The intuitive definition of a limit given in Section 2.2 is inadequate for some purposes.

- This is because such phrases as ‘$x$ is close to 2’ and ‘$f(x)$ gets closer and closer to $L$’ are vague.
- In order to be able to prove conclusively that

$$\lim_{x \to 0} \left( x^3 + \frac{\cos 5x}{10,000} \right) = 0.0001 \quad \text{or} \quad \lim_{x \to 0} \frac{\sin x}{x} = 1$$

we must make the definition of a limit precise.
2.4
The Precise Definition of a Limit

In this section, we will:
Define a limit precisely.
To motivate the precise definition of a limit, consider the function

\[ f(x) = \begin{cases} 
2x - 1 & \text{if } x \neq 3 \\
6 & \text{if } x = 3 
\end{cases} \]

- Intuitively, it is clear that when \( x \) is close to 3 but \( x \neq 3 \), then \( f(x) \) is close to 5—and so \( \lim_{x \to 3} f(x) = 5 \).
To obtain more detailed information about how $f(x)$ varies when $x$ is close to 3, we ask the following question.

- How close to 3 does $x$ have to be so that $f(x)$ differs from 5 by less than 0.1?
The distance from $x$ to 3 is $|x - 3|$ and the distance from $f(x)$ to 5 is $|f(x) - 5|$. So, our problem is to find a number $\delta$ such that

$$|f(x) - 5| < 0.1 \text{ if } |x - 3| < \delta \text{ but } x \neq 3$$
If $|x - 3| > 0$, then $x \neq 3$.
So, an equivalent formulation of our problem is to find a number $\delta$ such that

$|f(x) - 5| < 0.1$ if $0 < |x - 3| < \delta$
Notice that, if \(0 < |x - 3| < (0.1) / 2 = 0.05\), then

\[
|f(x) - 5| = |(2x - 1) - 5| = |2x - 6| = 2|x - 3| < 0.1
\]

That is,

\[
|f(x) - 5| < 0.1 \text{ if } 0 < |x - 3| < 0.05
\]

- Thus, an answer to the problem is given by \(\delta = 0.05\).
- That is, if \(x\) is within a distance of 0.05 from 3, then \(f(x)\) will be within a distance of 0.1 from 5.
If we change the number 0.1 in our problem to the smaller number 0.01, then, by using the same method, we find that $f(x)$ will differ from 5 by less than 0.01—provided that $x$ differs from 3 by less than $(0.01)/2 = 0.005$. 

$$|f(x) - 5| < 0.01 \text{  if  } 0 < |x - 3| < 0.005$$
Similarly,

\[ |f(x) - 5| < 0.001 \text{ if } 0 < |x - 3| < 0.0005 \]

- The numbers 0.1, 0.01, and 0.001 that we have considered are ‘error tolerances’ that we might allow.
For 5 to be the precise limit of $f(x)$ as $x$ approaches 3, we must not only be able to bring the difference between $f(x)$ and 5 below each of these three numbers, we must be able to bring it below any positive number.

- By the same reasoning, we can.
If we write $\varepsilon$ for an arbitrary positive number, then we find as before that:

- This is a precise way of saying that $f(x)$ is close to 5 when $x$ is close to 3.
- This is because (1) states that we can make the values of $f(x)$ within an arbitrary distance $\varepsilon$ from 5 by taking the values of $x$ within a distance $\frac{\varepsilon}{2}$ from 3 (but $x \neq 3$).
Note that Definition 1 can be rewritten as follows.

If \(3 - \delta < x < 3 + \delta \) \((x \neq 3)\)
then \(5 - \varepsilon < f(x) < 5 + \varepsilon\)

This is illustrated in the figure.
By taking the values of $x \neq 3$ to lie in the interval $(3 - \delta, 3 + \delta)$, we can make the values of $f(x)$ lie in the interval $(5 - \varepsilon, 5 + \varepsilon)$.
Using (1) as a model, we give a precise definition of a limit.
Let $f$ be a function defined on some open interval that contains the number $a$, except possibly at $a$ itself.

