3

DIFFERENTIATION RULES

Before starting this section, you might need to review the trigonometric functions.

In particular, it is important to remember that, when we talk about the function *f* defined for all real numbers *x* by $f(x) = \sin x$, it is understood that $\sin x$ means the sine of the angle whose radian measure is *x*.

A similar convention holds for the other trigonometric functions cos, tan, csc, sec, and cot.

- Recall from Section 2.5 that all the trigonometric functions are continuous at every number in their domains.

3.6

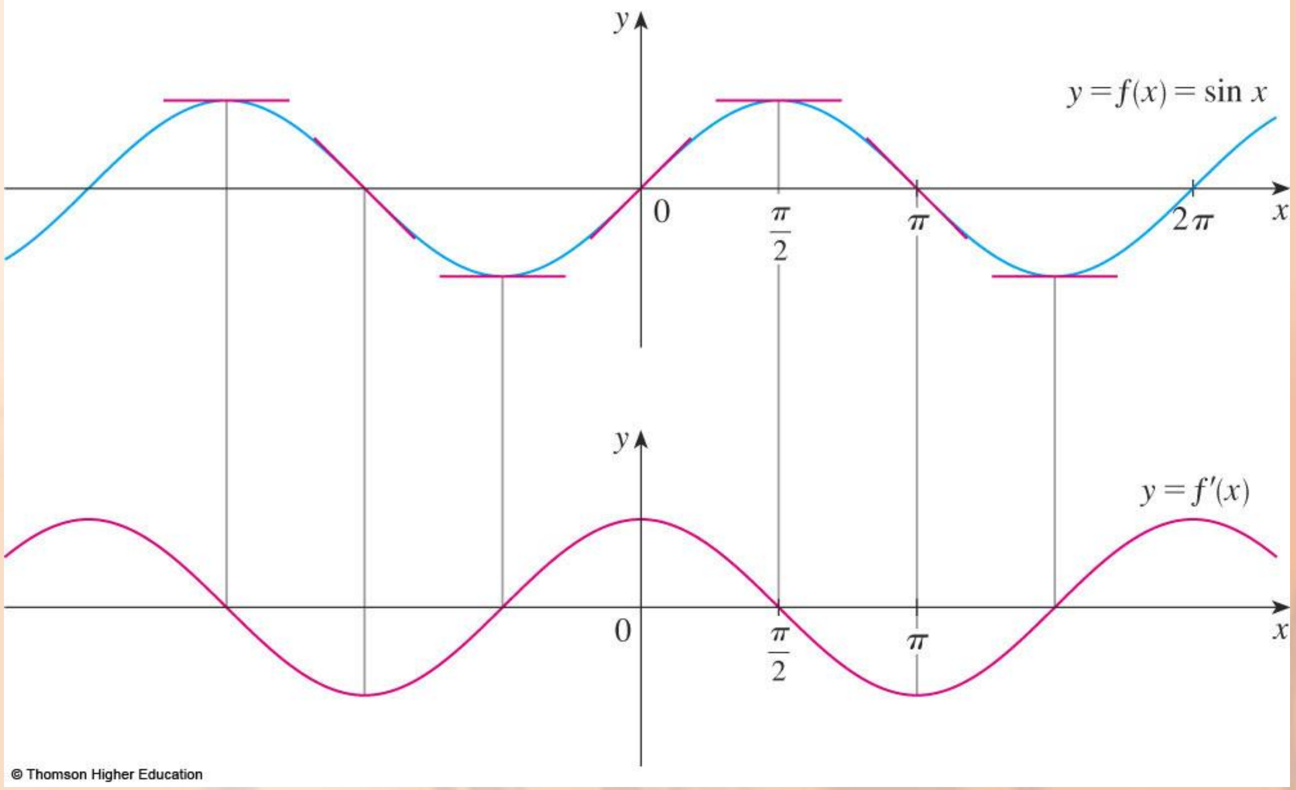
Derivatives of Trigonometric Functions

In this section, we will learn about:
Derivatives of trigonometric functions
and their applications.

