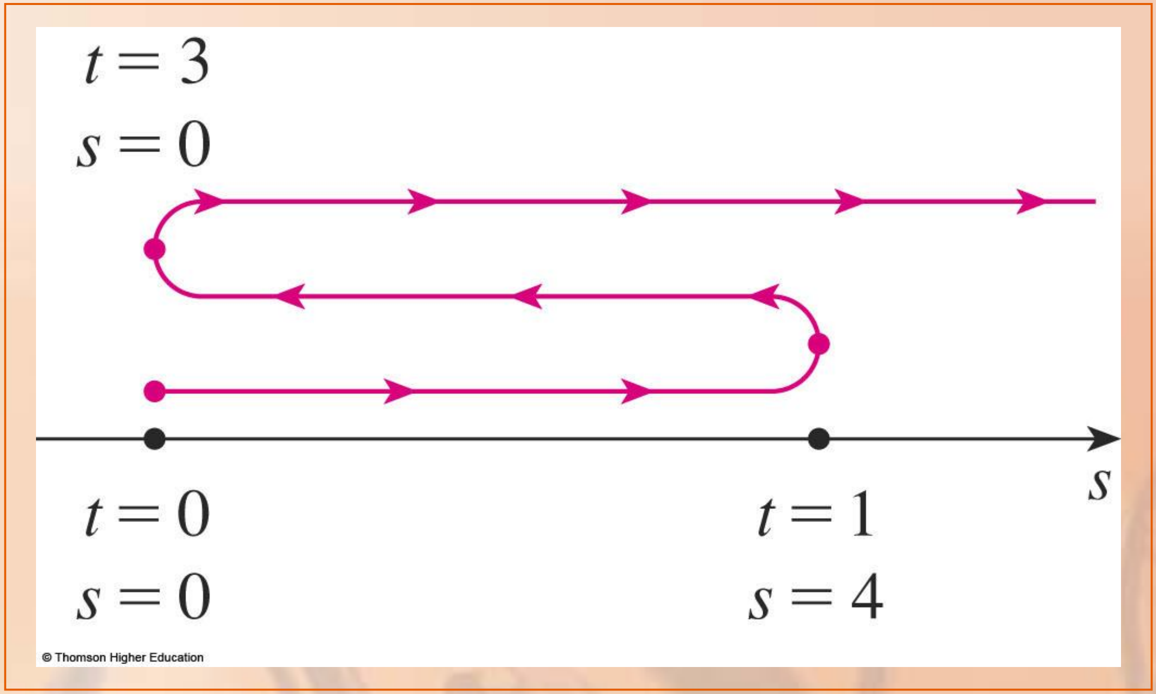
The particle moves in the positive direction when $v(t) > 0$, that is,

$$3t^2 - 12t + 9 = 3(t - 1)(t - 3) > 0$$

- This inequality is true when both factors are positive ($t > 3$) or when both factors are negative ($t < 1$).
- Thus the particle moves in the positive direction in the time intervals $t < 1$ and $t > 3$.
- It moves backward (in the negative direction) when $1 < t < 3$.

Example 1 e

Using the information from (d), we make a schematic sketch of the motion of the particle back and forth along a line (the *s* -axis).



Due to what we learned in (d) and (e), we need to calculate the distances traveled during the time intervals [0, 1], [1, 3], and [3, 5] separately.

The distance traveled in the first second is:

$$|f(1) - f(0)| = |4 - 0| = 4 \text{ m}$$

From $t = 1$ to $t = 3$, it is:

$$|f(3) - f(1)| = |0 - 4| = 4 \text{ m}$$

From $t = 3$ to $t = 5$, it is:

