# 3

# DIFFERENTIATION RULES

The functions that we have met so far can be described by expressing one variable explicitly in terms of another variable.

- For example, $y = \sqrt{x^3 + 1}$, or $y = x \sin x$, or in general $y = f(x)$.

However, some functions are defined implicitly.

# 3.5

# Implicit Differentiation

In this section, we will learn:

How functions are defined implicitly.

**Equations 1 and 2**

Some examples of implicit functions are:

$$x^2 + y^2 = 25$$

$$x^3 + y^3 = 6xy$$

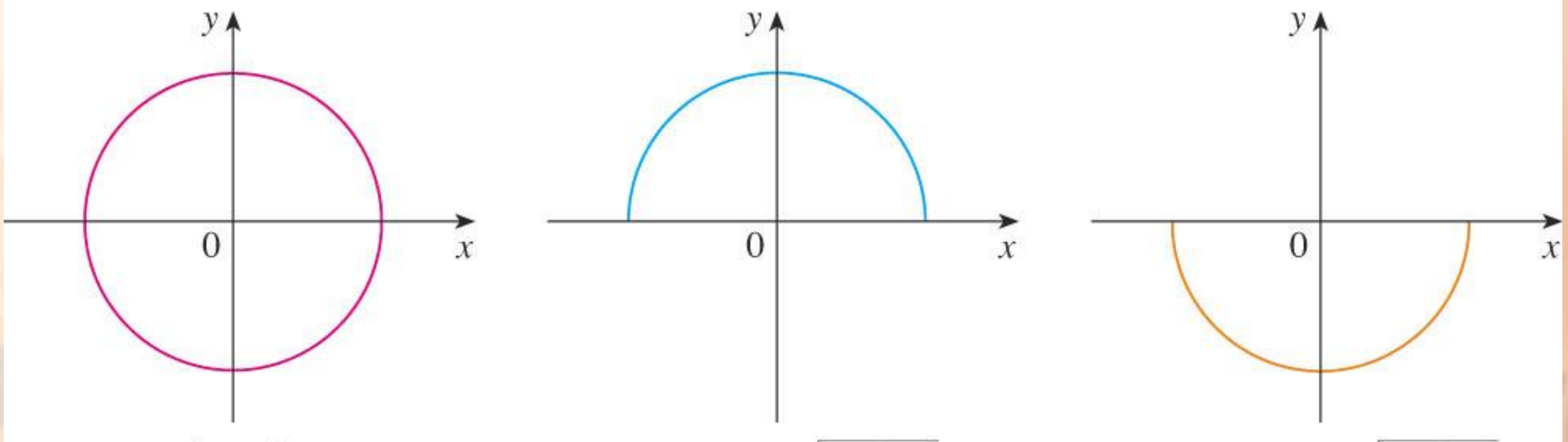
## IMPLICIT DIFFERENTIATION

In some cases, it is possible to solve such an equation for *y* as an explicit function (or several functions) of *x*.

- For instance, if we solve Equation 1 for *y*, we get $y = \pm\sqrt{25 - x^2}$
- So, two of the functions determined by the implicit Equation 1 are $f(x) = \sqrt{25 - x^2}$ and $g(x) = -\sqrt{25 - x^2}$

# IMPLICIT DIFFERENTIATION

The graphs of *f* and *g* are the upper and lower semicircles of the circle $x^2 + y^2 = 25$.