Then, we say that the limit of $f(x)$ as $x$ approaches $a$ is $L$, and we write $\lim_{x \to a} f(x) = L$ if, for every number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \text{then} \quad |f(x) - L| < \varepsilon$$
Since $|x - a|$ is the distance from $x$ to $a$ and $|f(x) - L|$ is the distance from $f(x)$ to $L$, and since $\varepsilon$ can be arbitrarily small, the definition can be expressed in words as follows.

- $\lim_{x \to a} f(x) = L$ means that the distance between $f(x)$ and $L$ can be made arbitrarily small by taking the distance from $x$ to $a$ sufficiently small (but not 0).
- Alternatively, $\lim_{x \to a} f(x) = L$ means that the values of $f(x)$ can be made as close as we please to $L$ by taking $x$ close enough to $a$ (but not equal to $a$).
We can also reformulate Definition 2 in terms of intervals by observing that the inequality \( |x - a| < \delta \) is equivalent to \( -\delta < x - a < \delta \), which in turn can be written as \( a - \delta < x < a + \delta \).
Also, $0 < |x - a|$ is true if and only if $x - a \neq 0$, that is, $x \neq a$.

Similarly, the inequality $|f(x) - L| < \varepsilon$ is equivalent to the pair of inequalities $L - \varepsilon < f(x) < L + \varepsilon$. 
Therefore, in terms of intervals, Definition 2 can be stated as follows.

\[ \lim_{x \to a} f(x) = L \] means that, for every \( \varepsilon > 0 \) (no matter how small \( \varepsilon \) is), we can find \( \delta > 0 \) such that, if \( x \) lies in the open interval \((a - \delta, a + \delta)\) and \( x \neq a \), then \( f(x) \) lies in the open interval \((L - \varepsilon, L + \varepsilon)\).
We interpret this statement geometrically by representing a function by an arrow diagram as in the figure, where $f$ maps a subset of $\mathbb{R}$ onto another subset of $\mathbb{R}$. 
The definition of limit states that, if any small interval \((L - \varepsilon, L + \varepsilon)\) is given around \(L\), then we can find an interval \((a - \delta, a + \delta)\) around \(a\) such that \(f\) maps all the points in \((a - \delta, a + \delta)\) (except possibly \(a\)) into the interval \((L - \varepsilon, L + \varepsilon)\).
Another geometric interpretation of limits can be given in terms of the graph of a function.
If \( \varepsilon > 0 \) is given, then we draw the horizontal lines \( y = L + \varepsilon \) and \( y = L - \varepsilon \) and the graph of \( f \).
If \( \lim_{x \to a} f(x) = L \), then we can find a number \( \delta > 0 \) such that, if we restrict \( x \) to lie in the interval \((a - \delta, a + \delta)\) and take \( x \neq a \), then the curve \( y = f(x) \) lies between the lines \( y = L - \varepsilon \) and \( y = L + \varepsilon \).
You can see that, if such a $\delta$ has been found, then any smaller $\delta$ will also work.
It is important to realize that the process illustrated in the figures must work for every positive number $\varepsilon$, no matter how small it is chosen.
The third figure shows that, if a smaller $\varepsilon$ is chosen, then a smaller $\delta$ may be required.
Use a graph to find a number $\delta$ such that

$$\text{if } |x - 1| < \delta \text{ then } \left| \left( x^3 - 5x + 6 \right) - 2 \right| < 0.2$$

- In other words, find a number $\delta$ that corresponds to $\varepsilon = 0.2$ in the definition of a limit for the function $f(x) = x^3 - 5x + 6$ with $a = 1$ and $L = 2$. 
A graph of $f$ is shown. We are interested in the region near the point $(1, 2)$. 
Notice that we can rewrite the inequality
\[ \left| (x^3 - 5x + 6) - 2 \right| < 0.2 \]
as
\[ 1.8 < x^3 - 5x + 6 < 2.2 \]

- So, we need to determine the values of \( x \) for which the curve \( y = x^3 - 5x + 6 \) lies between the horizontal lines \( y = 1.8 \) and \( y = 2.2 \).
Therefore, we graph the curves $y = x^3 - 5x + 6$, $y = 1.8$, and $y = 2.2$ near the point $(1, 2)$. Then, we use the cursor to estimate that the $x$-coordinate of the point of intersection of the line $y = 2.2$ and the curve $y = x^3 - 5x + 6$ is about 0.911.
Similarly, \( y = x^3 - 5x + 6 \) intersects the line \( y = 1.8 \) when \( x \approx 1.124 \)

So, rounding to be safe, we can say that if \( 0.92 < x < 1.12 \) then \( 1.8 < x^3 - 5x + 6 < 2.2 \)
This interval \((0.92, 1.12)\) is not symmetric about \(x = 1\).