Let’s sketch the graph of the function $f(x) = \sin x$ and use the interpretation of $f'(x)$ as the slope of the tangent to the sine curve in order to sketch the graph of f'.

DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

Then, it looks as if the graph of *f'* may be the same as the cosine curve.



Let's try to confirm

our guess that, if $f(x) = \sin x$,

then $f'(x) = \cos x$.

From the definition of a derivative, we have:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin \mathrm{h} - \sin x}{h}$$

$$= \lim_{h\to 0} \left[\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right]$$

$$= \lim_{h\to 0} \left[\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right]$$

$$= \lim_{h\to 0} \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \lim_{h\to 0} \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h}$$

$$\lim_{h\to 0}\sin x\cdot\lim_{h\to 0}\frac{\cos h-1}{h}+\lim_{h\to 0}\cos x\cdot\lim_{h\to 0}\frac{\sin h}{h}$$

Two of these four limits are easy to evaluate.

Since we regard *x* as a constant when computing a limit as $h \to 0$, we have:

$$\lim_{h \to 0} \sin x = \sin x$$

$$\lim_{h \to 0} \cos x = \cos x$$

 Equation 2

The limit of (sin *h*)/*h* is not so obvious.

In Example 3 in Section 2.2, we made the guess—on the basis of numerical and graphical evidence—that:

$$\lim_{\theta \to 0} = \frac{\sin \theta}{\theta} = 1$$

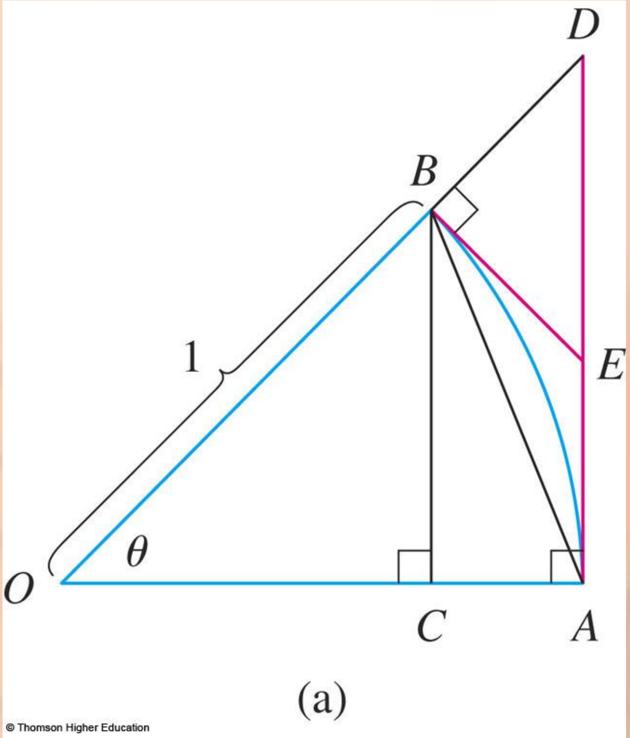
We now use a geometric argument to prove Equation 2.

- Assume first that θ lies between 0 and $\pi/2$.

Proof

The figure shows a sector of a circle with center O, central angle θ, and radius 1. BC is drawn perpendicular to OA.

- By the definition of radian measure, we have arc $AB = \theta$.
- Also, $|BC| = |OB| \sin\theta = \sin\theta$.