$$|f(5) - f(3)| = |20 - 0| = 20 \text{ m}$$

- The total distance is 4 + 4 + 20 = 28 m

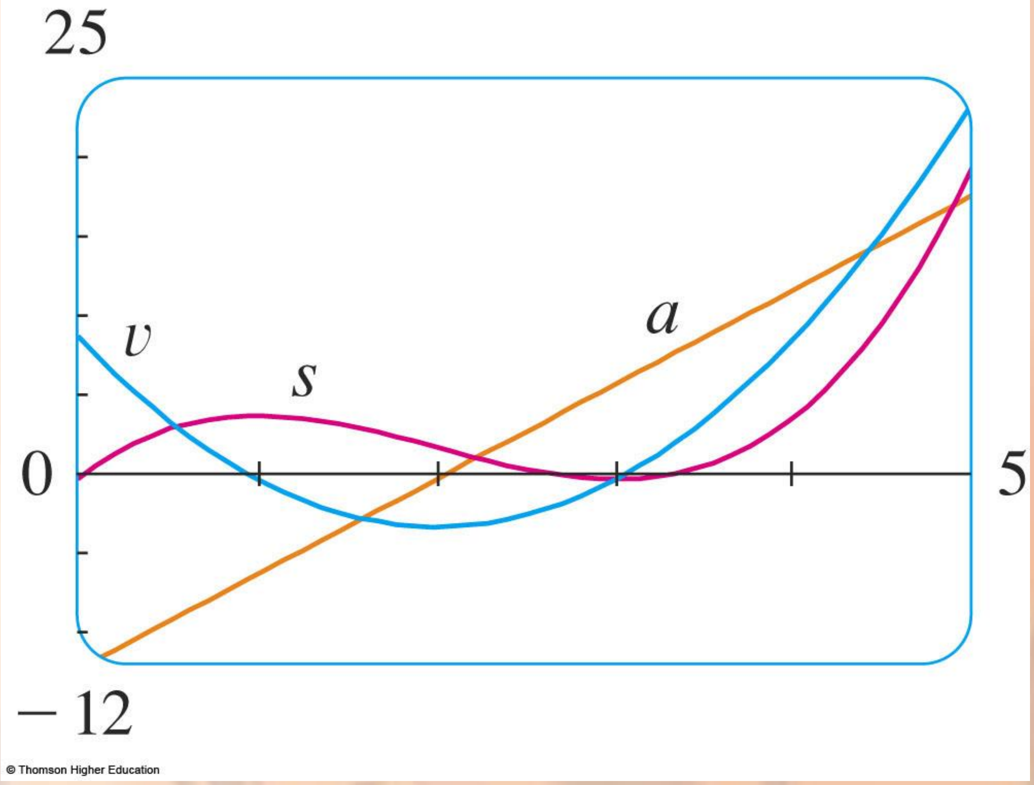
Example 1 g

The acceleration is the derivative of the velocity function:

$$a(t) = \frac{d^2 s}{dt^2} = \frac{dv}{dt} = 6t - 12$$

$$a(4) = 6(4) - 12 = 12 \ \ m/s^2$$

The figure shows the graphs of *s*, *v*, and *a*.

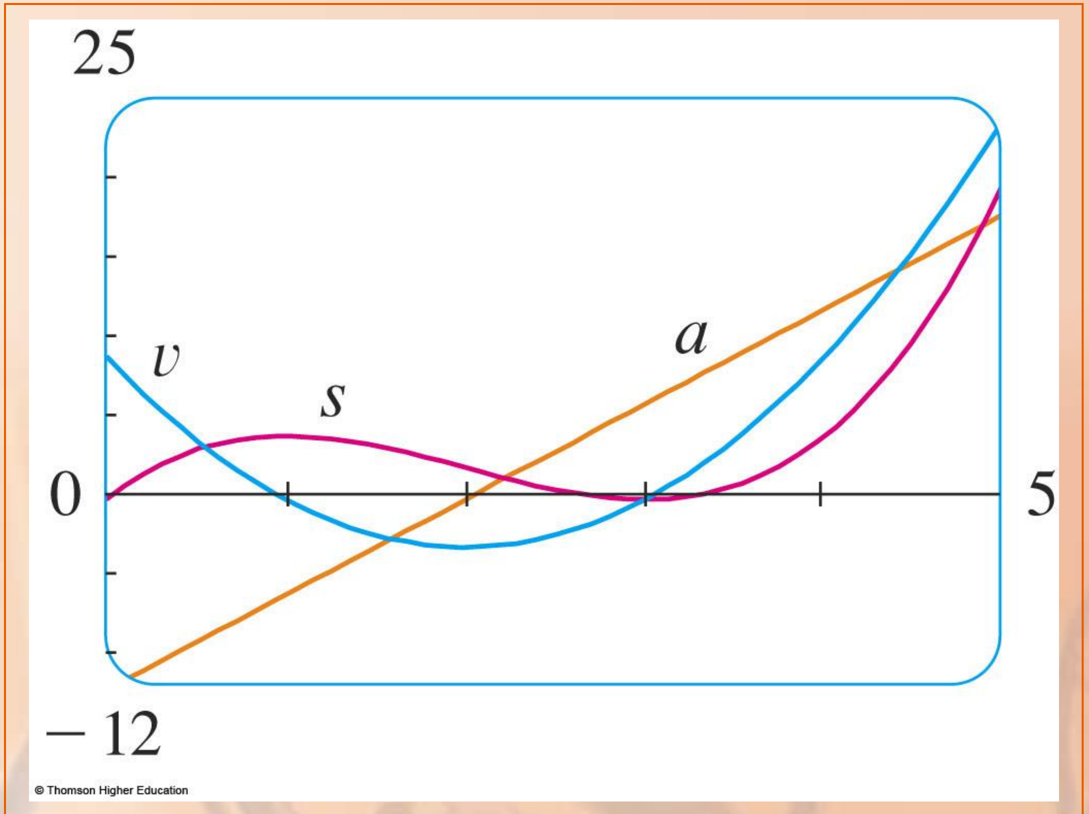


Example 1 i

The particle speeds up when the velocity is positive and increasing (*v* and *a* are both positive) and when the velocity is negative and decreasing (*v* and *a* are both negative).

