3

DIFFERENTIATION RULES

3.6
Derivatives of Logarithmic Functions

In this section, we:

use implicit differentiation to find the derivatives of the logarithmic functions and, in particular, the natural logarithmic function.

DERIVATIVES OF LOGARITHMIC FUNCTIONS

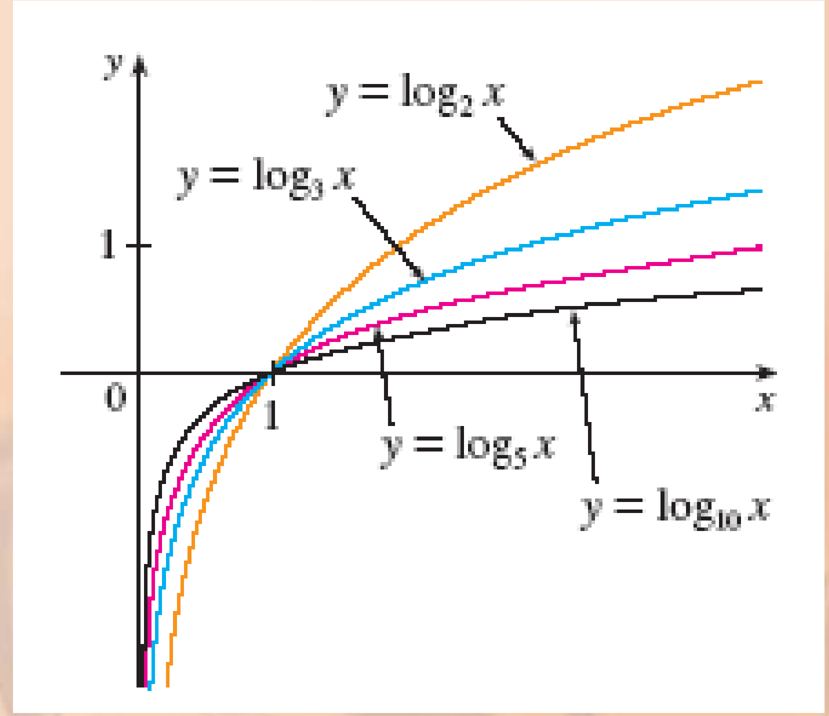
An example of a logarithmic function is: $y = \log_a x$

An example of a natural logarithmic function is: $y = \ln x$

DERIVATIVES OF LOG FUNCTIONS

It can be proved that logarithmic functions are differentiable.

- This is certainly plausible from their graphs.



DERIVATIVES OF LOG FUNCTIONS **Formula 1—Proof**

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

- Let $y = \log_a x$.
- Then, $a^y = x$.
- Differentiating this equation implicitly with respect to x, using Formula 5 in Section 3.4, we get: $$a^y (\ln a)\frac{dy}{dx} = 1$$
- So, $$\frac{dy}{dx} = \frac{1}{a^y \ln a} = \frac{1}{x \ln a}$$

DERIVATIVES OF LOG FUNCTIONS **Formula 2**

If we put $a = e$ in Formula 1, then the factor on the right side becomes $\ln e = 1$ and we get the formula for the derivative of the natural logarithmic function $\log_e x = \ln x$.

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

DERIVATIVES OF LOG FUNCTIONS

By comparing Formulas 1 and 2, we see one of the main reasons why natural logarithms (logarithms with base *e*) are used in calculus:

- The differentiation formula is simplest when $a = e$ because $\ln e = 1$.

 Example 1

Differentiate $y = \ln(x^3 + 1)$.

- To use the Chain Rule, we let $u = x^3 + 1$.
- Then $y = \ln u$.
- So, $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u}\frac{du}{dx} = \frac{1}{x^3+1}(3x^2) = \frac{3x^2}{x^3+1}$$

DERIVATIVES OF LOG FUNCTIONS **Formula 3**

In general, if we combine Formula 2 with the Chain Rule, as in Example 1, we get:

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx} \quad \text{or} \quad \frac{d}{dx}\ln g(x) = \frac{g'(x)}{g(x)}$$

Example 2

Find $\frac{d}{dx} \ln(\sin x)$.