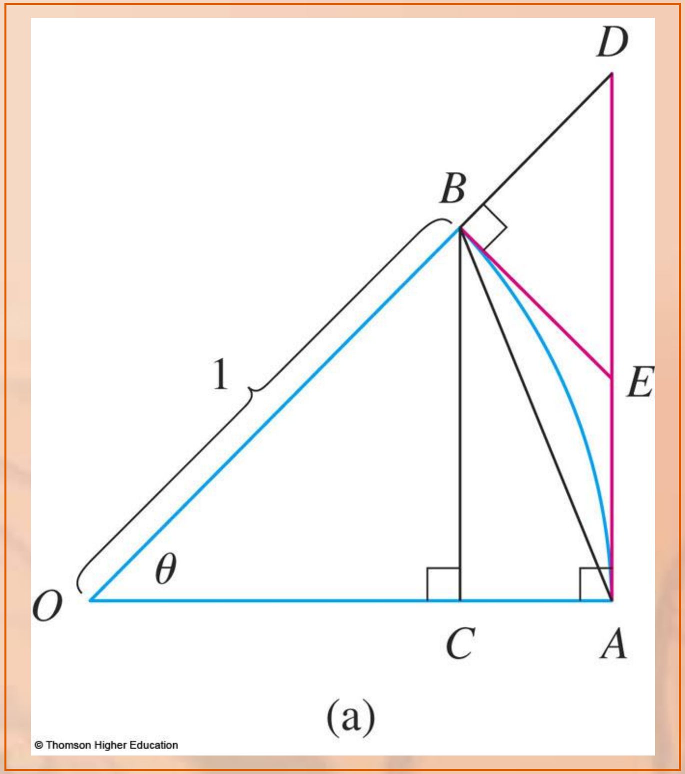
 Proof

We see that

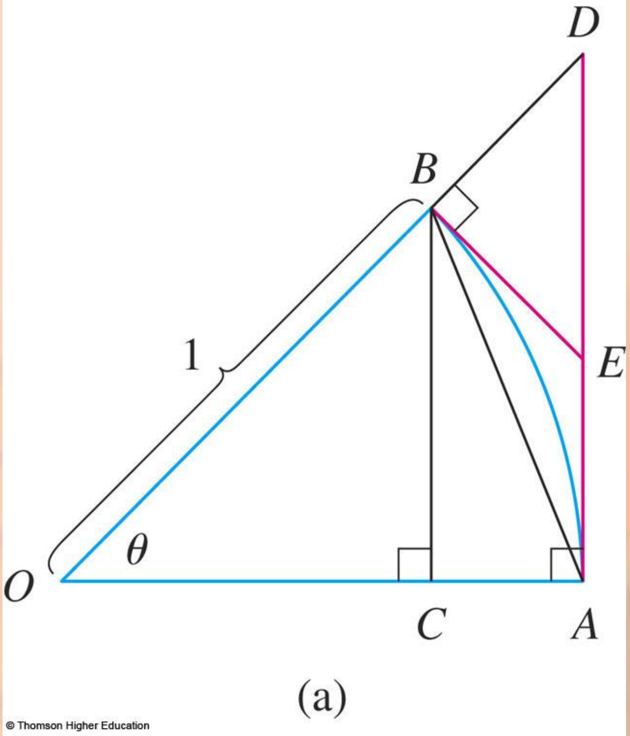
$$|BC| < |AB| < \text{arc } AB$$

Thus,

$$\sin\theta < \theta \quad \text{so} \quad \frac{\sin\theta}{\theta} < 1$$

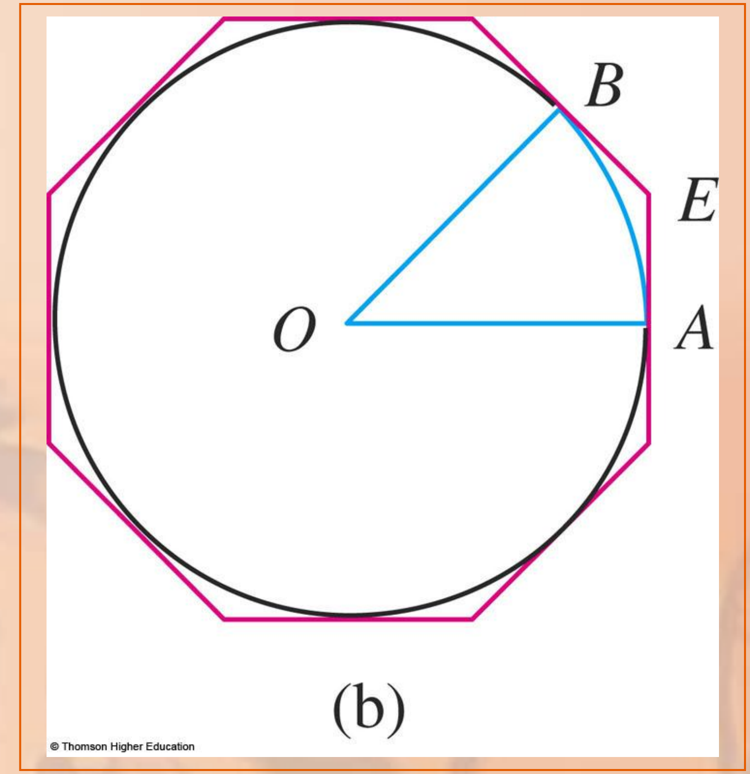


Let the tangent lines at *A* and *B* intersect at *E*.



Proof

You can see from this figure that the circumference of a circle is smaller than the length of a circumscribed polygon.



So,

arc *AB* < |*AE*| + |*EB*|

Proof

Thus,

$$\theta = \text{arc } AB < |AE| + |EB|$$

$$< |AE| + |ED|$$

$$= |AD| = |OA| \tan \theta$$

$$= \tan \theta$$

Therefore, we have:

$$\theta < \frac{\sin\theta}{\cos\theta}$$

So, $\cos\theta < \dfrac{\sin\theta}{\theta} < 1$

Proof

We know that $\lim_{\theta \to 0} 1 = 1$ and $\lim_{\theta \to 0} \cos\theta = 1$.

So, by the Squeeze Theorem, we have:

$$\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1$$

Proof

However, the function (sin *θ*)/*θ* is an even function.

So, its right and left limits must be equal.

Hence, we have:

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

We can deduce the value of the remaining limit in Equation 1 as follows.

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta}$$

$$= \lim_{\theta \to 0} \left(\frac{\cos\theta - 1}{\theta} \cdot \frac{\cos\theta + 1}{\cos\theta + 1} \right)$$

$$= \lim_{\theta \to 0} \frac{\cos^2\theta - 1}{\theta(\cos\theta + 1)}$$

Equation 3

$$= \lim_{\theta\to 0} \frac{-\sin^2\theta}{\theta(\cos\theta+1)}$$

$$= -\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\cdot\frac{\sin\theta}{\cos\theta+1}\right)$$

$$= -\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to 0}\frac{\sin\theta}{\cos\theta+1} = -1\cdot\left(\frac{0}{1+1}\right) = 0$$