(a) $x^2 + y^2 = 25$ (b) $f(x) = \sqrt{25 - x^2}$ (c) $g(x) = -\sqrt{25 - x^2}$

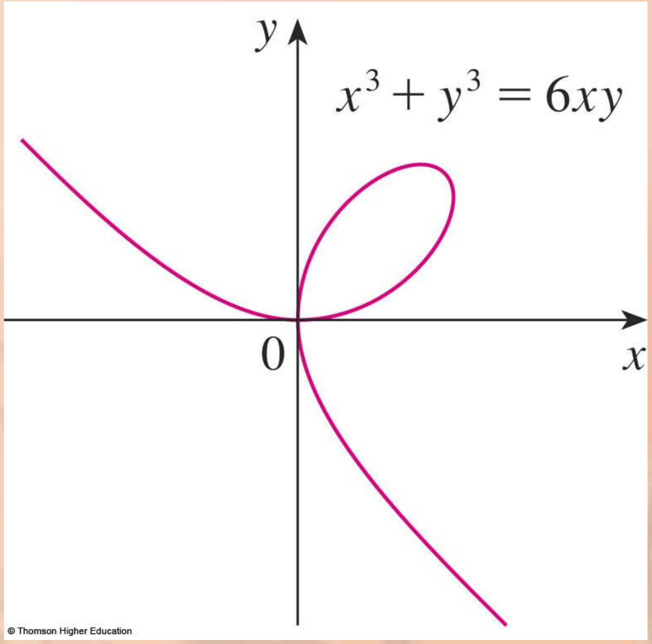
# IMPLICIT DIFFERENTIATION

It’s not easy to solve Equation 2 for *y* explicitly as a function of *x* by hand.

- A computer algebra system has no trouble.
- However, the expressions it obtains are very complicated.

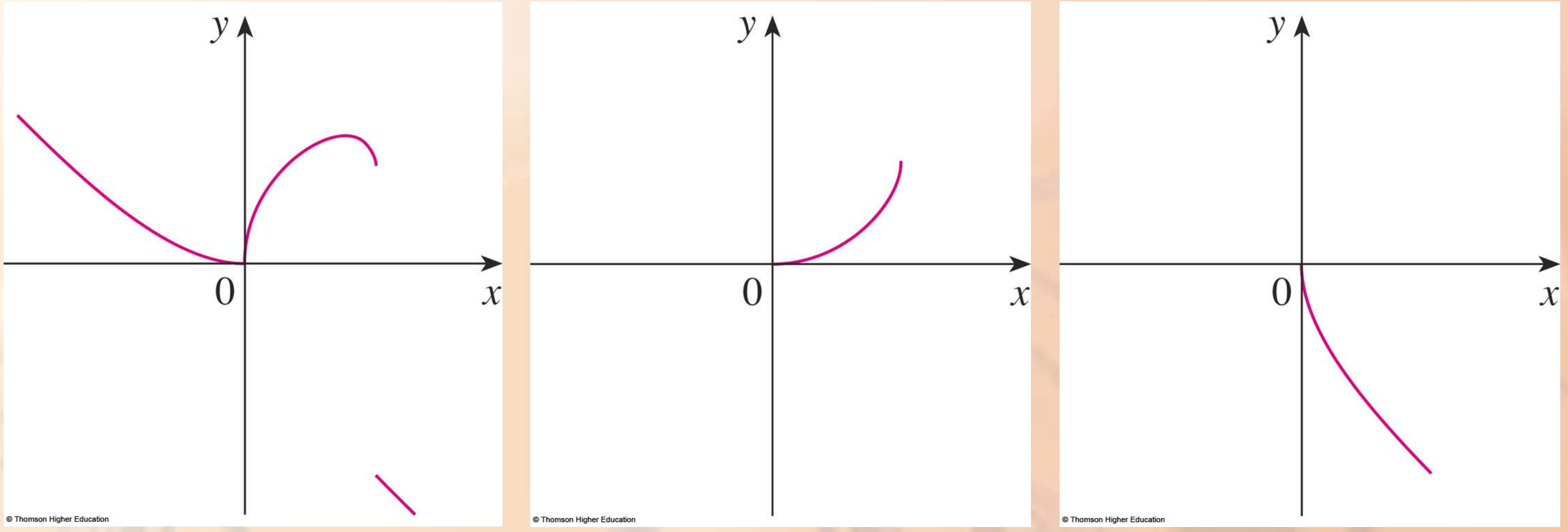
## FOLIUM OF DESCARTES

Nonetheless, Equation 2 is the equation of a curve called the folium of Descartes shown here and it implicitly defines *y* as several functions of *x*.



# FOLIUM OF DESCARTES

The graphs of three functions defined by the folium of Descartes are shown.



## IMPLICIT DIFFERENTIATION

When we say that *f* is a function defined implicitly by Equation 2, we mean that the equation $x^3 + [f(x)]^3 = 6x\, f(x)$ is true for all values of *x* in the domain of *f*.

## IMPLICIT DIFFERENTIATION

Fortunately, we don't need to solve an equation for $y$ in terms of $x$ to find the derivative of $y$.

## IMPLICIT DIFFERENTIATION METHOD

Instead, we can use the method of implicit differentiation.

