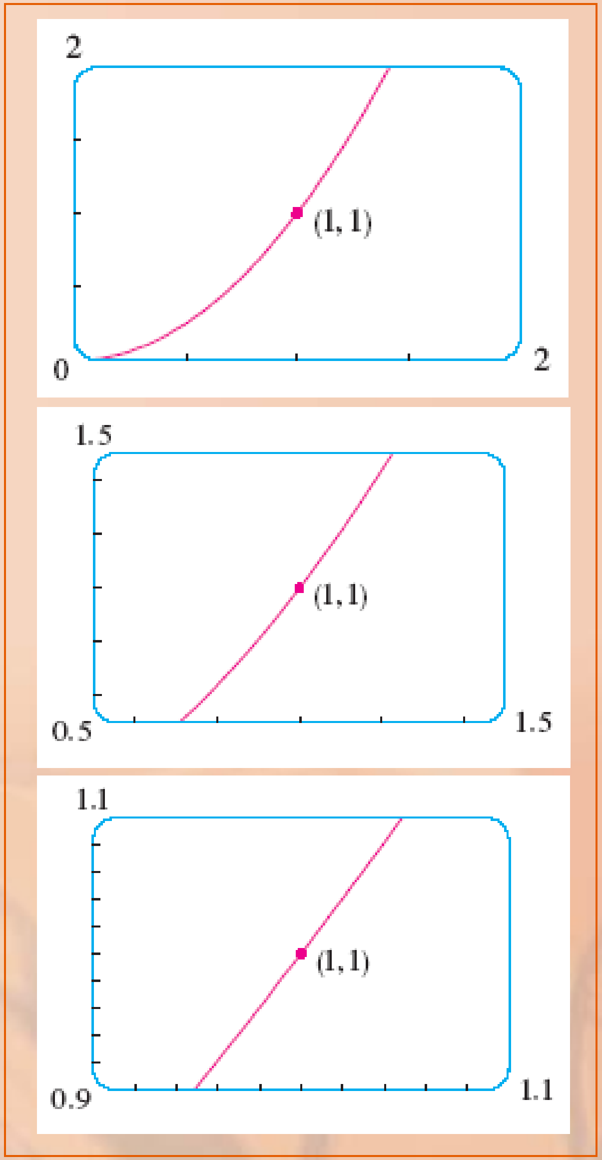
3

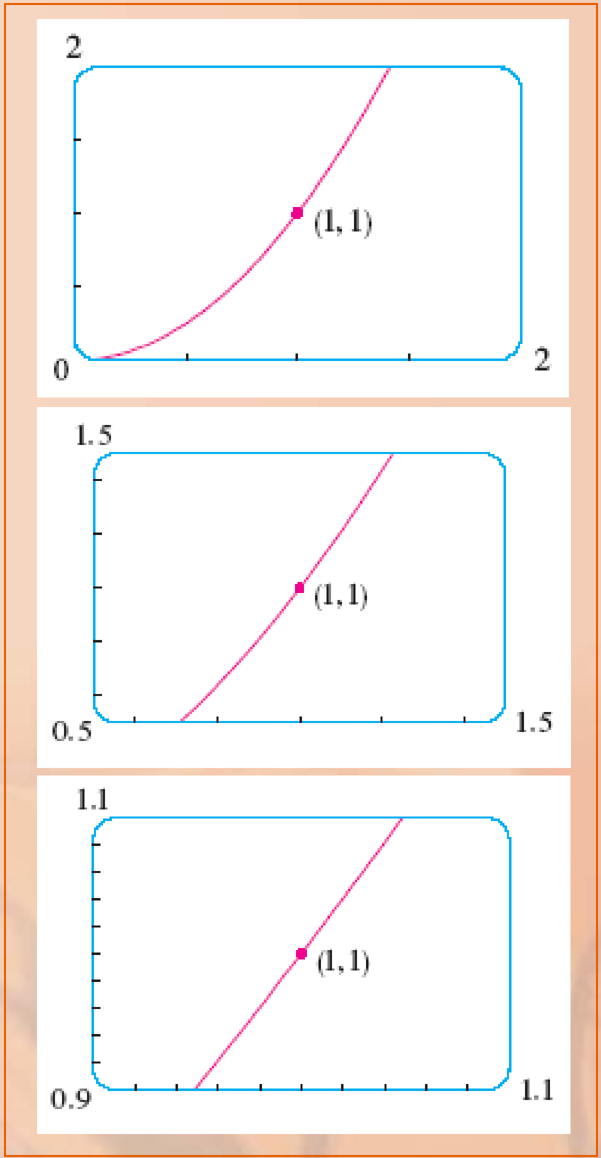
DIFFERENTIATION RULES

We have seen that a curve lies very close to its tangent line near the point of tangency.