- The distance from \(x = 1\) to the left endpoint is \(1 - 0.92 = 0.08\) and the distance to the right endpoint is \(0.12\).
We can choose $\delta$ to be the smaller of these numbers—that is, $\delta = 0.08$

- Then, we can rewrite our inequalities in terms of distances: if $|x - 1| < 0.08$ then $\left| (x^3 - 5x + 6) - 2 \right| < 0.2$
This just says that, by keeping $x$ within 0.08 of 1, we are able to keep $f(x)$ within 0.2 of 2.

- Though we chose $\delta = 0.08$, any smaller, positive value of $\delta$ would also have worked.
The graphical procedure in the example gives an illustration of the definition for $\varepsilon = 0.2$

However, it does not prove that the limit is equal to 2.

- A proof has to provide a $\delta$ for every $\varepsilon$. 
In proving limit statements, it may be helpful to think of the definition of limit as a challenge.

- First, it challenges you with a number $\varepsilon$.
- Then, you must be able to produce a suitable $\delta$.
- You have to be able to do this for every $\varepsilon > 0$, not just a particular $\varepsilon$. 
Imagine a contest between two people, A and B.

Imagine yourself to be B.

- A stipulates that the fixed number $L$ should be approximated by the values of $f(x)$ to within a degree of accuracy $\varepsilon$ (say 0.01).

- Then, B responds by finding a number $\delta$ such that, if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$. 

**PRECISE DEFINITION OF LIMIT**
Then, A may become more exacting and challenge B with a smaller value of $\varepsilon$ (say, 0.0001).

Again, B has to respond by finding a corresponding $\delta$.

Usually, the smaller the value of $\varepsilon$, the smaller the corresponding value of $\delta$ must be.

If B always wins, no matter how small A makes $\varepsilon$, then $\lim_{x \to a} f(x) = L$. 

**PRECISE DEFINITION OF LIMIT**
Prove that:

\[ \lim_{x \to 3} (4x - 5) = 7 \]
The first step is the preliminary analysis—guessing a value for $\delta$.

- Let $\varepsilon$ be a given positive number.
- We want to find a number $\delta$ such that
  
  $0 < |x - 3| < \delta$ then $|(4x - 5) - 7| < \varepsilon$

- However, $|(4x - 5) - 7| = |4x - 12| = |4(x - 3)| = 4|x - 3|$
Therefore, we want

\[
\text{if } 0 < |x - 3| < \delta \text{ then } 4|x - 3| < \varepsilon
\]

That is,

\[
\text{if } 0 < |x - 3| < \delta \text{ then } |x - 3| < \frac{\varepsilon}{4}
\]

This suggests that we should choose \( \delta = \frac{\varepsilon}{4} \).
The second step is the proof—showing that this $\delta$ works.

- Given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{4}$.
- If $0 < |x - 3| < \delta$, then
  \[
  \left| (4x - 5) - 7 \right| = |4x - 12| = 4|x - 3| < 4\delta = 4\left(\frac{\varepsilon}{4}\right) = \varepsilon
  \]
- Thus, if $0 < |x - 3| < \delta$ then $\left| (4x - 5) - 7 \right| < \varepsilon$
- Therefore, by the definition of a limit, $\lim_{x \to 3} (4x - 5) = 7$
The example is illustrated by the figure.
Note that, in the solution of the example, there were two stages—guessing and proving.

- We made a preliminary analysis that enabled us to guess a value for $\delta$.
- In the second stage, though, we had to go back and prove in a careful, logical fashion that we had made a correct guess.
This procedure is typical of much of mathematics.