- This consists of differentiating both sides of the equation with respect to *x* and then solving the resulting equation for *y*'.

# IMPLICIT DIFFERENTIATION METHOD

In the examples, it is always assumed that the given equation determines $y$ implicitly as a differentiable function of $x$ so that the method of implicit differentiation can be applied.

**Example 1**

a. If $x^2 + y^2 = 25$, find $\frac{dy}{dx}$ .

b. Find an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point (3, 4).

**Example 1 a**

Differentiate both sides of the equation

$x^2 + y^2 = 25$:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

**Example 1 a**

Remembering that *y* is a function of *x* and using the Chain Rule, we have:

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\frac{dy}{dx} = 2y\frac{dy}{dx}$$

$$2x + 2y\frac{dy}{dx} = 0$$

Then, we solve this equation for $\frac{dy}{dx}$: $\frac{dy}{dx} = -\frac{x}{y}$

## E. g. 1 b—Solution 1

At the point (3, 4) we have $x = 3$ and $y = 4$.

So, $$\frac{dy}{dx} = -\frac{3}{4}$$

- Thus, an equation of the tangent to the circle at (3, 4) is: $y - 4 = -\frac{3}{4}(x - 3)$ or $3x + 4y = 25$.

## E. g. 1 b—Solution 2

Solving the equation $x^2 + y^2 = 25$,

we get: $y = \pm\sqrt{25 - x^2}$

- The point (3, 4) lies on the upper semicircle $y = \sqrt{25 - x^2}$
- So, we consider the function $f(x) = \sqrt{25 - x^2}$

Differentiating *f* using the Chain Rule, we have:

$$f'(x) = \tfrac{1}{2}(25 - x^2)^{-1/2} \frac{d}{dx}(25 - x^2)$$

$$= \tfrac{1}{2}(25 - x^2)^{-1/2}(-2x)$$

$$= -\frac{x}{\sqrt{25 - x^2}}$$

## E. g. 1 b—Solution 2

So,

$$f\,'(3) = -\frac{3}{\sqrt{25-3^2}} = -\frac{3}{4}$$

- As in Solution 1, an equation of the tangent is $3x + 4y = 25$.

## NOTE 1

The expression $dy/dx = -x/y$ in Solution 1 gives the derivative in terms of both $x$ and $y$.

It is correct no matter which function $y$ is determined by the given equation.

## NOTE 1

For instance, for $y = f(x) = \sqrt{25 - x^2}$,

we have: $$\frac{dy}{dx} = -\frac{x}{y} = -\frac{x}{\sqrt{25 - x^2}}$$

However, for $y = g(x) = -\sqrt{25 - x^2}$,

we have: $$\frac{dy}{dx} = -\frac{x}{y} = -\frac{x}{-\sqrt{25 - x^2}} = \frac{x}{\sqrt{25 - x^2}}$$

**Example 2**

a. Find $y'$ if $x^3 + y^3 = 6xy$.

b. Find the tangent to the folium of Descartes $x^3 + y^3 = 6xy$ at the point (3, 3).

c. At what points in the first quadrant is the tangent line horizontal?

**Example 2 a**

Differentiating both sides of $x^3 + y^3 = 6xy$ with respect to $x$, regarding $y$ as a function of $x$, and using the Chain Rule on $y^3$ and the Product Rule on $6xy$, we get:

$$3x^2 + 3y^2y' = 6xy' + 6y$$

or $$x^2 + y^2y' = 2xy' + 2y$$

Now, we solve for *y*':