In fact, by zooming in toward a point on the graph of a differentiable function, we noticed that the graph looks more and more like its tangent line.



This observation is the basis for a method of finding approximate values of functions.



3.10

Linear Approximations and Differentials

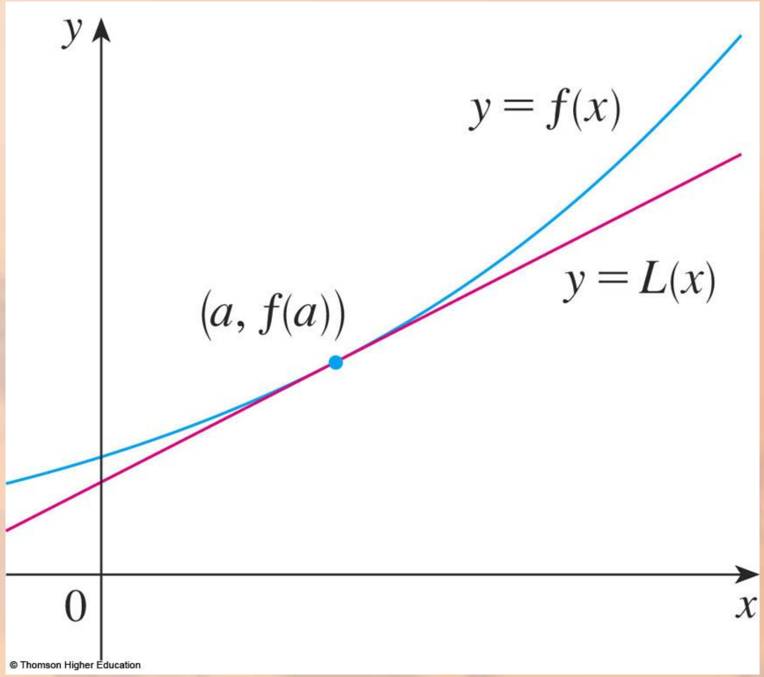
In this section, we will learn about:

Linear approximations and differentials and their applications.

LINEAR APPROXIMATIONS

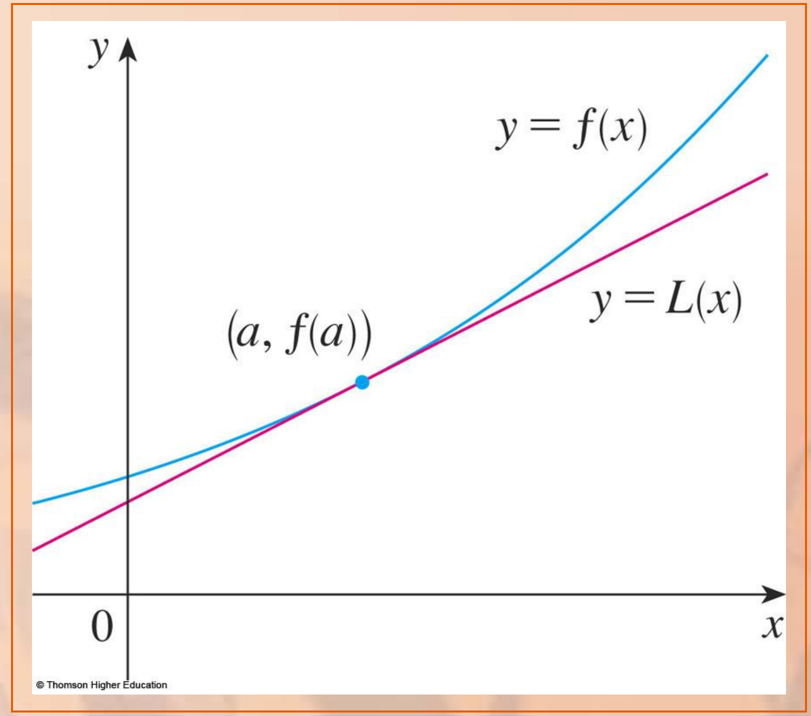
The idea is that it might be easy to calculate a value *f*(*a*) of a function, but difficult (or even impossible) to compute nearby values of *f*.

- So, we settle for the easily computed values of the linear function *L* whose graph is the tangent line of *f* at (*a*, *f*(*a*)).



In other words, we use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a.

- An equation of this tangent line is $y = f(a) + f'(a)(x - a)$



LINEAR APPROXIMATION

Equation 1

The approximation

$$f(x) \approx f(a) + f'(a)(x - a)$$

is called the linear approximation or tangent line approximation of *f* at *a*.