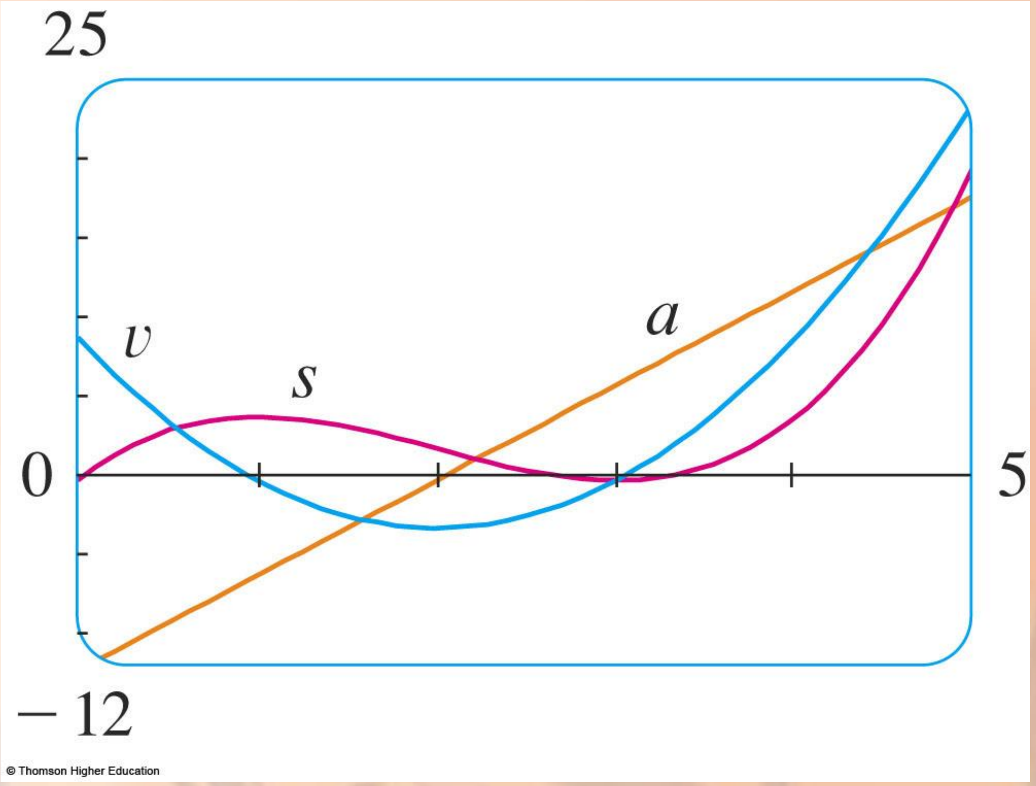
- In other words, the particle speeds up when the velocity and acceleration have the same sign.
- The particle is pushed in the same direction it is moving.

From the figure, we see that this happens when $1 < t < 2$ and when $t > 3$.

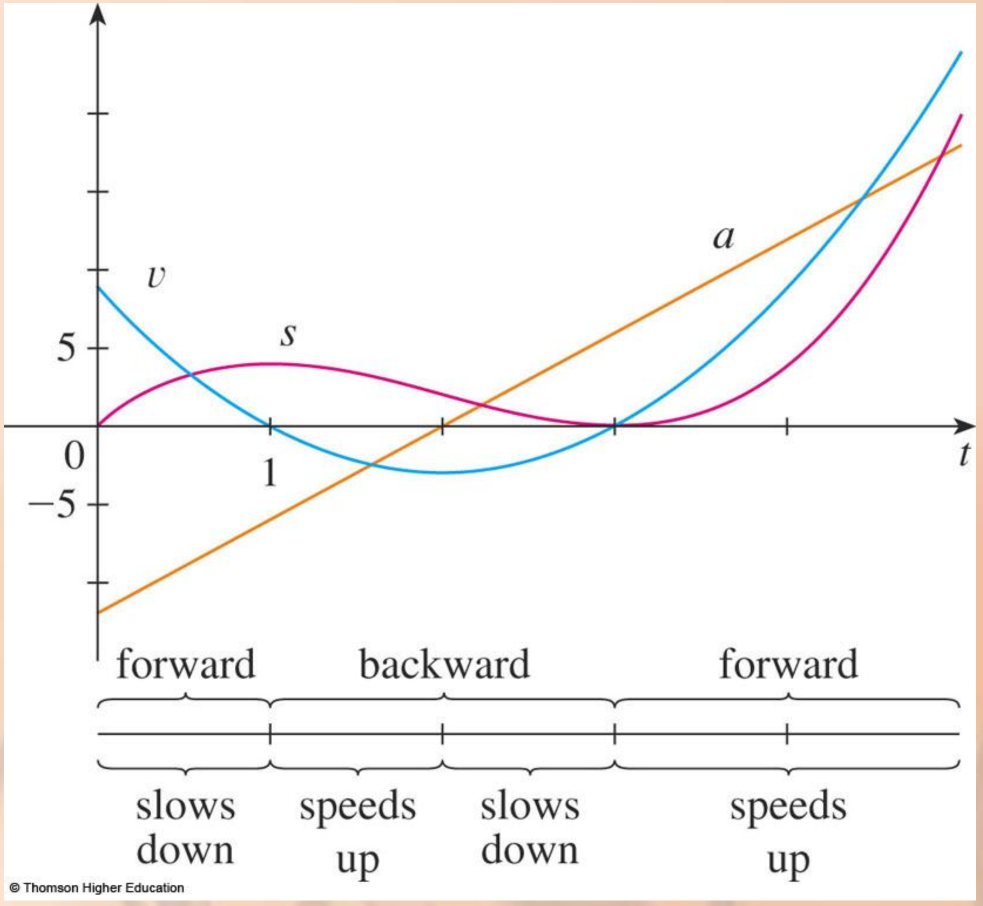


Example 1 i

The particle slows down when v and a have opposite signs—that is, when $0 \leq t < 1$ and when $2 < t < 3$.