$$\boxed{\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta} = 0}$$

If we put the limits (2) and (3) in (1), we get:

$$f'(x) = \lim_{h\to 0} \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \lim_{h\to 0} \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h}$$

$$= (\sin x)\cdot 0 + (\cos x)\cdot 1$$

$$= \cos x$$

 Formula 4

So, we have proved the formula for the derivative of the sine function:

$$\frac{d}{dx}(\sin x) = \cos x$$

Example 1

Differentiate $y = x^2 \sin x$.

- Using the Product Rule and Formula 4, we have:

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x^2)$$

$$= x^2 \cos x + 2x \sin x$$

DERIV. OF COSINE FUNCTION

Formula 5

Using the same methods as in the proof of Formula 4, we can prove:

$$\frac{d}{dx}(\cos x) = -\sin x$$

DERIV. OF TANGENT FUNCTION

The tangent function can also be differentiated by using the definition of a derivative.

However, it is easier to use the Quotient Rule together with Formulas 4 and 5—as follows.

DERIV. OF TANGENT FUNCTION

Formula 6

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

$$\boxed{\frac{d}{dx}(\tan x) = \sec^2 x}$$

DERIVS. OF TRIG. FUNCTIONS

The derivatives of the remaining trigonometric functions—csc, sec, and cot—can also be found easily using the Quotient Rule.

We have collected all the differentiation formulas for trigonometric functions here.

- Remember, they are valid only when *x* is measured in radians.

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

 Example 2

Differentiate $f(x) = \dfrac{\sec x}{1 + \tan x}$

For what values of *x* does the graph of *f* have a horizontal tangent?