LINEARIZATION

Equation 2

The linear function whose graph is this tangent line, that is,

$$L(x) = f(a) + f'(a)(x - a)$$

is called the linearization of *f* at *a*.

Example 1

Find the linearization of the function $f(x)=\sqrt{x+3}$ at a = 1 and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$

Are these approximations overestimates or underestimates?

Example 1

The derivative of $f(x) = (x + 3)^{1/2}$ is:

$$f'(x) = \tfrac{1}{2}(x+3)^{-1/2} = \frac{1}{2\sqrt{x+3}}$$

So, we have $f(1) = 2$ and $f'(1) = ¼$.

Example 1

Putting these values into Equation 2, we see that the linearization is:

$$\begin{aligned} L(x) &= f(1) + f'(1)(x-1) \\ &= 2 + \frac{1}{4}(x-1) \\ &= \frac{7}{4} + \frac{x}{4} \end{aligned}$$

The corresponding linear approximation is:

$$\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}$$ (when *x* is near 1)

In particular, we have:

$$\sqrt{3.98} \approx \frac{7}{4} + \frac{0.98}{4} = 1.995$$

and

$$\sqrt{4.05} \approx \frac{7}{4} + \frac{1.05}{4} = 2.0125$$

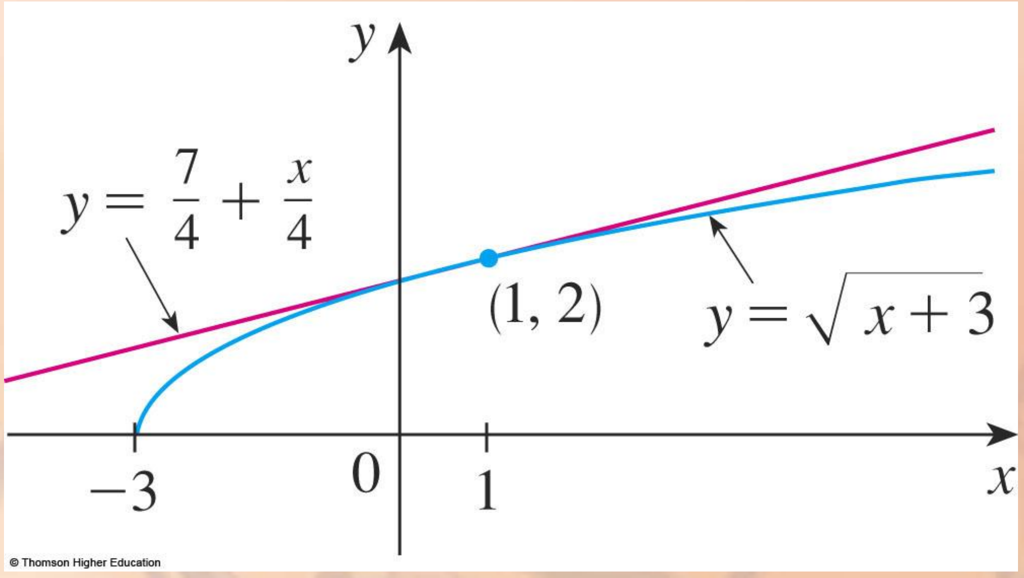
LINEAR APPROXIMATIONS

Example 1

The linear approximation is illustrated here.

We see that:

- The tangent line approximation is a good approximation to the given function when *x* is near 1.
- Our approximations are overestimates, because the tangent line lies above the curve.



Example 1

Of course, a calculator could give us approximations for $\sqrt{3.98}$ and $\sqrt{4.05}$

The linear approximation, though, gives an approximation over an entire interval.

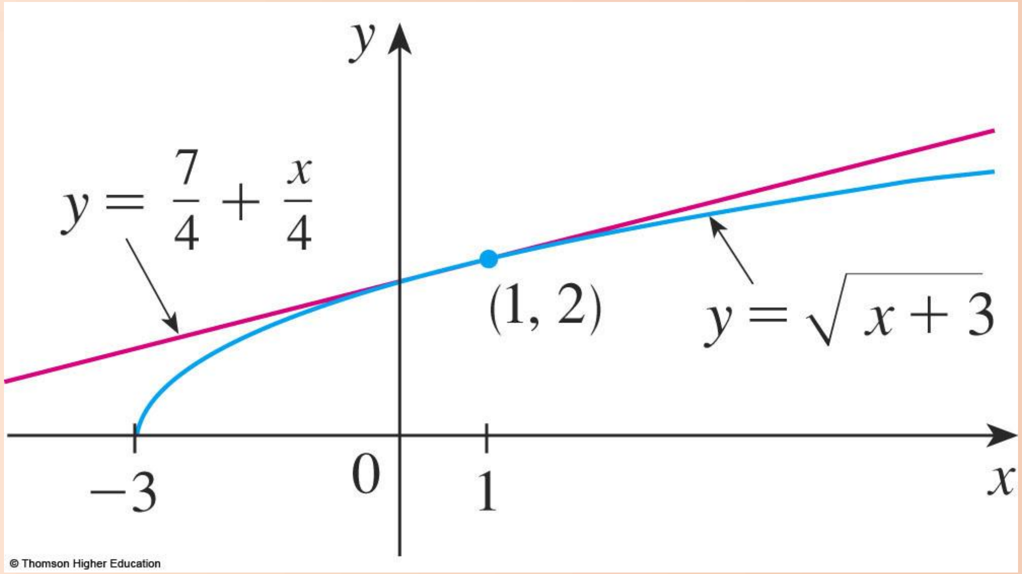
In the table, we compare the estimates from the linear approximation in Example 1 with the true values.