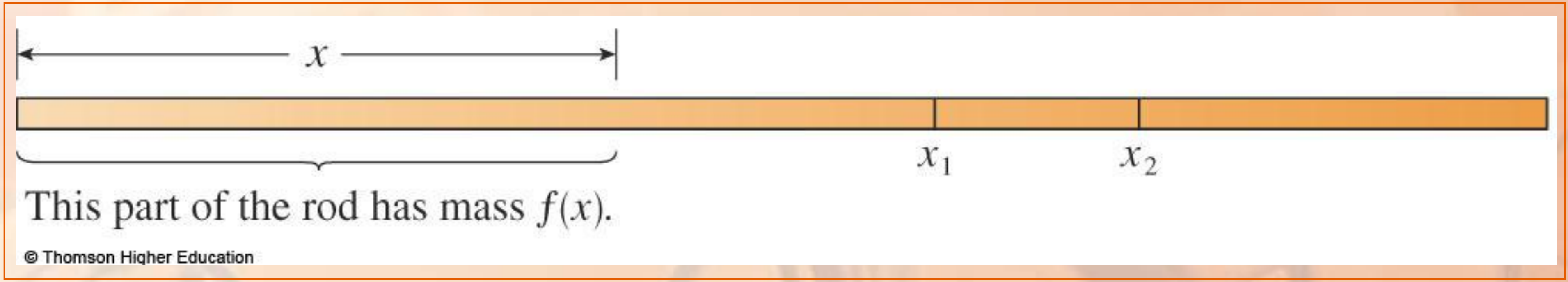
This figure summarizes the motion of the particle.



Example 2

If a rod or piece of wire is homogeneous, then its linear density is uniform and is defined as the mass per unit length ($\rho = m/l$) and measured in kilograms per meter.

However, suppose that the rod is not homogeneous but that its mass measured from its left end to a point x is $m = f(x)$.



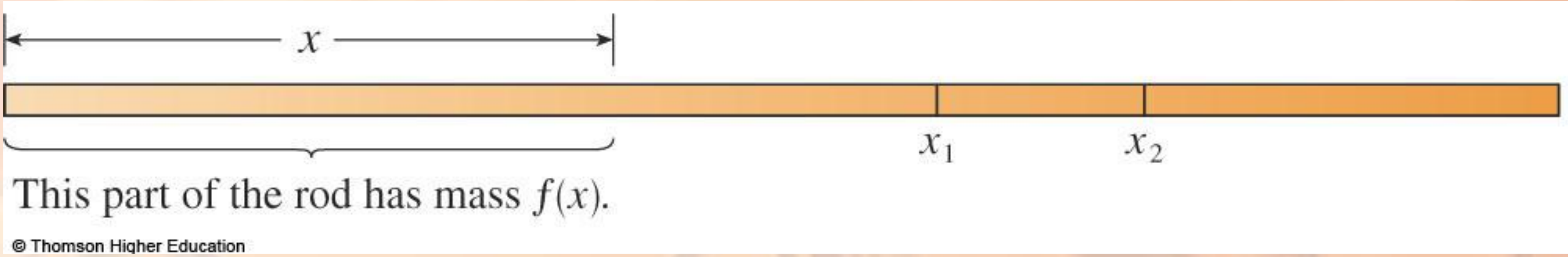
Example 2

The mass of the part of the rod that lies between $x = x_1$ and $x = x_2$ is given by

$\Delta m = f(x_2) - f(x_1)$

So, the average density of that part is:

$$\text{average density} = \frac{\Delta m}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$



Example 2

If we now let $\Delta x \to 0$ (that is, $x_2 \to x_1$), we are computing the average density over smaller and smaller intervals.

LINEAR DENSITY

The linear density ρ at x_1 is the limit of these average densities as $\Delta x \to 0$.

- That is, the linear density is the rate of change of mass with respect to length.

Symbolically,

$$\rho = \lim_{\Delta x \to 0} \frac{\Delta m}{\Delta x} = \frac{dm}{dx}$$

- Thus, the linear density of the rod is the derivative of mass with respect to length.

Example 2

For instance, if $m = f(x) = \sqrt{x}$, where x is measured in meters and m in kilograms, then the average density of the part of the rod given by $1 \leq x \leq 1.2$ is:

