3

DIFFERENTIATION RULES

3.2

The Product and Quotient Rules

In this section, we will learn about:

Formulas that enable us to differentiate new functions formed from old functions by multiplication or division.

THE PRODUCT RULE

By analogy with the Sum and Difference Rules, one might be tempted to guess—as Leibniz did three centuries ago—that the derivative of a product is the product of the derivatives.

- However, we can see that this guess is wrong by looking at a particular example.

THE PRODUCT RULE

Let $f(x) = x$ and $g(x) = x^2$.

- Then, the Power Rule gives $f'(x) = 1$ and $g'(x) = 2x$.
- However, $(fg)(x) = x^3$.
- So, $(fg)'(x) = 3x^2$.
- Thus, $(fg)' \neq f'g'$.

THE PRODUCT RULE

The correct formula was discovered by Leibniz (soon after his false start) and is called the Product Rule.

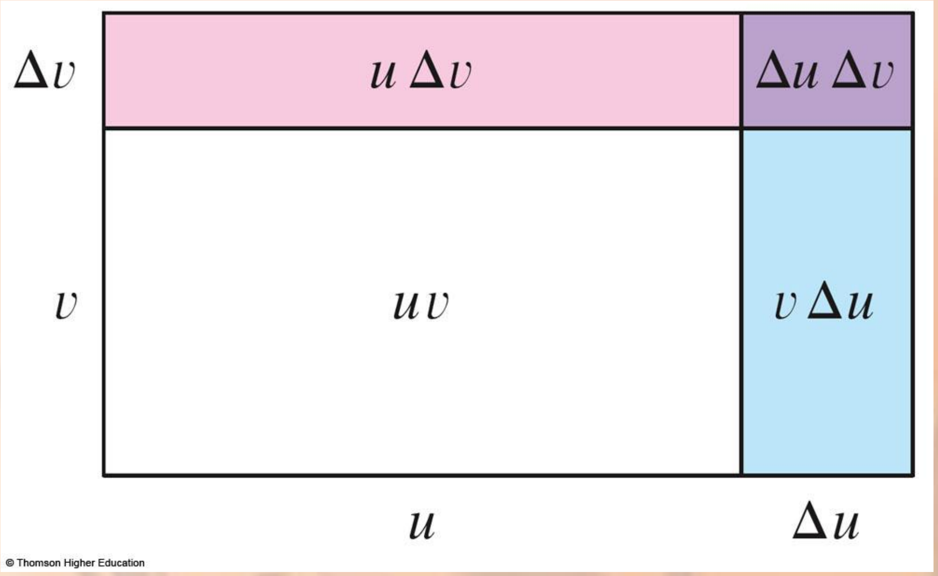
THE PRODUCT RULE

Before stating the Product Rule, let's see how we might discover it.

We start by assuming that $u = f(x)$ and $v = g(x)$ are both positive differentiable functions.

THE PRODUCT RULE

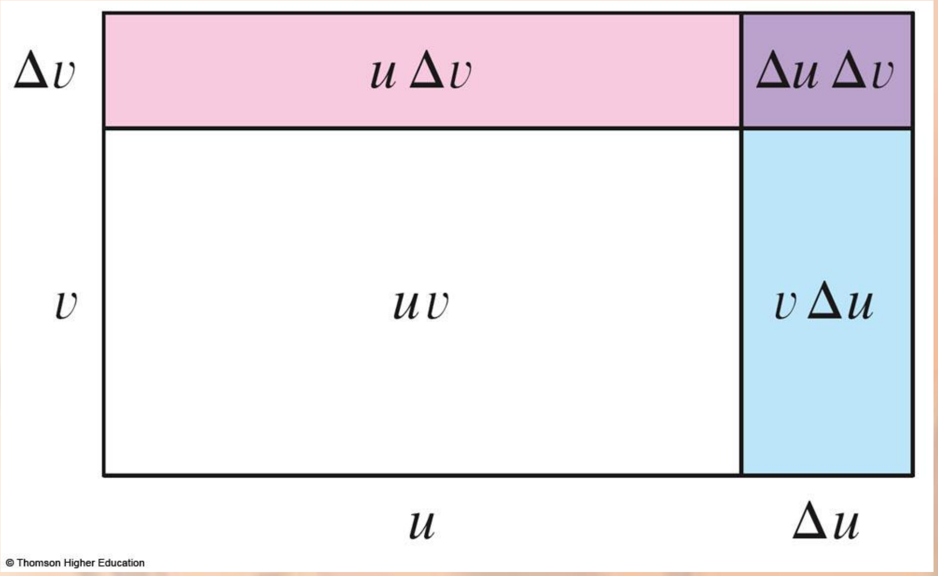
Then, we can interpret the product *uv* as an area of a rectangle.