	x	From $L(x)$	Actual value
$\sqrt{3.9}$	0.9	1.975	1.97484176 . . .
$\sqrt{3.98}$	0.98	1.995	1.99499373 . . .
$\sqrt{4}$	1	2	2.00000000 . . .
$\sqrt{4.05}$	1.05	2.0125	2.01246117 . . .
$\sqrt{4.1}$	1.1	2.025	2.02484567 . . .
$\sqrt{5}$	2	2.25	2.23606797 . . .
$\sqrt{6}$	3	2.5	2.44948974 . . .

LINEAR APPROXIMATIONS

Look at the table and the figure.

- The tangent line approximation gives good estimates if *x* is close to 1.
- However, the accuracy decreases when *x* is farther away from 1.



	x	From $L(x)$	Actual value
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$\sqrt{5}$	2	2.25	2.23606797 . . .
$\sqrt{6}$	3	2.5	2.44948974 . . .

How good is the approximation that we obtained in Example 1?

- The next example shows that, by using a graphing calculator or computer, we can determine an interval throughout which a linear approximation provides a specified accuracy.

Example 2

For what values of *x* is the linear approximation $\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}$ accurate to within 0.5?

What about accuracy to within 0.1?

Example 2

Accuracy to within 0.5 means that the functions should differ by less than 0.5:

$$\left| \sqrt{x+3} - \left(\frac{7}{4} + \frac{x}{4} \right) \right| < 0.5$$

Example 2

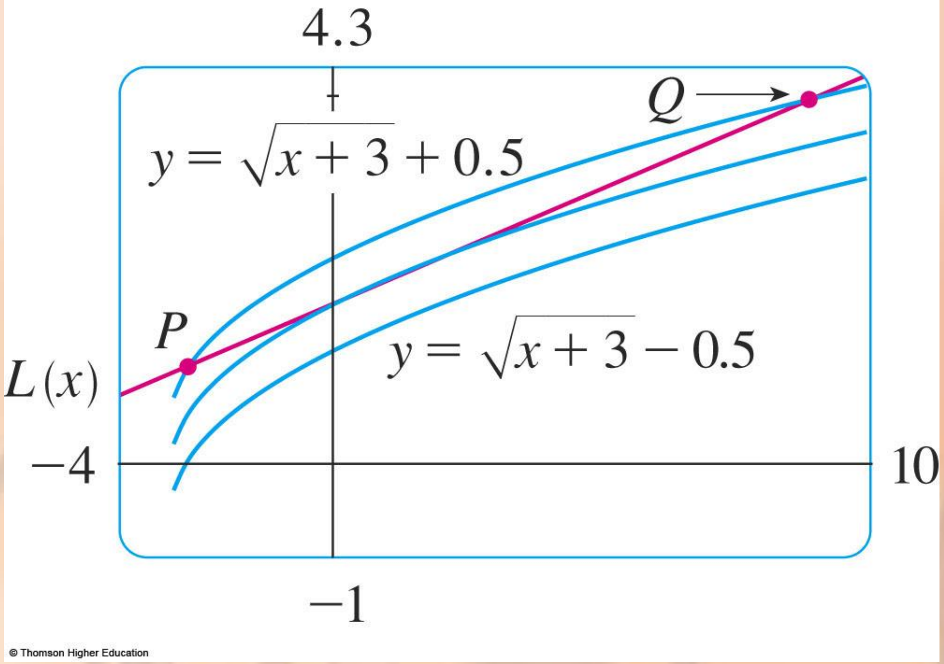
Equivalently, we could write:

$$\sqrt{x+3}-0.5<\frac{7}{4}+\frac{x}{4}<\sqrt{x+3}+0.5$$

- This says that the linear approximation should lie between the curves obtained by shifting the curve $y=\sqrt{x+3}$ upward and downward by an amount 0.5.