$$\frac{\Delta m}{\Delta x} = \frac{f(1.2) - f(1)}{1.2 - 1}$$

$$= \frac{\sqrt{1.2} - 1}{0.2}$$

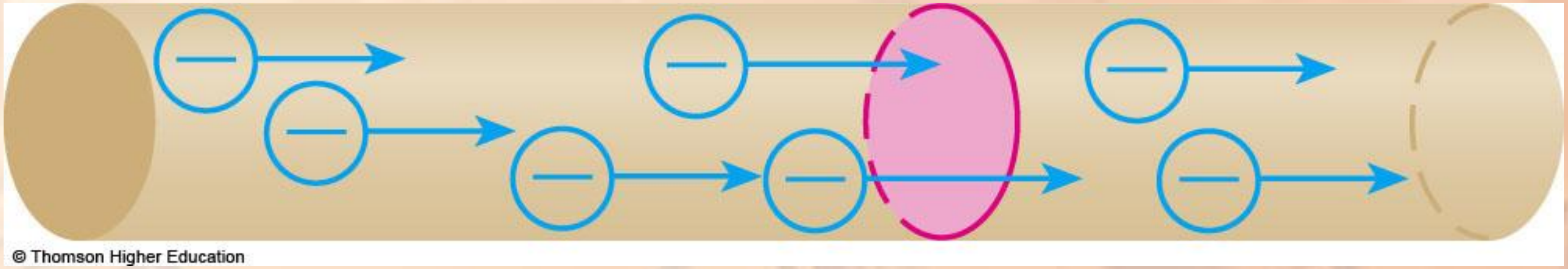
$$\approx 0.48\ kg/m$$

The density right at $x = 1$ is:

$$\rho = \left.\frac{dm}{dx}\right|_{x=1} = \left.\frac{1}{2\sqrt{x}}\right|_{x=1} = 0.50\ kg/m$$

A current exists whenever electric charges move.

- The figure shows part of a wire and electrons moving through a shaded plane surface.



If ΔQ is the net charge that passes through this surface during a time period Δt, then the average current during this time interval is defined as:

$$\text{average current} = \frac{\Delta Q}{\Delta t} = \frac{Q_2 - Q_1}{t_2 - t_1}$$

If we take the limit of this average current over smaller and smaller time intervals, we get what is called the current I at a given time t_1:

$$I = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt}$$

- Thus, the current is the rate at which charge flows through a surface.
- It is measured in units of charge per unit time (often coulombs per second—called amperes).

Velocity, density, and current are not the only rates of change important in physics.

Others include:

- Power (the rate at which work is done)
- Rate of heat flow
- Temperature gradient (the rate of change of temperature with respect to position)
- Rate of decay of a radioactive substance in nuclear physics

A chemical reaction results in the formation of one or more substances (products) from one or more starting materials (reactants).

- For instance, the ‘equation’

$$2H_2 + O_2 \rightarrow 2H_2O$$

indicates that two molecules of hydrogen and one molecule of oxygen form two molecules of water.

Example 4

Let's consider the reaction $A + B \rightarrow C$ where A and B are the reactants and C is the product.

- The concentration of a reactant A is the number of moles (6.022×10^{23} molecules) per liter and is denoted by [A].
- The concentration varies during a reaction.
- So, [A], [B], and [C] are all functions of time (*t*).

The average rate of reaction of the product C over a time interval $t_1 \le t \le t_2$ is:

$$\frac{\Delta[C]}{\Delta t} = \frac{[C](t_2) - [C](t_1)}{t_2 - t_1}$$

However, chemists are more interested in the instantaneous rate of reaction.

- This is obtained by taking the limit of the average rate of reaction as the time interval Δt approaches 0:

$$\text{rate of reaction} = \lim_{\Delta t \to 0} \frac{\Delta[C]}{\Delta t} = \frac{d[C]}{dt}$$

Since the concentration of the product increases as the reaction proceeds, the derivative $d[C]/dt$ will be positive.

- So, the rate of reaction of C is positive.

However, the concentrations of the reactants decrease during the reaction.

- So, to make the rates of reaction of A and B positive numbers, we put minus signs in front of the derivatives *d*[A]/*dt* and *d*[B]/*dt.*

Since [A] and [B] each decrease at the same rate that [C] increases, we have:

$$\text{rate of reaction} = \frac{d[C]}{dt} = -\frac{d[A]}{dt} = -\frac{d[B]}{dt}$$

More generally, it turns out that for a reaction of the form

$$aA + bB \rightarrow cC + dD$$

we have

$$-\frac{1}{a}\frac{d[A]}{dt} = -\frac{1}{b}\frac{d[B]}{dt} = \frac{1}{c}\frac{d[C]}{dt} = \frac{1}{d}\frac{d[D]}{dt}$$

The rate of reaction can be determined from data and graphical methods.

- In some cases, there are explicit formulas for the concentrations as functions of time—which enable us to compute the rate of reaction.

One of the quantities of interest in thermodynamics is compressibility.

- If a given substance is kept at a constant temperature, then its volume V depends on its pressure P.
- We can consider the rate of change of volume with respect to pressure—namely, the derivative dV/dP.
- As P increases, V decreases, so $dV/dP < 0$.

The compressibility is defined by introducing a minus sign and dividing this derivative by the volume *V*:

$$\text{isothermal compressibility} = \beta = -\frac{1}{V}\frac{dV}{dP}$$

- Thus, *β* measures how fast, per unit volume, the volume of a substance decreases as the pressure on it increases at constant temperature.