The Quotient Rule gives:

$$f'(x) = \frac{(1+\tan x)\frac{d}{dx}(\sec x) - \sec x \frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$= \frac{(1+\tan x)\sec x \tan x - \sec x \cdot \sec^2 x}{(1+\tan x)^2}$$

$$= \frac{\sec x(\tan x + \tan^2 x - \sec^2 x)}{(1+\tan x)^2}$$

$$= \frac{\sec x(\tan x - 1)}{(1+\tan x)^2}$$

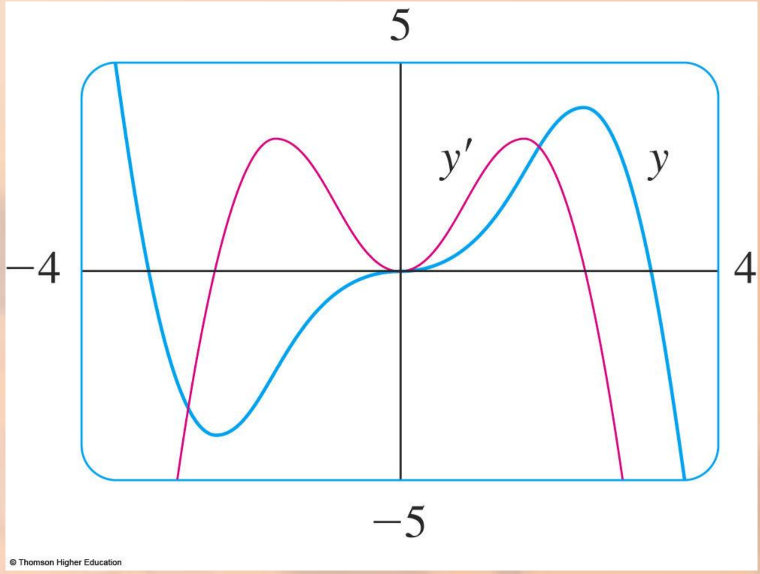
Example 2

In simplifying the answer, we have used the identity

$\tan^2 x + 1 = \sec^2 x$.

Since sec *x* is never 0, we see that *f'*(*x*) when tan *x* = 1.

- This occurs when $x = n\pi + \pi/4$, where *n* is an integer.



APPLICATIONS

Trigonometric functions are often used in modeling real-world phenomena.

- In particular, vibrations, waves, elastic motions, and other quantities that vary in a periodic manner can be described using trigonometric functions.

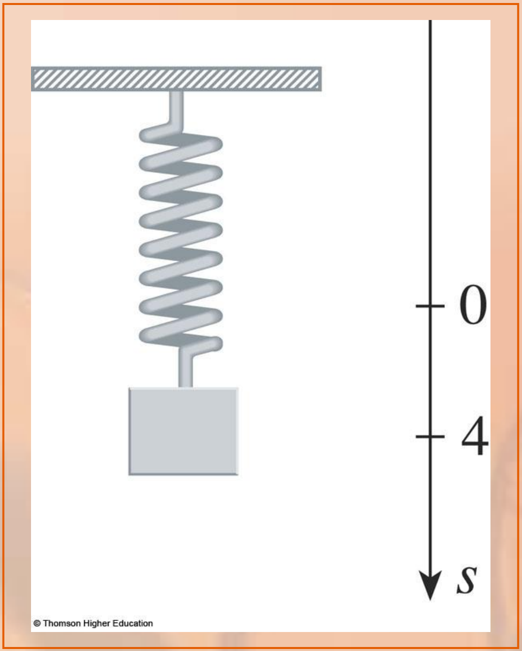
- In the following example, we discuss an instance of simple harmonic motion.

APPLICATIONS

Example 3

An object at the end of a vertical spring is stretched 4 cm beyond its rest position and released at time $t = 0$.

- In the figure, note that the downward direction is positive.
- Its position at time t is
  $s = f(t) = 4 \cos t$
- Find the velocity and acceleration at time t and use them to analyze the motion of the object.