$$y^2 y' - 2xy' = 2y - x^2$$

$$(y^2 - 2x)y' = 2y - x^2$$

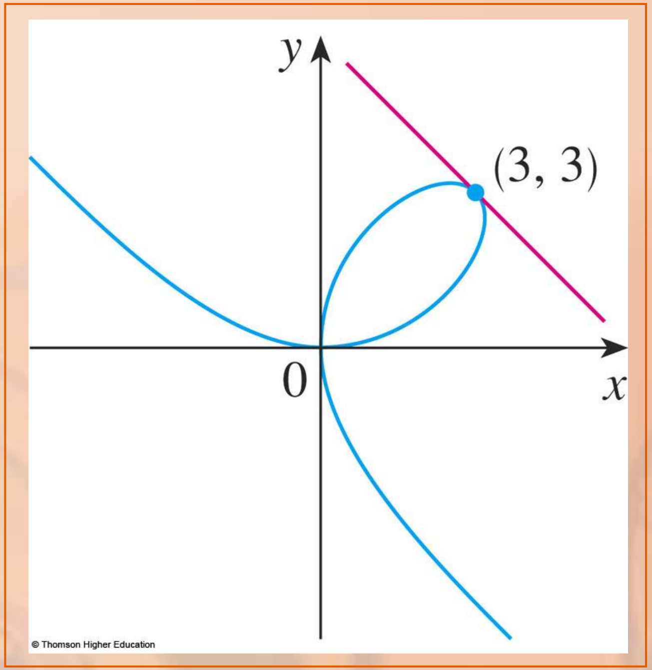
$$y' = \frac{2y - x^2}{y^2 - 2x}$$

**Example 2 b**

When $x = y = 3$,

$$y' = \frac{2 \cdot 3 - 3^2}{3^2 - 2 \cdot 3} = -1$$

- A glance at the figure confirms that this is a reasonable value for the slope at (3, 3).
- So, an equation of the tangent to the folium at (3, 3) is:
  $y - 3 = -1(x - 3)$ or $x + y = 6$.



**Example 2 c**

# The tangent line is horizontal if $y'$ = 0.

- Using the expression for $y'$ from (a), we see that $y' = 0$ when $2y - x^2 = 0$ (provided that $y^2 - 2x \neq 0$).
- Substituting $y = \frac{1}{2}x^2$ in the equation of the curve, we get $x^3 + (\frac{1}{2}x^2)^3 = 6x(\frac{1}{2}x^2)$ which simplifies to $x^6 = 16x^3$.

**Example 2 c**

Since $x \neq 0$ in the first quadrant,

we have $x^3 = 16$.

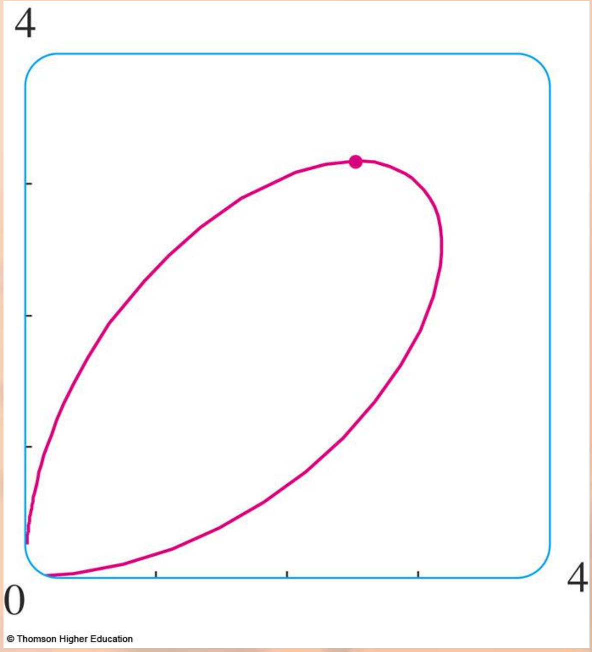
If $x = 16^{1/3} = 2^{4/3}$, then $y = \frac{1}{2}(2^{8/3}) = 2^{5/3}$.

## IMPLICIT DIFFERENTIATION

**Example 2 c**

Thus, the tangent is horizontal at (0, 0) and at $(2^{4/3}, 2^{5/3})$, which is approximately (2.5198, 3.1748).

- Looking at the figure, we see that our answer is reasonable.



## NOTE 2

There is a formula for the three roots of a cubic equation that is like the quadratic formula, but much more complicated.

## NOTE 2

If we use this formula (or a computer algebra system) to solve the equation $x^3 + y^3 = 6xy$ for $y$ in terms of $x$, we get three functions determined by the following equation.