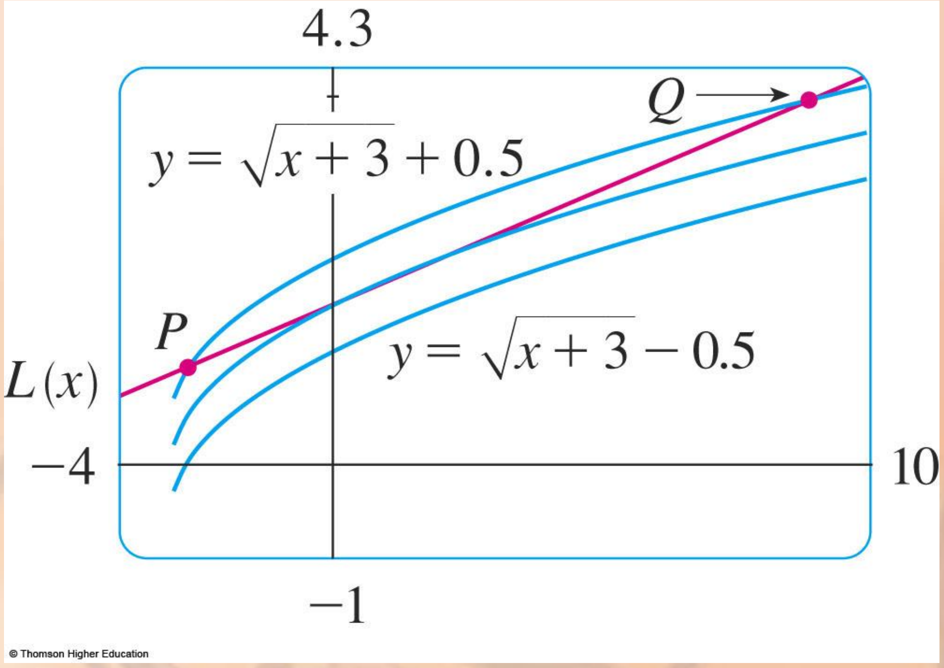
Example 2

The figure shows the tangent line $y = (7 + x) / 4$ intersecting the upper curve $y = \sqrt{x+3} + 0.5$ at *P* and *Q*.



Example 2

Zooming in and using the cursor, we estimate that the *x*-coordinate of *P* is about -2.66 and the *x*-coordinate of *Q* is about 8.66

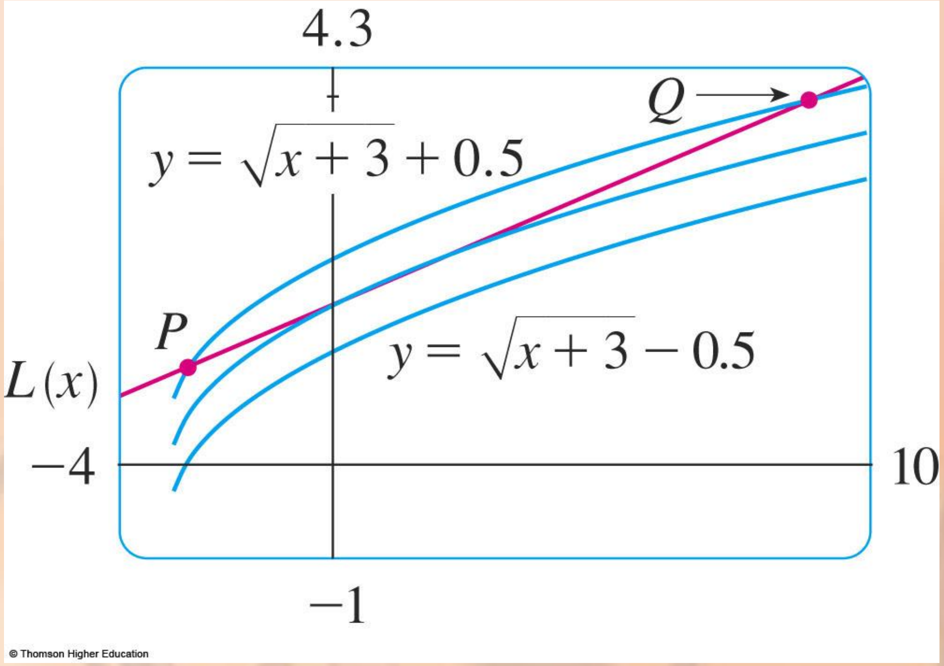


Example 2

Thus, we see from the graph that the approximation $\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}$ is accurate to within 0.5 when