- Using Formula 3, we have:

$$\frac{d}{dx} \ln(\sin x) = \frac{1}{\sin x} \frac{d}{dx} (\sin x)$$

$$= \frac{1}{\sin x} \cos x = \cot x$$

Differentiate $f(x)=\sqrt{\ln x}$.

- This time, the logarithm is the inner function.
- So, the Chain Rule gives:

$$f'(x)=\tfrac{1}{2}(\ln x)^{-1/2}\frac{d}{dx}(\ln x)$$

$$=\frac{1}{2\sqrt{\ln x}}\cdot\frac{1}{x}=\frac{1}{2x\sqrt{\ln x}}$$

 Example 4

Differentiate $f(x) = \log_{10}(2 + \sin x)$.

- Using Formula 1 with $a = 10$, we have:

$$f'(x) = \frac{d}{dx}\log_{10}(2+\sin x)$$

$$= \frac{1}{(2+\sin x)\ln 10}\frac{d}{dx}(2+\sin x)$$

$$= \frac{\cos x}{(2+\sin x)\ln 10}$$

E. g. 5—Solution 1

Find $\dfrac{d}{dx}\ln\dfrac{x+1}{\sqrt{x-2}}$.

$$\frac{d}{dx}\ln\frac{x+1}{\sqrt{x-2}} = \frac{1}{\dfrac{x+1}{\sqrt{x-2}}}\frac{d}{dx}\frac{x+1}{\sqrt{x-2}}$$

$$= \frac{\sqrt{x-2}}{x+1}\,\frac{\sqrt{x-2}\cdot 1-(x+1)\left(\frac{1}{2}\right)(x-2)^{-1/2}}{x-2}$$

$$= \frac{x-2-\frac{1}{2}(x+1)}{(x+1)(x-2)} = \frac{x-5}{2(x+1)(x-2)}$$

E. g. 5—Solution 2

If we first simplify the given function using the laws of logarithms, then the differentiation becomes easier:

$$\frac{d}{dx}\ln\frac{x+1}{\sqrt{x-2}} = \frac{d}{dx}\left[\ln(x+1) - \tfrac{1}{2}\ln(x-2)\right]$$

$$= \frac{1}{x+1} - \frac{1}{2}\left(\frac{1}{x-2}\right)$$

- This answer can be left as written.
- However, if we used a common denominator, it would give the same answer as in Solution 1.

 Example 6

Find $f'(x)$ if $f(x) = \ln |x|$.

- Since $f(x) = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$

  it follows that $f'(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 0 \\ \dfrac{1}{-x}(-1) = \dfrac{1}{x} & \text{if } x < 0 \end{cases}$

- Thus, $f'(x) = 1/x$ for all $x \neq 0$.

Equation 4

The result of Example 6 is worth remembering:

$$\frac{d}{dx}\ln|x| = \frac{1}{x}$$

LOGARITHMIC DIFFERENTIATION

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms.

- The method used in the following example is called logarithmic differentiation.

 Example 7

Differentiate

$$y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$$

- We take logarithms of both sides of the equation and use the Laws of Logarithms to simplify:

$$\ln y = \tfrac{3}{4}\ln x + \tfrac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

- Differentiating implicitly with respect to *x* gives:

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+1} - 5\cdot\frac{3}{3x+2}$$

- Solving for *dy* / *dx*, we get:

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

- Since we have an explicit expression for *y*, we can substitute and write:

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

STEPS IN LOGARITHMIC DIFFERENTIATION

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to simplify.

2. Differentiate implicitly with respect to x.

3. Solve the resulting equation for y'.

LOGARITHMIC DIFFERENTIATION

If $f(x) < 0$ for some values of x, then $\ln f(x)$ is not defined.