THE PRODUCT RULE

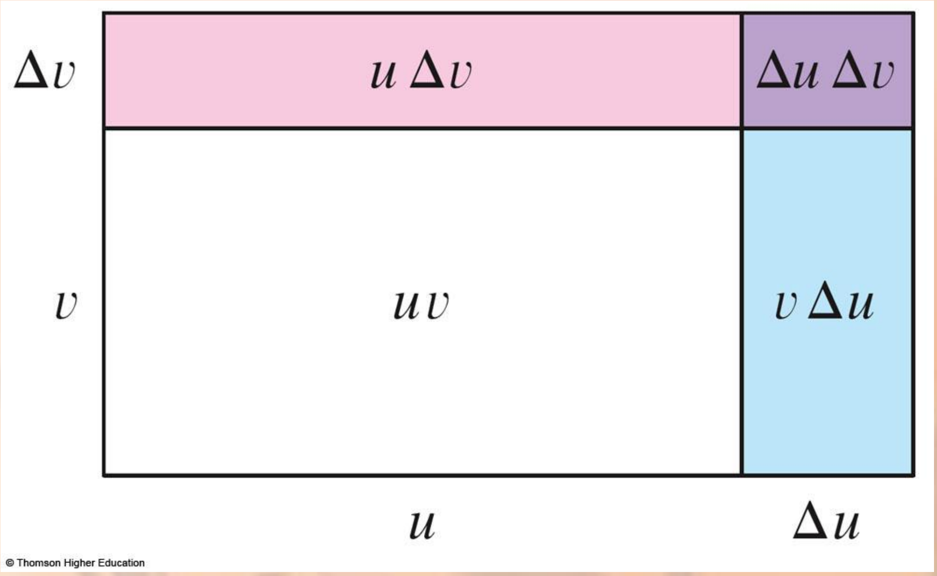
If x changes by an amount Δx, then the corresponding changes in u and v are:

- $\Delta u = f(x + \Delta x) - f(x)$
- $\Delta v = g(x + \Delta x) - g(x)$



THE PRODUCT RULE

The new value of the product, $(u + \Delta u)(v + \Delta v)$, can be interpreted as the area of the large rectangle in the figure—provided that Δu and Δv happen to be positive.



Equation 1

The change in the area of the rectangle is:

$$\begin{aligned}\Delta(uv) &= (u+\Delta u)(v+\Delta v) - uv \\ &= u\Delta v + v\Delta u + \Delta u \Delta v \\ &= \text{the sum of the three shaded areas}\end{aligned}$$

If we divide by Δx, we get:

$$\frac{\Delta(uv)}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}$$

THE PRODUCT RULE

If we let $\Delta x \to 0$, we get the derivative of *uv*:

$$\frac{d}{dx}(uv) = \lim_{\Delta x \to 0} \frac{\Delta(uv)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x} \right)$$

$$= u\lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} + v\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} \Delta u \left(\lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} \right)$$

$$= u\frac{dv}{dx} + v\frac{du}{dx} + 0.\frac{dv}{dx}$$

THE PRODUCT RULE

Equation 2

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

- Notice that $\Delta u \to 0$ as $\Delta x \to 0$ since *f* is differentiable and therefore continuous.

THE PRODUCT RULE

Though we began by assuming (for the geometric interpretation) that all quantities are positive, we see Equation 1 is always true.

- The algebra is valid whether u, v, Δu, and Δv are positive or negative.

- So, we have proved Equation 2—known as the Product Rule—for all differentiable functions u and v.

THE PRODUCT RULE

If *f* and *g* are both differentiable, then:

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)\frac{d}{dx}\left[g(x)\right] + g(x)\frac{d}{dx}\left[f(x)\right]$$

In words, the Product Rule says:

- The derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.

Example 1

a. If $f(x) = xe^x$, find $f'(x)$.

b. Find the nth derivative, $f^{(n)}(x)$

Example 1 a

By the Product Rule, we have:

$$f'(x) = \frac{d}{dx}(xe^x)$$

$$= x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x)$$

$$= xe^x + e^x \cdot 1$$

$$= (x+1)e^x$$

Example 1 b

Using the Product Rule again, we get:

$$f''(x) = \frac{d}{dx}\left[(x+1)e^x\right]$$

$$= (x+1)\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x+1)$$

$$= (x+1)e^x + e^x \cdot 1$$

$$= (x+2)e^x$$

Example 1 b

Further applications of the Product Rule give:

$$f'''(x) = (x+3)e^x$$

$$f^4(x) = (x+4)e^x$$

In fact, each successive differentiation adds another term e^x.