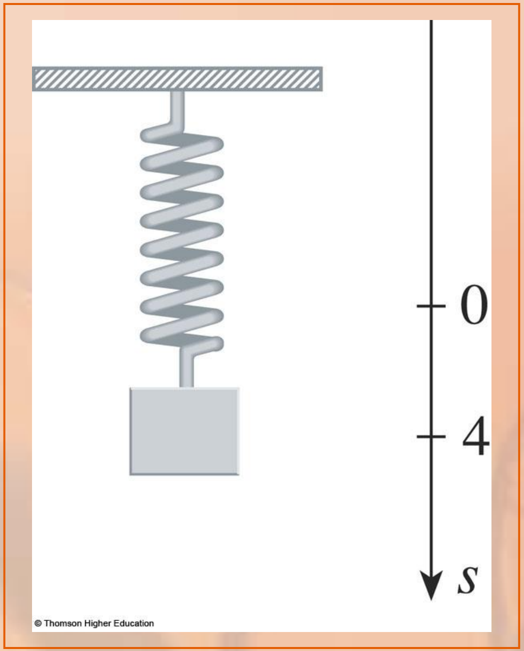
The velocity and acceleration are:

$$v = \frac{ds}{dt} = \frac{d}{dt}(4\cos t) = 4\frac{d}{dt}(\cos t) = -4\sin t$$

$$a = \frac{dv}{dt} = \frac{d}{dt}(-4\sin t) = -4\frac{d}{dt}(\sin t) = -4\cos t$$

The object oscillates from the lowest point (s = 4 cm) to the highest point (s = -4 cm).

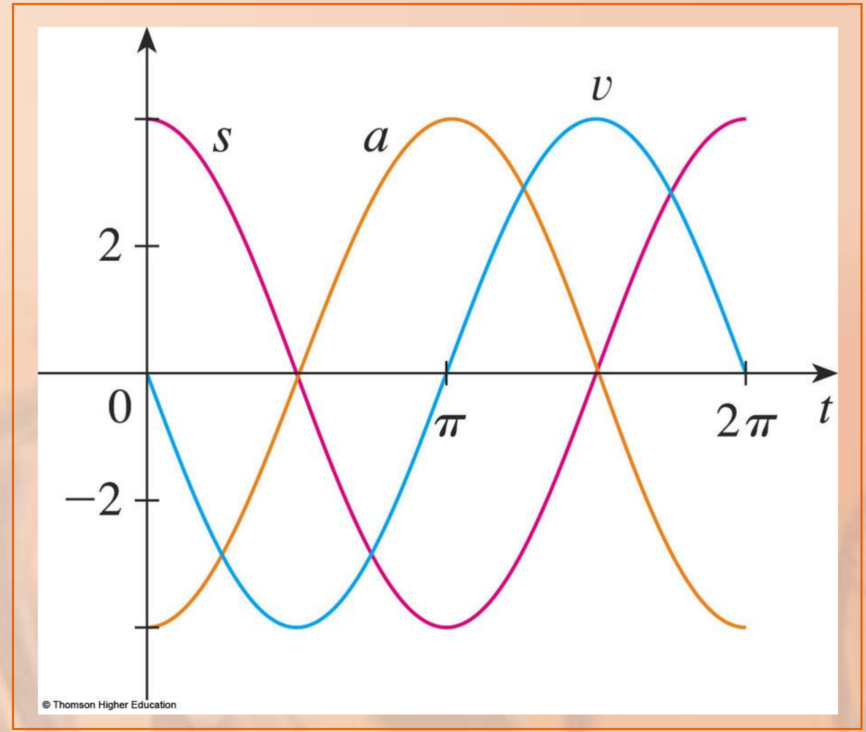
The period of the oscillation is 2π, the period of cos t.