However, we can write $|y| = |f(x)|$ and use Equation 4.

- We illustrate this procedure by proving the general version of the Power Rule—as promised in Section 3.1.

THE POWER RULE

PROOF

If n is any real number and $f(x) = x^n$,

then

$$f'(x) = nx^{n-1}$$

- Let $y = x^n$ and use logarithmic differentiation:

$$\ln|y| = \ln|x|^n = n\ln|x| \quad x \neq 0$$

- Thus, $\dfrac{y'}{y} = \dfrac{n}{x}$

- Hence, $y' = n\dfrac{y}{x} = n\dfrac{x^n}{x} = nx^{n-1}$

You should distinguish carefully between:

- The Power Rule $[(x^n)' = nx^{n-1}]$, where the base is variable and the exponent is constant
- The rule for differentiating exponential functions $[(a^x)' = a^x \ln a]$, where the base is constant and the exponent is variable

LOGARITHMIC DIFFERENTIATION

In general, there are four cases for exponents and bases:

1. $\frac{d}{dx}(a^b) = 0$ $\quad$ a and b are constants
2. $\frac{d}{dx}\left[f(x)\right]^b = b\left[f(x)\right]^{b-1} f'(x)$
3. $\frac{d}{dx}\left[a^{g(x)}\right] = a^{g(x)}(\ln a)g'(x)$
4. To find $(d/dx)[f(x)]^{g(x)}$, logarithmic differentiation can be used, as in the next example.

 E. g. 8—Solution 1

Differentiate $y = x^{\sqrt{x}}$.

- Using logarithmic differentiation, we have:

$$\ln y = \ln x^{\sqrt{x}} = \sqrt{x}\ln x$$

$$\frac{y'}{y} = \sqrt{x} \cdot \frac{1}{x} + (\ln x)\frac{1}{2\sqrt{x}}$$

$$y' = y\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{2 + \ln x}{2\sqrt{x}}\right)$$

E. g. 8—Solution 2

Another method is to write $x^{\sqrt{x}} = (e^{\ln x})^{\sqrt{x}}$.

$$\frac{d}{dx}(x^{\sqrt{x}}) = \frac{d}{dx}(e^{\sqrt{x}\ln x})$$

$$= e^{\sqrt{x}\ln x}\frac{d}{dx}(\sqrt{x}\ln x)$$

$$= x^{\sqrt{x}}\left(\frac{2+\ln x}{2\sqrt{x}}\right)$$

THE NUMBER *e* AS A LIMIT

We have shown that, if $f(x) = \ln x$,

then $f'(x) = 1/x$.

Thus, $f'(1) = 1$.

- Now, we use this fact to express the number *e* as a limit.

THE NUMBER *e* AS A LIMIT

From the definition of a derivative as a limit, we have:

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{x \to 0} \frac{f(1+x) - f(1)}{x}$$

$$= \lim_{x \to 0} \frac{\ln(1+x) - \ln 1}{x} = \lim_{x \to 0} \frac{1}{x} \ln(1+x)$$