## NOTE 2

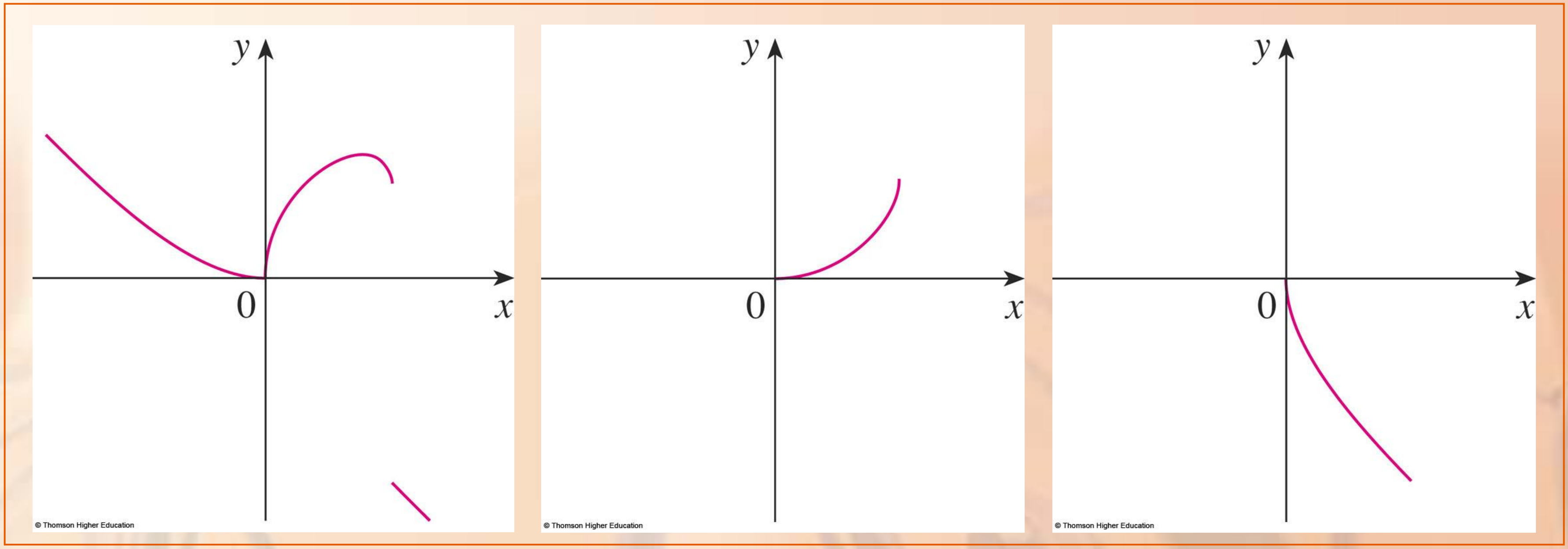
$$y = f(x) = \sqrt[3]{-\tfrac{1}{2}x^3 + \sqrt{\tfrac{1}{4}x^6 - 8x^3}} + \sqrt[3]{-\tfrac{1}{2}x^3 - \sqrt{\tfrac{1}{4}x^6 - 8x^3}}$$

and

$$y = \tfrac{1}{2}\left[-f(x) \pm \sqrt{-3}\left(\sqrt[3]{-\tfrac{1}{2}x^3 + \sqrt{\tfrac{1}{4}x^6 - 8x^3}} - \sqrt[3]{-\tfrac{1}{2}x^3 - \sqrt{\tfrac{1}{4}x^6 - 8x^3}}\right)\right]$$

