3
DIFFERENTIATION RULES
In this section, we will learn about:
Formulas that enable us to differentiate new functions formed from old functions by multiplication or division.
By analogy with the Sum and Difference Rules, one might be tempted to guess—as Leibniz did three centuries ago—that the derivative of a product is the product of the derivatives.

- However, we can see that this guess is wrong by looking at a particular example.
Let \( f(x) = x \) and \( g(x) = x^2 \).

- Then, the Power Rule gives \( f'(x) = 1 \) and \( g'(x) = 2x \).
- However, \( (fg)(x) = x^3 \).
- So, \( (fg)'(x) = 3x^2 \).
- Thus, \( (fg)' \neq f' g' \).
The correct formula was discovered by Leibniz (soon after his false start) and is called the Product Rule.
Before stating the Product Rule, let’s see how we might discover it.

We start by assuming that \( u = f(x) \) and \( v = g(x) \) are both positive differentiable functions.
Then, we can interpret the product $uv$ as an area of a rectangle.
THE PRODUCT RULE

If \( x \) changes by an amount \( \Delta x \), then the corresponding changes in \( u \) and \( v \) are:

- \( \Delta u = f(x + \Delta x) - f(x) \)
- \( \Delta v = g(x + \Delta x) - g(x) \)
THE PRODUCT RULE

The new value of the product, \((u + \Delta u)(v + \Delta v)\), can be interpreted as the area of the large rectangle in the figure—provided that \(\Delta u\) and \(\Delta v\) happen to be positive.
The change in the area of the rectangle is:

$$\Delta(uv) = (u + \Delta u)(v + \Delta v) - uv$$

$$= u\Delta v + v\Delta u + \Delta u \Delta v$$

= the sum of the three shaded areas
If we divide by $\Delta x$, we get:

$$\frac{\Delta (uv)}{\Delta x} = u \frac{\Delta v}{\Delta x} + v \frac{\Delta u}{\Delta x} + \Delta u \frac{\Delta v}{\Delta x}$$
If we let $\Delta x \to 0$, we get the derivative of $uv$:

$$\frac{d}{dx} (uv) = \lim_{\Delta x \to 0} \frac{\Delta (uv)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left( u \frac{\Delta v}{\Delta x} + v \frac{\Delta u}{\Delta x} + \Delta u \frac{\Delta v}{\Delta x} \right)$$

$$= u \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} + v \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} \Delta u \left( \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} \right)$$

$$= u \frac{dv}{dx} + v \frac{du}{dx} + 0 \frac{dv}{dx}$$
Notice that $\Delta u \to 0$ as $\Delta x \to 0$ since $f$ is differentiable and therefore continuous.
Though we began by assuming (for the geometric interpretation) that all quantities are positive, we see Equation 1 is always true.

- The algebra is valid whether $u$, $v$, $\Delta u$, and $\Delta v$ are positive or negative.

- So, we have proved Equation 2—known as the Product Rule—for all differentiable functions $u$ and $v$. 

THE PRODUCT RULE
THE PRODUCT RULE

If \( f \) and \( g \) are both differentiable, then:

\[
\frac{d}{dx} f(x)g(x) = f(x) \frac{d}{dx} g(x) + g(x) \frac{d}{dx} f(x)
\]

In words, the Product Rule says:

- The derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.
a. If \( f(x) = xe^x \), find \( f'(x) \).

b. Find the \( n \)th derivative, \( f^{(n)}(x) \).
By the Product Rule, we have:

\[
f'(x) = \frac{d}{dx}(xe^x) \\
= x \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(x) \\
= xe^x + e^x \cdot 1 \\
= (x + 1)e^x
\]
Using the Product Rule again, we get:

\[
\begin{align*}
  f''(x) &= \frac{d}{dx} \left[ (x + 1)e^x \right] \\
  &= (x + 1) \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(x + 1) \\
  &= (x + 1)e^x + e^x \cdot 1 \\
  &= (x + 2)e^x
\end{align*}
\]
Further applications of the Product Rule give:

\[ f^{''}(x) = (x + 3)e^x \]

\[ f^{(4)}(x) = (x + 4)e^x \]
In fact, each successive differentiation adds another term $e^x$.

So:

$$f^n(x) = (x + n)e^x$$
Differentiate the function

\[ f(t) = \sqrt{t} (a + bt) \]
Using the Product Rule, we have:

\[ f'(t) = \sqrt{t} \frac{d}{dt} (a + bt) + (a + bt) \frac{d}{dt} \sqrt{t} \]

\[ = \sqrt{t} \cdot b + (a + bt) \cdot \frac{1}{2} t^{-1/2} \]

\[ = b\sqrt{t} + \frac{(a + bt)}{2\sqrt{t}} = \frac{(a + 3bt)}{2\sqrt{t}} \]
If we first use the laws of exponents to rewrite \( f(t) \), then we can proceed directly without using the Product Rule.

\[
    f(t) = a\sqrt{t} + bt\sqrt{t} = at^{1/2} + bt^{3/2}
\]

\[
    f'(t) = \frac{1}{2}at^{-1/2} + \frac{3}{2}bt^{1/2}
\]

- This is equivalent to the answer in Solution 1.
Example 2 shows that it is sometimes easier to simplify a product of functions than to use the Product Rule.