So:

$$f^n(x) = (x + n)e^x$$

Example 2

Differentiate the function

$$f(t) = \sqrt{t}(a + bt)$$

Using the Product Rule, we have:

$$f'(t) = \sqrt{t}\,\frac{d}{dt}(a+bt) + (a+bt)\frac{d}{dt}\sqrt{t}$$

$$= \sqrt{t}\cdot b + (a+bt)\cdot \tfrac{1}{2}t^{-1/2}$$

$$= b\sqrt{t} + \frac{(a+bt)}{2\sqrt{t}} = \frac{(a+3bt)}{2\sqrt{t}}$$

E. g. 2—Solution 2

If we first use the laws of exponents to rewrite *f*(*t*), then we can proceed directly without using the Product Rule.

$$f(t) = a\sqrt{t} + bt\sqrt{t} = at^{1/2} + bt^{3/2}$$

$$f'(t) = \tfrac{1}{2}at^{-1/2} + \tfrac{3}{2}bt^{1/2}$$

- This is equivalent to the answer in Solution 1.

THE PRODUCT RULE

Example 2 shows that it is sometimes easier to simplify a product of functions than to use the Product Rule.

In Example 1, however, the Product Rule is the only possible method.

If $f(x) = \sqrt{x}g(x)$, where $g(4) = 2$ and $g'(4) = 3$, find $f'(4)$.

Example 3

Applying the Product Rule, we get:

$$f'(x) = \frac{d}{dx}\left[\sqrt{x}g(x)\right] = \sqrt{x}\frac{d}{dx}\ g(x)\ + g(x)\frac{d}{dx}\left[\sqrt{x}\right]$$

$$= \sqrt{x}g'(x) + g(x)\cdot\tfrac{1}{2}x^{-1/2} = \sqrt{x}g'(x) + \frac{g(x)}{2\sqrt{x}}$$

So,

$$f'(4) = \sqrt{4}g'(4) + \frac{g(4)}{2\sqrt{4}} = 2\cdot 3 + \frac{2}{2\cdot 2} = 6.5$$

THE QUOTIENT RULE

We find a rule for differentiating the quotient of two differentiable functions $u = f(x)$ and $v = g(x)$ in much the same way that we found the Product Rule.

THE QUOTIENT RULE

If *x*, *u*, and *v* change by amounts *Δx, Δu*, and *Δv,* then the corresponding change in the quotient *u / v* is:

$$\Delta\left(\frac{u}{v}\right) = \frac{u+\Delta u}{v+\Delta v} - \frac{u}{v}$$

$$= \frac{u+\Delta u \;\; v - u \;\; v+\Delta v}{v \;\; v+\Delta v} = \frac{v\Delta u - u\Delta v}{v \;\; v+\Delta v}$$

So,

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \lim_{\Delta x \to 0} \frac{\Delta\ \ u/v}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{v\dfrac{\Delta u}{\Delta x} - u\dfrac{\Delta v}{\Delta x}}{v\ \ v + \Delta v}$$

As $\Delta x \to 0$, $\Delta v \to 0$ also—because $v = g(x)$ is differentiable and therefore continuous.

Thus, using the Limit Laws, we get:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \lim\limits_{\Delta x \to 0} \dfrac{\Delta u}{\Delta x} - u \lim\limits_{\Delta x \to 0} \dfrac{\Delta v}{\Delta x}}{v \lim\limits_{\Delta x \to 0} \left(v + \Delta v\right)} = \frac{v \dfrac{du}{dx} - u \dfrac{dv}{dx}}{v^2}$$

THE QUOTIENT RULE

If *f* and *g* are differentiable, then:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}\,f(x) - f(x)\frac{d}{dx}\,g(x)}{\left[g(x)^{2}\right]}$$

In words, the Quotient Rule says:

- The derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

THE QUOTIENT RULE

The Quotient Rule and the other differentiation formulas enable us to compute the derivative of any rational function—as the next example illustrates.