For instance, the volume *V* (in cubic meters) of a sample of air at 25°C was found to be related to the pressure *P* (in kilopascals) by the equation

$$V = \frac{5.3}{P}$$

The rate of change of *V* with respect to *P* when *P* = 50 kPa is:

$$\frac{dV}{dP}=\Big|_{P=50}=-\frac{5.3}{P^2}\Big|_{P=50}$$

$$=-\frac{5.3}{2500}$$

$$=-0.00212\ \ m^3/kPa$$

The compressibility at that pressure is:

$$\beta = -\frac{1}{V}\frac{dV}{dP}\bigg|_{P=50} = \frac{0.00212}{\frac{5.3}{50}}$$

$$= 0.02(m^3/kPa)/m^3$$

Let $n = f(t)$ be the number of individuals in an animal or plant population at time t.

- The change in the population size between the times $t = t_1$ and $t = t_2$ is $\Delta n = f(t_2) - f(t_1)$

AVERAGE RATE

So, the average rate of growth during the time period $t_1 \le t \le t_2$ is:

$$\text{average rate of growth} = \frac{\Delta n}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

The instantaneous rate of growth is obtained from this average rate of growth by letting the time period Δt approach 0:

$$\text{growth rate} = \lim_{\Delta t \to 0} \frac{\Delta n}{\Delta t} = \frac{dn}{dt}$$

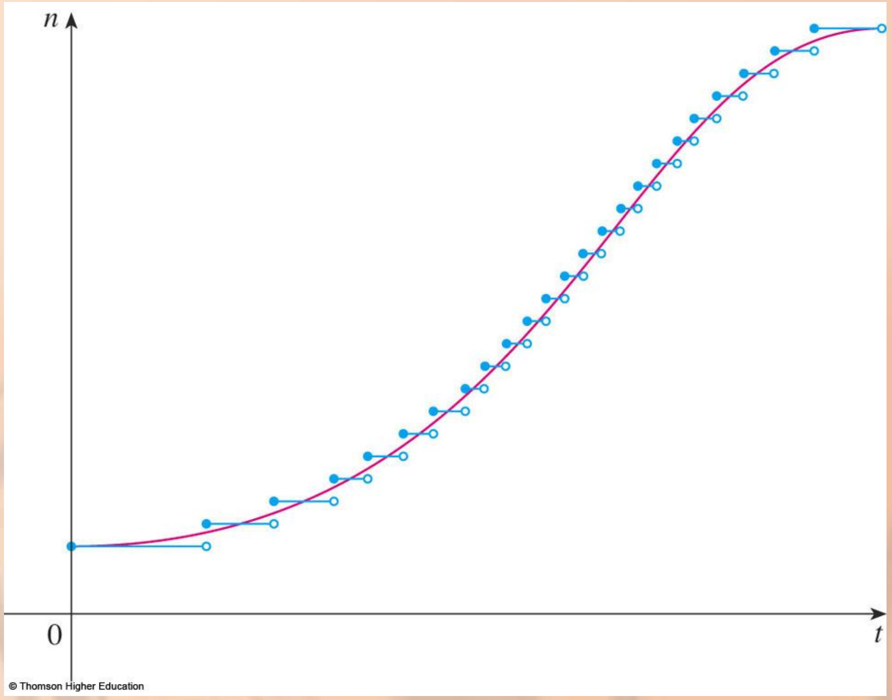
Strictly speaking, this is not quite accurate.

- This is because the actual graph of a population function $n = f(t)$ would be a step function that is discontinuous whenever a birth or death occurs and, therefore, not differentiable.

BIOLOGY

Example 6

However, for a large animal or plant population, we can replace the graph by a smooth approximating curve.



To be more specific, consider
a population of bacteria in
a homogeneous nutrient medium.

- Suppose that, by sampling the population at certain intervals, it is determined that the population doubles every hour.

If the initial population is n_0 and the time t is measured in hours, then

$$f(1) = 2f(0) = 2n_0$$

$$f(2) = 2f(1) = 2^2 n_0$$

$$f(3) = 2f(2) = 2^3 n_0$$

and, in general, $f(t) = 2^t n_0$

- The population function is $n = n_0 2^t$

In Section 3.4, we showed that:

$$\frac{d}{dx}(a^x) = a^x \ln a$$

So, the rate of growth of the bacteria population at time *t* is:

$$\frac{dn}{dt} = \frac{d}{dt}(n_0 2^t) = n_0 2^t \ln 2$$

For example, suppose that we start with an initial population of $n_0 = 100$ bacteria.

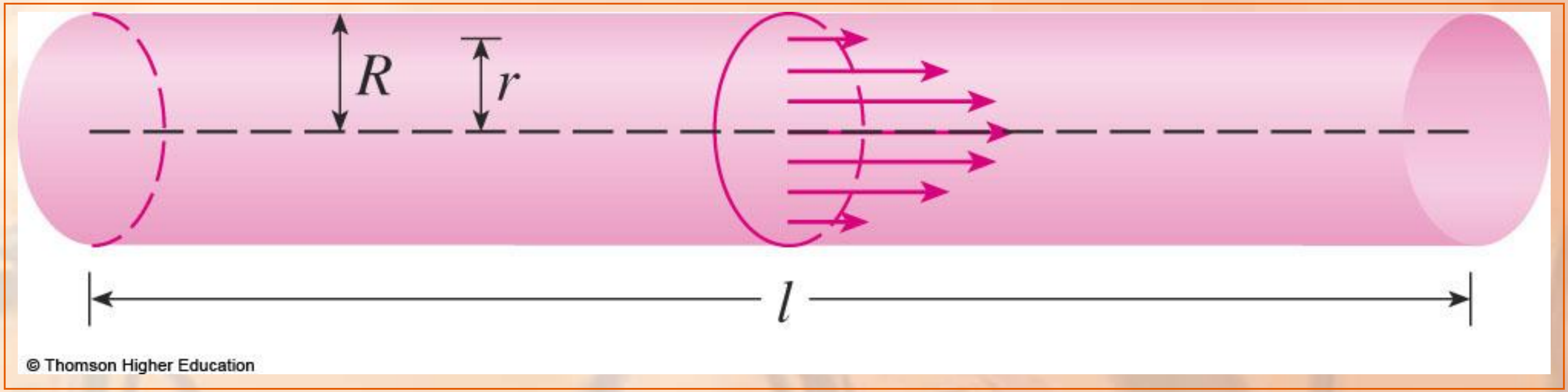
- Then, the rate of growth after 4 hours is:

$$\left.\frac{dn}{dt}\right|_{t=4} = 100 \cdot 2^4 \ln 2 = 1600 \ln 2 \approx 1109$$

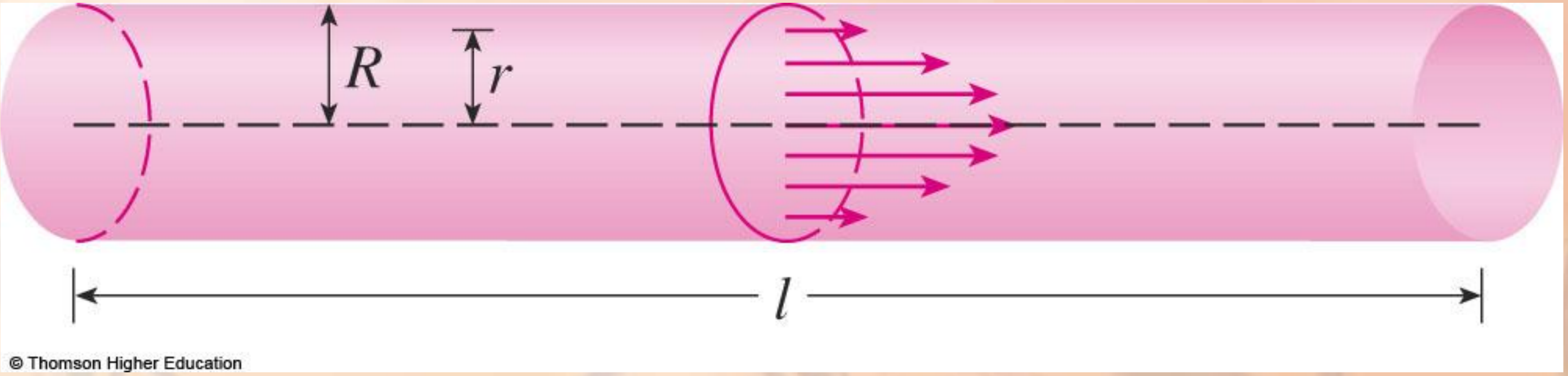
- This means that, after 4 hours, the bacteria population is growing at a rate of about 1109 bacteria per hour.

Example 7

When we consider the flow of blood through a blood vessel, such as a vein or artery, we can model the shape of the blood vessel by a cylindrical tube with radius *R* and length *l*.



Due to friction at the walls of the tube, the velocity *v* of the blood is greatest along the central axis of the tube and decreases as the distance *r* from the axis increases until *v* becomes 0 at the wall.



The relationship between *v* and *r* is given by the law of laminar flow discovered by the French physician Jean-Louis-Marie Poiseuille in 1840.

LAW OF LAMINAR FLOW

E. g. 7—Eqn. 1

The law states that

$$v = \frac{P}{4\eta l}(R^2 - r^2)$$

where η is the viscosity of the blood and P is the pressure difference between the ends of the tube.

- If P and l are constant, then v is a function of r with domain $[0, R]$.

The average rate of change of the velocity as we move from $r = r_1$ outward to $r = r_2$ is given by:

$$\frac{\Delta v}{\Delta r} = \frac{v(r_2) - v(r_1)}{r_2 - r_1}$$

If we let $\Delta r \to 0$, we obtain the velocity gradient—that is, the instantaneous rate of change of velocity with respect to *r:*

$$\text{velocity gradient} = \lim_{\Delta r \to 0} \frac{\Delta v}{\Delta r} = \frac{dv}{dr}$$

Using Equation 1, we obtain:

$$\frac{dv}{dr} = \frac{P}{4\eta l}(0 - 2r) = -\frac{Pr}{2\eta l}$$

For one of the smaller human arteries, we can take η = 0.027, R = 0.008 cm, l = 2 cm, and P = 4000 dynes/cm^2.

This gives: $$v = \frac{4000}{4(0.027)2}(0.000064 - r^2)$$

$$\approx 1.85 \times 10^4 (6.4 \times 10^{-5} - r^2)$$

At r = 0.002 cm, the blood is flowing at:

$$v(0.002) \approx 1.85 \times 10^4 (64 \times 10^{-6} - 4 \times 10^{-6})$$
$$= 1.11 \text{cm}/\text{s}$$

The velocity gradient at that point is:

$$\left.\frac{dv}{dr}\right|_{r=0.002} = -\frac{4000(0.002)}{2(0.027)2} = -74 (\text{cm}/\text{s})/\text{cm}$$

To get a feeling of what this statement means, let’s change our units from centimeters to micrometers (1 cm = 10,000 μm).