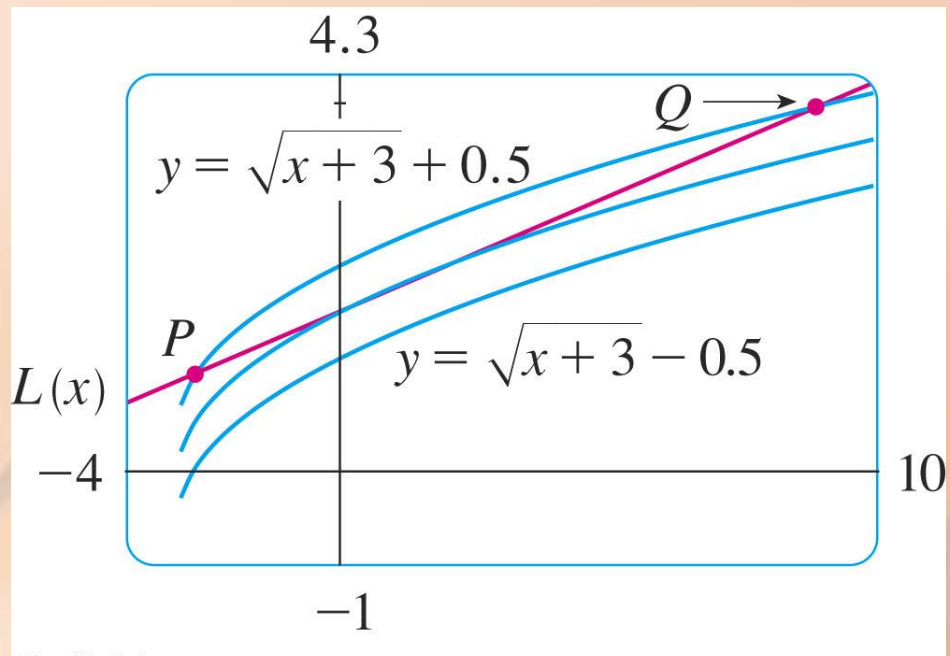
-2.6 < *x* < 8.6

- We have rounded to be safe.



Example 2

Similarly, from this figure, we see that the approximation is accurate to within 0.1 when -1.1 < *x* < 3.9



Linear approximations are often used in physics.

- In analyzing the consequences of an equation, a physicist sometimes needs to simplify a function by replacing it with its linear approximation.

APPLICATIONS TO PHYSICS

For instance, in deriving a formula for the period of a pendulum, physics textbooks obtain the expression $a_T = -g \sin \theta$ for tangential acceleration and then replace $\sin \theta$ by θ with the remark that $\sin \theta$ is very close to θ if θ is not too large.

APPLICATIONS TO PHYSICS

You can verify that the linearization of the function $f(x) = \sin x$ at $a = 0$ is $L(x) = x$.

So, the linear approximation at 0 is:

$$\sin x \approx x$$

So, in effect, the derivation of the formula for the period of a pendulum uses the tangent line approximation for the sine function.

APPLICATIONS TO PHYSICS

Another example occurs in the theory of optics, where light rays that arrive at shallow angles relative to the optical axis are called paraxial rays.

In paraxial (or Gaussian) optics, both $\sin \theta$ and $\cos \theta$ are replaced by their linearizations.

- In other words, the linear approximations

  $$\sin \theta \approx \theta \qquad \text{and} \qquad \cos \theta \approx 1$$

  are used because θ is close to 0.

APPLICATIONS TO PHYSICS

The results of calculations made with these approximations became the basic theoretical tool used to design lenses.

APPLICATIONS TO PHYSICS

In Section 11.11, we will present several other applications of the idea of linear approximations to physics.

DIFFERENTIALS

The ideas behind linear approximations are sometimes formulated in the terminology and notation of differentials.

DIFFERENTIALS

If $y = f(x)$, where f is a differentiable function, then the differential dx is an independent variable.

- That is, dx can be given the value of any real number.