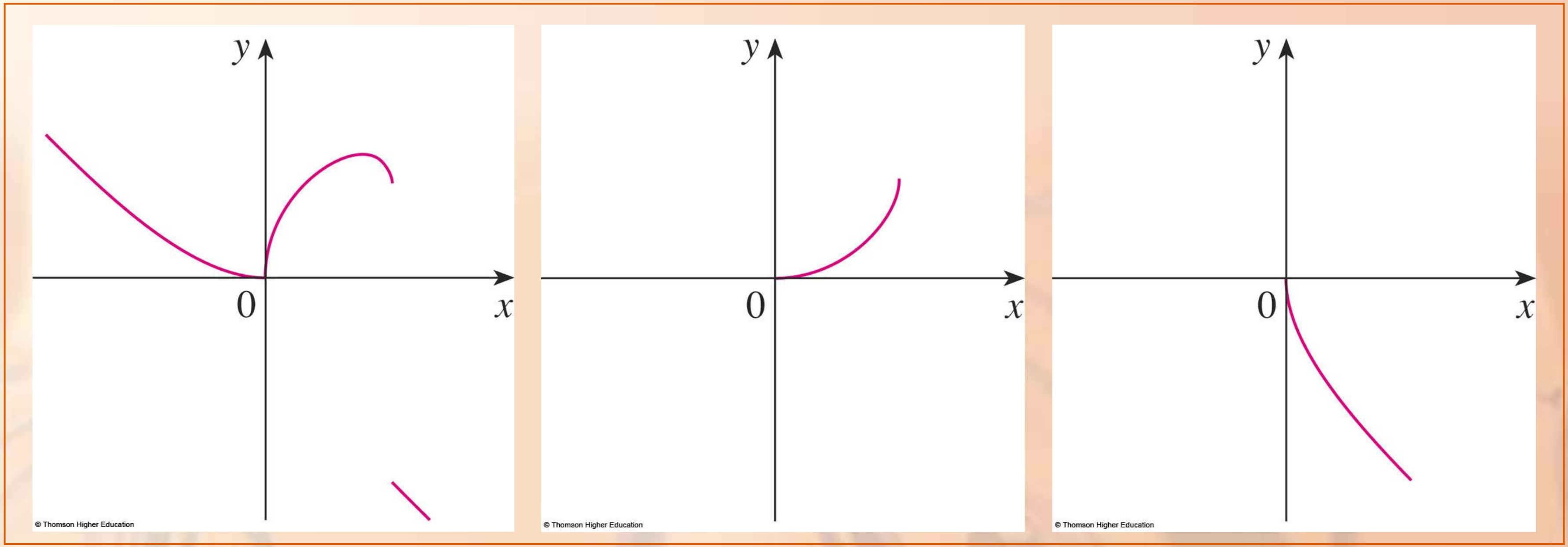
## NOTE 2

These are the three functions whose graphs are shown in the earlier figure.



## NOTE 2

You can see that the method of implicit differentiation saves an enormous amount of work in cases such as this.



## NOTE 2

Moreover, implicit differentiation works just as easily for equations such as

$$y^5 + 3x^2y^2 + 5x^4 = 12$$

for which it is impossible to find a similar expression for $y$ in terms of $x$.

**Example 3**

# Find $y'$ if $\sin(x + y) = y^2 \cos x$.

- Differentiating implicitly with respect to *x* and remembering that *y* is a function of *x*, we get:

$$\cos(x+y)\cdot(1+y') = y^2(-\sin x) + (\cos x)(2yy')$$

- Note that we have used the Chain Rule on the left side and the Product Rule and Chain Rule on the right side.

**Example 3**

If we collect the terms that involve *y*', we get:

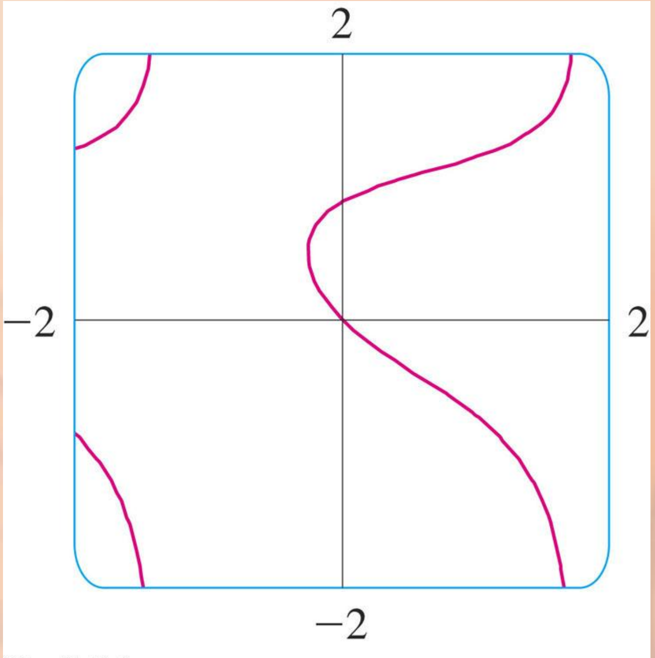
$$\cos(x+y)+y^2\sin x=(2y\cos x)y'-\cos(x+y)\cdot y'$$

So, $y'=\dfrac{y^2\sin x+\cos(x+y)}{2y\cos x-\cos(x+y)}$

**Example 3**

The figure, drawn with the implicit-plotting command of a computer algebra system, shows part of the curve $\sin(x + y) = y^2 \cos x$.