$$= \lim_{x \to 0} \ln(1+x)^{1/x}$$

Formula 5

As *f* ’(1) = 1, we have $\lim_{x \to 0} \ln(1+x)^{1/x} = 1$

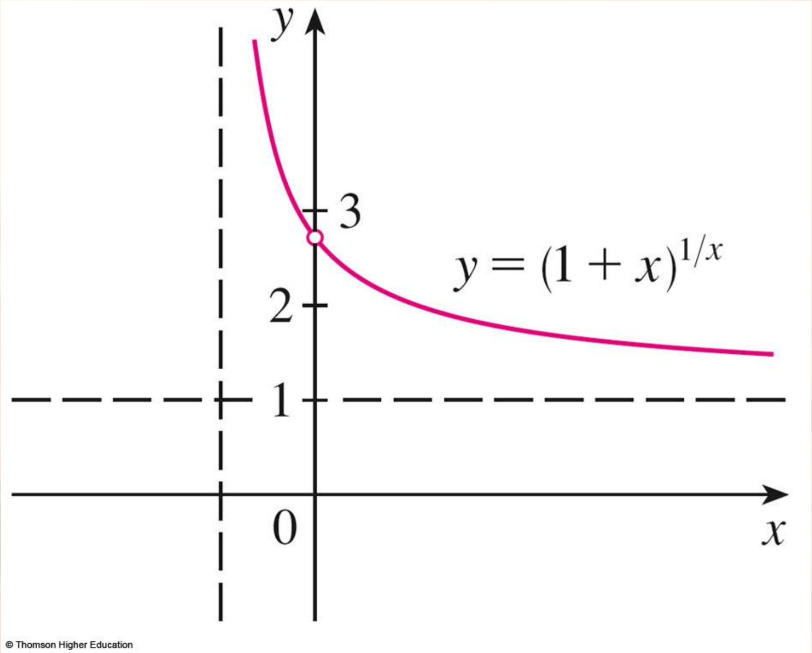
- Then, by Theorem 8 in Section 2.5 and the continuity of the exponential function, we have:

$$e = e^1 = e^{\lim_{x \to 0} \ln(1+x)^{1/x}} = \lim_{x \to 0} e^{\ln(1+x)^{1/x}} = \lim_{x \to 0} (1+x)^{1/x}$$

$$e = \lim_{x \to 0} (1+x)^{1/x}$$

THE NUMBER *e* AS A LIMIT

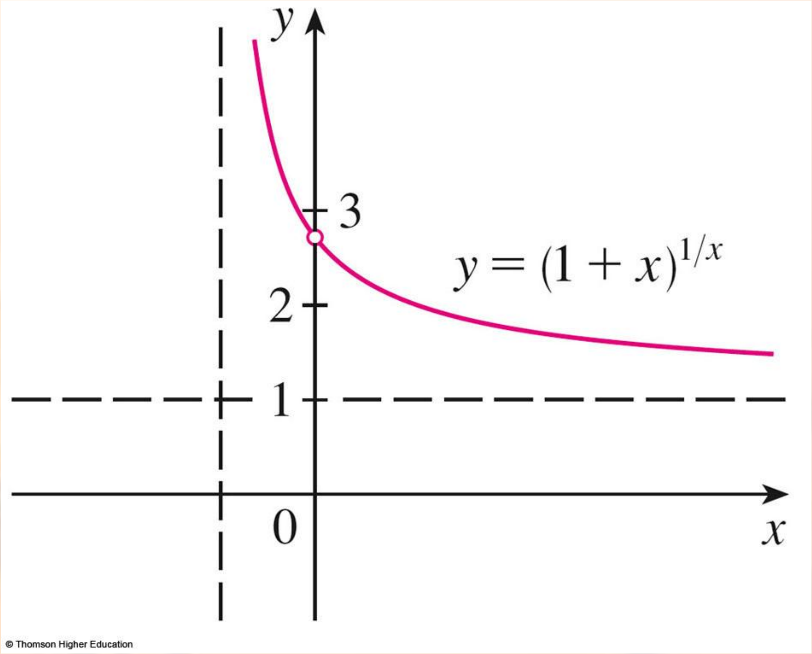
Formula 5 is illustrated by the graph of the function $y = (1 + x)^{1/x}$ here and a table of values for small values of x.



x	$(1 + x)^{1/x}$
0.1	2.59374246
0.01	2.70481383
0.001	2.71692393
0.0001	2.71814593
0.00001	2.71826824
0.000001	2.71828047
0.0000001	2.71828169
0.00000001	2.71828181

THE NUMBER *e* AS A LIMIT

This illustrates the fact that, correct to seven decimal places, $e \approx 2.7182818$



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Formula 6

If we put $n = 1/x$ in Formula 5, then $n \to \infty$ as $x \to 0^+$.

So, an alternative expression for *e* is:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$