- Then, the radius of the artery is 80 μm.
- The velocity at the central axis is 11,850 μm/s, which decreases to 11,110 μm/s at a distance of r = 20 μm.

The fact that $dv/dr = -74$ (μm/s)/μm means that, when $r = 20$ μm, the velocity is decreasing at a rate of about 74 μm/s for each micrometer that we proceed away from the center.

Suppose $C(x)$ is the total cost that a company incurs in producing x units of a certain commodity.

The function C is called a cost function.

If the number of items produced is increased from x_1 to x_2, then the additional cost is $\Delta C = C(x_2) - C(x_1)$ and the average rate of change of the cost is:

$$\frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1} = \frac{C(x_1 + \Delta x) - C(x_1)}{\Delta x}$$

The limit of this quantity as $\Delta x \to 0$, that is, the instantaneous rate of change of cost with respect to the number of items produced, is called the marginal cost by economists:

$$\text{marginal cost} = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = \frac{dC}{dx}$$

As x often takes on only integer values, it may not make literal sense to let Δx approach 0.

- However, we can always replace $C(x)$ by a smooth approximating function—as in Example 6.

Taking $\Delta x = 1$ and n large (so that Δx is small compared to n), we have:

$$C'(n) \approx C(n + 1) - C(n)$$

- Thus, the marginal cost of producing n units is approximately equal to the cost of producing one more unit [the $(n + 1)$st unit].

It is often appropriate to represent a total cost function by a polynomial

$$C(x) = a + bx + cx^2 + dx^3$$

where a represents the overhead cost (rent, heat, and maintenance) and the other terms represent the cost of raw materials, labor, and so on.

The cost of raw materials may be proportional to *x*.

However, labor costs might depend partly on higher powers of *x* because of overtime costs and inefficiencies involved in large-scale operations.

For instance, suppose a company has estimated that the cost (in dollars) of producing x items is:

$$C(x) = 10,000 + 5x + 0.01x^2$$

- Then, the marginal cost function is:

$$C'(x) = 5 + 0.02x$$

The marginal cost at the production level of 500 items is:

$$C'(500) = 5 + 0.02(500) = \$15/\text{item}$$

- This gives the rate at which costs are increasing with respect to the production level when $x = 500$ and predicts the cost of the 501st item.

The actual cost of producing the 501st item is:

$C(501) - C(500) =$

$$[10{,}000 + 5(501) + 0.01(501)^2]$$
$$- [10{,}000 + 5(500) + 0.01(500)^2]$$
$$= \$15.01$$

- Notice that $C'(500) \approx C(501) - C(500)$

Economists also study marginal demand, marginal revenue, and marginal profit—which are the derivatives of the demand, revenue, and profit functions.

- These will be considered in Chapter 4—after we have developed techniques for finding the maximum and minimum values of functions.

Rates of change occur in all the sciences.

- A geologist is interested in knowing the rate at which an intruded body of molten rock cools by conduction of heat into surrounding rocks.

- An engineer wants to know the rate at which water flows into or out of a reservoir.

GEOGRAPHY AND METEOROLOGY

- An urban geographer is interested in the rate of change of the population density in a city as the distance from the city center increases.

- A meteorologist is concerned with the rate of change of atmospheric pressure with respect to height.

In psychology, those interested in learning theory study the so-called learning curve.

- This graphs the performance $P(t)$ of someone learning a skill as a function of the training time t.
- Of particular interest is the rate at which performance improves as time passes—that is, dP/dt.

SOCIOLOGY

In sociology, differential calculus is used in analyzing the spread of rumors (or innovations or fads or fashions).

- If $p(t)$ denotes the proportion of a population that knows a rumor by time t, then the derivative dp/dt represents the rate of spread of the rumor.

A SINGLE IDEA, MANY INTERPRETATIONS

You have learned about many special cases of a single mathematical concept, the derivative.

- Velocity, density, current, power, and temperature gradient in physics
- Rate of reaction and compressibility in chemistry
- Rate of growth and blood velocity gradient in biology
- Marginal cost and marginal profit in economics
- Rate of heat flow in geology
- Rate of improvement of performance in psychology
- Rate of spread of a rumor in sociology

A SINGLE IDEA, MANY INTERPRETATIONS

This is an illustration of the fact that part of the power of mathematics lies in its abstractness.

- A single abstract mathematical concept (such as the derivative) can have different interpretations in each of the sciences.

A SINGLE IDEA, MANY INTERPRETATIONS

When we develop the properties of the mathematical concept once and for all, we can then turn around and apply these results to all the sciences.

- This is much more efficient than developing properties of special concepts in each separate science.

A SINGLE IDEA, MANY INTERPRETATIONS

The French mathematician Joseph Fourier (1768–1830) put it succinctly:

> "Mathematics compares the most diverse phenomena and discovers the secret analogies that unite them."