Example 4

Let

$$y = \frac{x^2 + x - 2}{x^3 + 6}$$

Then,

$$y' = \frac{\left(x^3+6\right)\frac{d}{dx}\left(x^2+x-2\right) - \left(x^2+x-2\right)\frac{d}{dx}\left(x^3+6\right)}{\left(x^3+6\right)^2}$$

$$= \frac{\left(x^3+6\right)\left(2x+1\right) - \left(x^2+x-2\right)\left(3x^2\right)}{\left(x^3+6\right)^2}$$

$$= \frac{\left(2x^4+x^3+12x+6\right) - \left(3x^4+3x^3-6x^2\right)}{\left(x^3+6\right)^2}$$

$$= \frac{-x^4-2x^3+6x^2+12x+6}{\left(x^3+6\right)^2}$$

Example 5

Find an equation of the tangent line to the curve $y = e^x / (1 + x^2)$ at the point $(1, \frac{1}{2}e)$.

THE QUOTIENT RULE

Example 5

According to the Quotient Rule, we have:

$$\frac{dy}{dx}=\frac{\left(1+x^{2}\right)\frac{d}{dx}\left(e^{x}\right)-e^{x}\frac{d}{dx}\left(1+x^{2}\right)}{\left(1+x^{2}\right)^{2}}$$

$$=\frac{\left(1+x^{2}\right)e^{x}-e^{x}\left(2x\right)}{\left(1+x^{2}\right)^{2}}=\frac{e^{x}\left(1-x\right)^{2}}{\left(1+x^{2}\right)^{2}}$$

Example 5

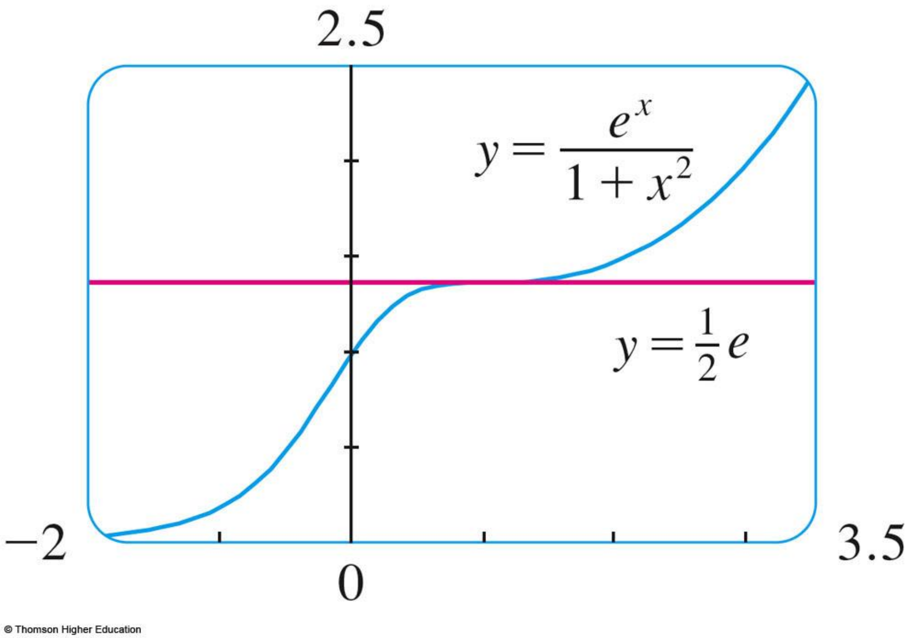
So, the slope of the tangent line at $(1, \frac{1}{2}e)$ is:

$$\left.\frac{dy}{dx}\right|_{x=1} = 0$$

- This means that the tangent line at $(1, \frac{1}{2}e)$ is horizontal and its equation is $y = \frac{1}{2}e$.

Example 5

In the figure, notice that the function is increasing and crosses its tangent line at $(1, \frac{1}{2}e)$.



NOTE

Don’t use the Quotient Rule every time you see a quotient.

- Sometimes, it’s easier to rewrite a quotient first to put it in a form that is simpler for the purpose of differentiation.

NOTE

For instance, though it is possible to differentiate the function

$$F(x) = \frac{3x^2 + 2\sqrt{x}}{x}$$

using the Quotient Rule, it is much easier to perform the division first and write the function as $F(x) = 3x + 2x^{-1/2}$ before differentiating.

DIFFERENTIATION FORMULAS

Here's a summary of the differentiation formulas we have learned so far.

$$\frac{d}{dx}\ c\ = 0 \qquad \frac{d}{dx}\ x^n\ = nx^{n-1} \qquad \frac{d}{dx}\ e^x\ = e^x$$

$$cf\ ' = cf' \qquad f+g\ ' = f' + g' \qquad f-g\ ' = f' - g'$$

$$fg\ ' = fg' + gf' \qquad \left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$