Example 3

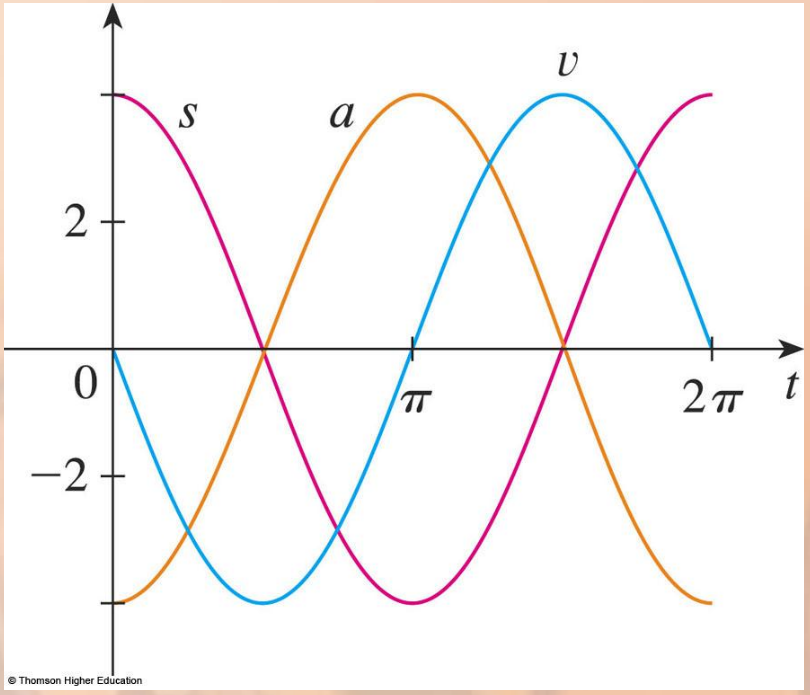
The speed is $|v| = 4|\sin t|$, which is greatest when $|\sin t| = 1$, that is, when $\cos t = 0$.

- So, the object moves fastest as it passes through its equilibrium position ($s = 0$).
- Its speed is 0 when $\sin t = 0$, that is, at the high and low points.



The acceleration $a = -4 \cos t = 0$ when $s = 0$.

It has greatest magnitude at the high and low points.



Find the 27th derivative of cos *x*.

- The first few derivatives of $f(x) = \cos x$ are as follows:

$$f'(x) = -\sin x$$

$$f''(x) = -\cos x$$

$$f'''(x) = \sin x$$

$$f^{(4)}(x) = \cos x$$

$$f^{(5)}(x) = -\sin x$$

Example 4

- We see that the successive derivatives occur in a cycle of length 4 and, in particular, $f^{(n)}(x) = \cos x$ whenever n is a multiple of 4.

- Therefore, $f^{(24)}(x) = \cos x$

- Differentiating three more times, we have:

$$f^{(27)}(x) = \sin x$$

Our main use for the limit in Equation 2 has been to prove the differentiation formula for the sine function.

- However, this limit is also useful in finding certain other trigonometric limits—as the following two examples show.

Example 5

Find $\lim_{x \to 0} \frac{\sin 7x}{4x}$

- In order to apply Equation 2, we first rewrite the function by multiplying and dividing by 7:

$$\frac{\sin 7x}{4x} = \frac{7}{4}\left(\frac{\sin 7x}{7x}\right)$$

If we let $\theta = 7x$, then $\theta \to 0$ as $x \to 0$.

So, by Equation 2, we have:

$$\lim_{x\to 0}\frac{\sin 7x}{4x} = \frac{7}{4}\lim_{x\to 0}\left(\frac{\sin 7x}{7x}\right)$$

$$= \frac{7}{4}\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)$$

$$= \frac{7}{4}\cdot 1 = \frac{7}{4}$$

 Example 6

Calculate $\lim_{x \to 0} x \cot x$.

- We divide the numerator and denominator by *x*:

$$\lim_{x \to 0} x \cot x = \lim_{x \to 0} \frac{x \cos x}{\sin x} = \lim_{x \to 0} \frac{\cos x}{\dfrac{\sin x}{x}}$$

$$= \frac{\lim_{x \to 0} \cos x}{\lim_{x \to 0} \dfrac{\sin x}{x}} = \frac{\cos 0}{1}$$

by the continuity of cosine and Eqn. 2

$$= 1$$