Equation 3

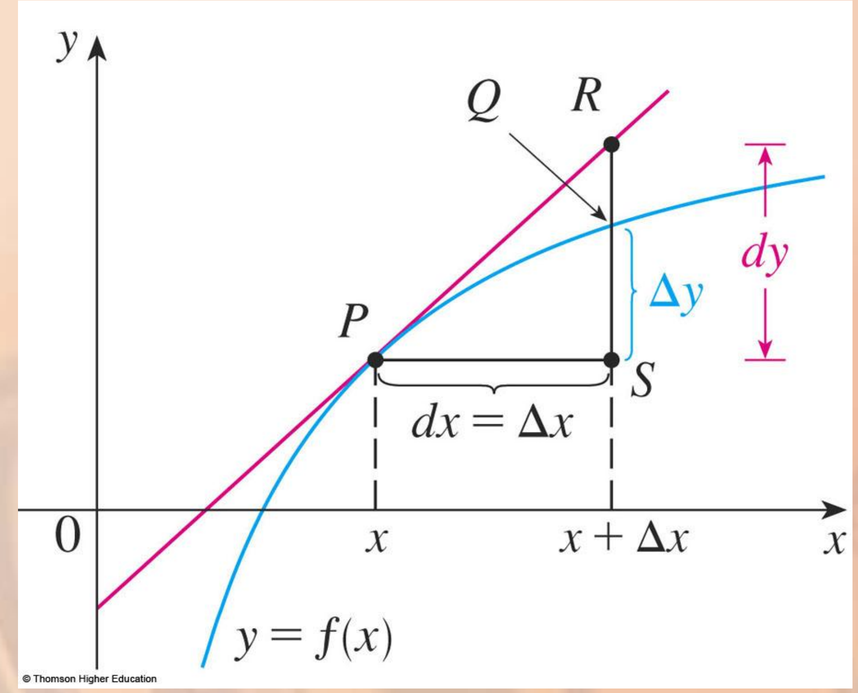
The differential *dy* is then defined in terms of *dx* by the equation

$$dy = f'(x)dx$$

- So, *dy* is a dependent variable—it depends on the values of *x* and *dx*.
- If *dx* is given a specific value and *x* is taken to be some specific number in the domain of *f*, then the numerical value of *dy* is determined.

The geometric meaning of differentials is shown here.

- Let $P(x, f(x))$ and $Q(x + \Delta x, f(x + \Delta x))$ be points on the graph of *f.*
- Let $dx = \Delta x$.

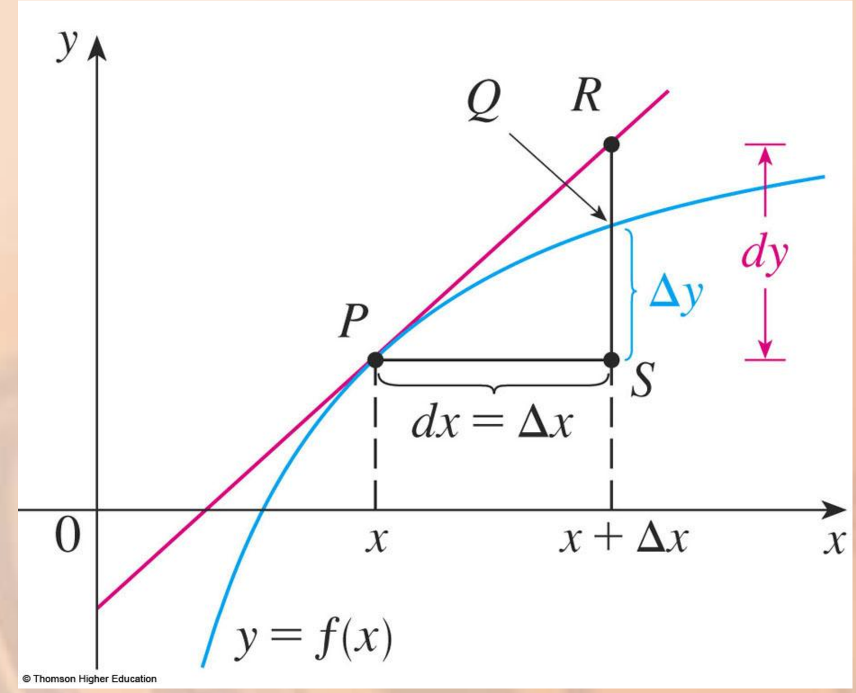


DIFFERENTIALS

The corresponding change in *y* is:

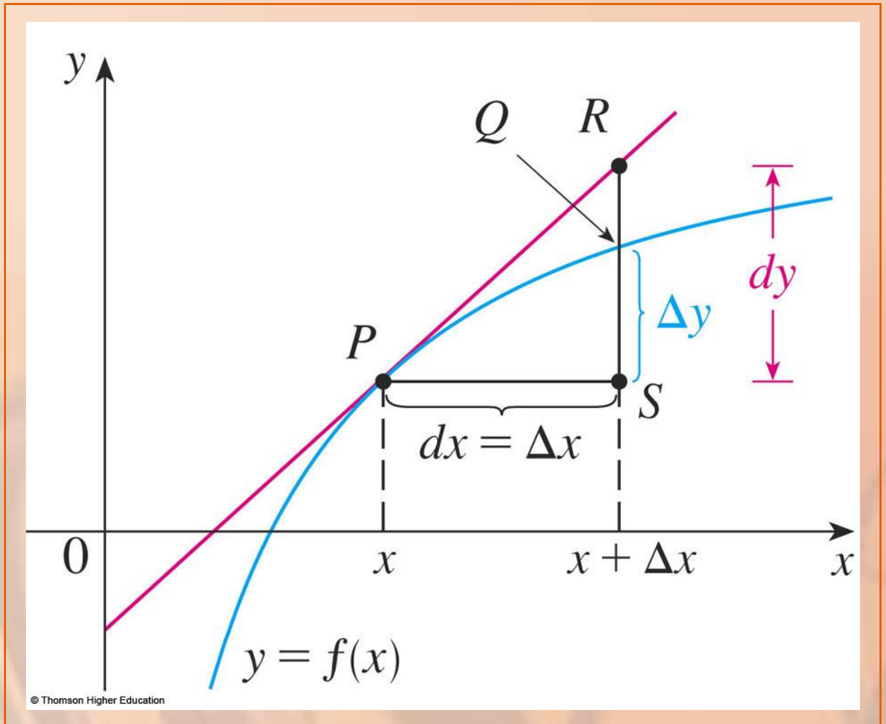
$$\Delta y = f(x + \Delta x) - f(x)$$

- The slope of the tangent line *PR* is the derivative *f'*(x).
- Thus, the directed distance from *S* to *R* is $f'(x)dx = dy$.



Therefore,

- *dy* represents the amount that the tangent line rises or falls (the change in the linearization).
- Δy represents the amount that the curve $y = f(x)$ rises or falls when changes by an amount *dx.*



Example 3

Compare the values of Δy and dy

if $y = f(x) = x^3 + x^2 - 2x + 1$

and x changes from:

a. 2 to 2.05

b. 2 to 2.01

Example 3 a

We have:

$$f(2) = 2^3 + 2^2 - 2(2) + 1 = 9$$

$$f(2.05) = (2.05)^3 + (2.05)^2 - 2(2.05) + 1$$

$$= 9.717625$$

$$\Delta y = f(2.05) - f(2) = 0.717625$$

In general,

$$dy = f'(x)dx = (3x^2 + 2x - 2)\, dx$$

When $x = 2$ and $dx = \Delta x$,

this becomes:

$$dy = [3(2)^2 + 2(2) - 2]0.05$$

$$= 0.7$$

Example 3 b

We have:

$$f(2.01) = (2.01)^3 + (2.01)^2 - 2(2.01) + 1$$

$$= 9.140701$$

$$\Delta y = f(2.01) - f(2) = 0.140701$$

When $dx = \Delta x = 0.01$,

$$dy = [3(2)^2 + 2(2) - 2]0.01 = 0.14$$

Notice that:

- The approximation $\Delta y \approx dy$ becomes better as Δx becomes smaller in the example.
- dy was easier to compute than Δy.

For more complicated functions, it may be impossible to compute Δy exactly.

- In such cases, the approximation by differentials is especially useful.

In the notation of differentials, the linear approximation can be written as:

$$f(a + dx) \approx f(a) + dy$$

For instance, for the function $f(x) = \sqrt{x+3}$ in Example 1, we have:

$$dy = f'(x)dx$$

$$= \frac{dx}{2\sqrt{x+3}}$$

If *a* = 1 and *dx* = Δx = 0.05, then

$$dy = \frac{0.05}{2\sqrt{1+3}} = 0.0125$$

and $\sqrt{4.05} = f(1.05) \approx f(1) + dy = 2.0125$

- This is just as we found in Example 1.

Our final example illustrates the use of differentials in estimating the errors that occur because of approximate measurements.

Example 4

The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm.

What is the maximum error in using this value of the radius to compute the volume of the sphere?

Example 4

If the radius of the sphere is *r*, then its volume is $V = 4/3\pi r^3$.

- If the error in the measured value of *r* is denoted by $dr = \Delta r$, then the corresponding error in the calculated value of *V* is ΔV.

Example 4

This can be approximated by the differential

$$dV = 4\pi\, r^2\, dr$$

When r = 21 and dr = 0.05, this becomes:

$$dV = 4\pi(21)^2\, 0.05 \approx 277$$

- The maximum error in the calculated volume is about 277 cm^3.

Note

Although the possible error in the example may appear to be rather large, a better picture of the error is given by the relative error.

RELATIVE ERROR

Note

Relative error is computed by dividing the error by the total volume:

$$\frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{4\pi r^2 dr}{\frac{4}{3}\pi r^3} = 3\frac{dr}{r}$$

- Thus, the relative error in the volume is about three times the relative error in the radius.

RELATIVE ERROR

Note

In the example, the relative error in the radius is approximately $dr/r = 0.05/21 \approx 0.0024$ and it produces a relative error of about 0.007 in the volume.

- The errors could also be expressed as percentage errors of 0.24% in the radius and 0.7% in the volume.