- As a check on our calculation, notice that *y’* = -1 when $x = y = 0$ and it appears that the slope is approximately -1 at the origin.



# IMPLICIT DIFFERENTIATION

The following example shows how to find the second derivative of a function that is defined implicitly.

**Example 4**

# Find $y''$ if $x^4 + y^4 = 16$.

- Differentiating the equation implicitly with respect to $x$, we get $4x^3 + 4y^3y' = 0$.

Solving for *y*' gives:

$$y' = -\frac{x^3}{y^3}$$

**Example 4**

To find *y*'', we differentiate this expression for *y*' using the Quotient Rule and remembering that *y* is a function of *x*:

$$y'' = \frac{d}{dx}\left(-\frac{x^3}{y^3}\right) = -\frac{y^3(d/dx)(x^3) - x^3(d/dx)(y^3)}{(y^3)^2}$$

$$= -\frac{y^3 \cdot 3x^2 - x^3(3y^2 y')}{y^6}$$

**Example 4**

If we now substitute Equation 3 into this expression, we get:

$$y'' = -\frac{3x^2y^3 - 3x^3y^2\left(-\dfrac{x^3}{y^3}\right)}{y^6}$$

$$= -\frac{3(x^2y^4 + x^6)}{y^7} = -\frac{3x^2(y^4 + x^4)}{y^7}$$

**Example 4**

However, the values of $x$ and $y$ must satisfy the original equation $x^4 + y^4 = 16$.

So, the answer simplifies to:

$$y'' = -\frac{3x^2(16)}{y^7} = -48\frac{x^2}{y^7}$$

# INVERSE TRIGONOMETRIC FUNCTIONS (ITFs)

The inverse trigonometric functions were reviewed in Section 1.6

- We discussed their continuity in Section 2.5 and their asymptotes in Section 2.6

## DERIVATIVES OF ITFs

Here, we use implicit differentiation to find the derivatives of the inverse trigonometric functions—assuming that these functions are differentiable.

# DERIVATIVES OF ITFs

In fact, if *f* is any one-to-one differentiable function, it can be proved that its inverse function $f^{-1}$ is also differentiable—except where its tangents are vertical.

- This is plausible because the graph of a differentiable function has no corner or kink.
- So, if we reflect it about $y = x$, the graph of its inverse function also has no corner or kink.

## DERIVATIVE OF ARCSINE FUNCTION

Recall the definition of the arcsine function:

$$y = \sin^{-1} x \text{ means } \sin y = x \text{ and } -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

Differentiating sin *y* = *x* implicitly with respect to *x*, we obtain:

$$\cos y \frac{dy}{dx} = 1 \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{\cos y}$$

## DERIVATIVE OF ARCSINE FUNCTION

Now, cos $y \geq 0$, since $-\pi/2 \leq y \leq \pi/2$.

So, $\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$

Thus, $\dfrac{dy}{dx} = \dfrac{1}{\cos y} = \dfrac{1}{\sqrt{1-x^2}}$

$$\boxed{\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}}$$

# DERIVATIVE OF ARCTANGENT FUNCTION

The formula for the derivative of the arctangent function is derived in a similar way.

- If $y = \tan^{-1} x$, then $\tan y = x$.
- Differentiating this latter equation implicitly with respect to $x$, we have:

$$\sec^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

$$\boxed{\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}}$$

**Example 5**

Differentiate:

a. $y = \dfrac{1}{\sin^{-1} x}$

b. $f(x) = x \arctan\sqrt{x}$

## DERIVATIVES OF ITFs

**Example 5 a**

$$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1} x)^{-1} = -(\sin^{-1} x)^{-2} \frac{d}{dx}(\sin^{-1} x)$$

$$= -\frac{1}{(\sin^{-1} x)^2 \sqrt{1 - x^2}}$$

**Example 5 b**

$$f'(x) = x\frac{1}{1+(\sqrt{x})^2}\left(\tfrac{1}{2}x^{-1/2}\right) + \arctan\sqrt{x}$$

$$= \frac{\sqrt{x}}{2(1+x)} + \arctan\sqrt{x}$$

# DERIVATIVES OF ITFs

The inverse trigonometric functions that occur most frequently are the ones that we have just discussed.

The derivatives of the remaining four are given in this table.

- The proofs of the formulas are left as exercises.

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$