In Example 1, however, the Product Rule is the only possible method.
If \( f(x) = \sqrt{x} g(x) \), where \( g(4) = 2 \) and \( g'(4) = 3 \), find \( f'(4) \).
THE PRODUCT RULE

Applying the Product Rule, we get:

\[ f'(x) = \frac{d}{dx} \left[ \sqrt{x} g(x) \right] = \sqrt{x} \frac{d}{dx} g(x) + g(x) \frac{d}{dx} \left[ \sqrt{x} \right] \]

\[ = \sqrt{x} g'(x) + g(x) \cdot \frac{1}{2} x^{-1/2} = \sqrt{x} g'(x) + \frac{g(x)}{2\sqrt{x}} \]

So,

\[ f'(4) = \sqrt{4} g'(4) + \frac{g(4)}{2\sqrt{4}} = 2 \cdot 3 + \frac{2}{2 \cdot 2} = 6.5 \]
THE QUOTIENT RULE

We find a rule for differentiating the quotient of two differentiable functions $u = f(x)$ and $v = g(x)$ in much the same way that we found the Product Rule.
THE QUOTIENT RULE

If \( x, u, \) and \( v \) change by amounts \( \Delta x, \Delta u, \) and \( \Delta v \), then the corresponding change in the quotient \( u / v \) is:

\[
\Delta \left( \frac{u}{v} \right) = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v}
\]

\[
= \frac{u + \Delta u}{v + \Delta v} - \frac{v - u}{v + \Delta v} = \frac{v \Delta u - u \Delta v}{v(v + \Delta v)}
\]
So,

\[ \frac{d}{dx} \left( \frac{u}{v} \right) = \lim_{\Delta x \to 0} \frac{\Delta \frac{u}{v}}{\Delta x} \]

\[ = \lim_{\Delta x \to 0} \frac{\frac{\Delta u}{v} - u \frac{\Delta v}{\Delta x}}{v + \Delta v} \]
As \( \Delta x \to 0 \), \( \Delta v \to 0 \) also—because \( v = g(x) \) is differentiable and therefore continuous.

Thus, using the Limit Laws, we get:

\[
\frac{d}{dx} \left( \frac{u}{v} \right) = v \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} - u \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}
\]
If $f$ and $g$ are differentiable, then:

$$
\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx} f(x) - f(x) \frac{d}{dx} g(x)}{g(x)^2}
$$

In words, the Quotient Rule says:

- The derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.
The Quotient Rule and the other differentiation formulas enable us to compute the derivative of any rational function—as the next example illustrates.
Let

\[ y = \frac{x^2 + x - 2}{x^3 + 6} \]
Then,

\[
y' = \frac{x^3 + 6}{x^3 + 6} \frac{d}{dx} \frac{x^2 + x - 2}{x^3 + 6} - \frac{x^2 + x - 2}{x^3 + 6} \frac{d}{dx} \frac{x^3 + 6}{x^3 + 6}^2
\]

\[
= \frac{x^3 + 6}{x^3 + 6} \frac{2x + 1}{x^3 + 6} - \frac{x^2 + x - 2}{x^3 + 6} \cdot 3x^2
\]

\[
= \frac{2x^4 + x^3 + 12x + 6}{x^3 + 6} - \frac{3x^4 + 3x^3 - 6x^2}{x^3 + 6}^2
\]

\[
= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{x^3 + 6}^2
\]
Find an equation of the tangent line to the curve \( y = \frac{e^x}{1 + x^2} \) at the point \( (1, \frac{1}{2}e) \).
According to the Quotient Rule, we have:

\[
\frac{dy}{dx} = \frac{1 + x^2}{1 + x^2} \cdot \frac{d}{dx} \left( e^x - e^x \frac{d}{dx} \right) \frac{1 + x^2}{2}
\]

\[
= \frac{1 + x^2}{1 + x^2} \cdot e^x - e^x \frac{2x}{1 + x^2} = \frac{e^x}{1 + x^2} \cdot \frac{1 - x}{1 + x^2}
\]
So, the slope of the tangent line at 
(1, \ ½e) is:

\[
\frac{dy}{dx} \bigg|_{x=1} = 0
\]

- This means that the tangent line at (1, \ ½e) 
is horizontal and its equation is \( y = \frac{1}{2}e \).
In the figure, notice that the function is increasing and crosses its tangent line at \((1, \frac{1}{2}e)\).
Don’t use the Quotient Rule every time you see a quotient.

- Sometimes, it’s easier to rewrite a quotient first to put it in a form that is simpler for the purpose of differentiation.
For instance, though it is possible to differentiate the function

\[ F(x) = \frac{3x^2 + 2\sqrt{x}}{x} \]

using the Quotient Rule, it is much easier to perform the division first and write the function as

\[ F(x) = 3x + 2x^{-1/2} \]

before differentiating.
Here’s a summary of the differentiation formulas we have learned so far.

<table>
<thead>
<tr>
<th>Formula</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{d}{dx} c = 0$</td>
<td>Constant Rule</td>
</tr>
<tr>
<td>$\frac{d}{dx} x^n = nx^{n-1}$</td>
<td>Power Rule</td>
</tr>
<tr>
<td>$\frac{d}{dx} e^x = e^x$</td>
<td>Exponential Rule</td>
</tr>
<tr>
<td>$cf' = cf'$</td>
<td>Constant Multiple Rule</td>
</tr>
<tr>
<td>$f + g' = f' + g'$</td>
<td>Sum Rule</td>
</tr>
<tr>
<td>$f - g' = f' - g'$</td>
<td>Difference Rule</td>
</tr>
<tr>
<td>$fg' = fg' + gf'$</td>
<td>Product Rule</td>
</tr>
<tr>
<td>$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$</td>
<td>Quotient Rule</td>
</tr>
</tbody>
</table>