- Sometimes, it is necessary to first make an intelligent guess about the answer to a problem and then prove that the guess is correct.
The intuitive definitions of one-sided limits given in Section 2.2 can be precisely reformulated as follows.
Left-hand limit is defined as follows.

\[
\lim_{x \to a^-} f(x) = L
\]

if, for every number \( \varepsilon > 0 \), there is a number \( \delta > 0 \) such that

if \( a - \delta < x < a \) then \( |f(x) - L| < \varepsilon \)

- Notice that Definition 3 is the same as Definition 2 except that \( x \) is restricted to lie in the left half \((a - \delta, a)\) of the interval \((a - \delta, a + \delta)\).
Right-hand limit is defined as follows.

\[
\lim_{{x \to a^+}} f(x) = L
\]

if, for every number \( \varepsilon > 0 \), there is a number \( \delta > 0 \) such that

\[\text{if } a < x < a + \delta \text{ then } |f(x) - L| < \varepsilon\]

- In Definition 4, \( x \) is restricted to lie in the right half \((a, a + \delta)\) of the interval \((a - \delta, a + \delta)\).
Use Definition 4
to prove that:

\[ \lim_{x \to 0^+} \sqrt{x} = 0 \]
Let $\varepsilon$ be a given positive number.

- Here, $a = 0$ and $L = 0$, so we want to find a number $\delta$ such that if $0 < x < \delta$ then $\left| \sqrt{x} - 0 \right| < \varepsilon$.
- That is, if $0 < x < \delta$ then $\sqrt{x} < \varepsilon$.
- Squaring both sides of the inequality $\sqrt{x} < \varepsilon$, we get if $0 < x < \delta$ then $x < \varepsilon^2$.
- This suggests that we should choose $\delta = \varepsilon^2$. 

Example 3

STEP 1: GUESSING THE VALUE
Given $\varepsilon > 0$, let $\delta = \varepsilon^2$.

- If $0 < x < \delta$, then $\sqrt{x} < \sqrt{\delta} < \sqrt{\varepsilon^2} = \varepsilon$.

- So, $|\sqrt{x} - 0| < \varepsilon$.

- According to Definition 4, this shows that $\lim_{x \to 0^+} \sqrt{x} = 0$. 
Prove that:

$$\lim_{x \to 3} x^2 = 9$$
Let $\varepsilon > 0$ be given.

- We have to find a number $\delta > 0$ such that
  
  \[
  \text{if } 0 < |x - 3| < \delta \text{ then } |x^2 - 9| < \varepsilon
  \]

- To connect $|x^2 - 9|$ with $|x - 3|$ we write
  
  \[
  |x^2 - 9| = |(x + 3)(x - 3)|
  \]

- Then, we want
  
  \[
  \text{if } 0 < |x - 3| < \delta \text{ then } |x + 3||x - 3| < \varepsilon
  \]
Notice that, if we can find a positive constant $C$ such that $|x + 3| < C$, then

$$|x + 3||x - 3| < C|x - 3|$$

and we can make $C|x - 3| < \varepsilon$ by taking $|x - 3| < \frac{\varepsilon}{C} = \delta$. 

Example 4

STEP 1: GUESSING THE VALUE
We can find such a number $C$ if we restrict $x$ to lie in some interval centered at 3.

- In fact, since we are interested only in values of $x$ that are close to 3, it is reasonable to assume that $x$ is within a distance 1 from 3, that is, $|x - 3| < 1$.
- Then, $2 < x < 4$, so $5 < x + 3 < 7$.
- Thus, we have $|x + 3| < 7$, and so $C = 7$ is a suitable choice for the constant.

Example 4

STEP 1: GUESSING THE VALUE
However, now, there are two restrictions on $|x - 3|$, namely

$$|x - 3| < 1 \quad \text{and} \quad |x - 3| < \frac{\varepsilon}{C} = \frac{\varepsilon}{7}$$

- To make sure that both inequalities are satisfied, we take $\delta$ to be the smaller of the two numbers 1 and $\varepsilon/7$.
- The notation for this is $\delta = \min\left\{1, \frac{\varepsilon}{7}\right\}$.
Given \( \varepsilon > 0 \), let \( \delta = \min \left\{ 1, \frac{\varepsilon}{7} \right\} \).

- If \( 0 < |x - 3| < \delta \), then \( |x - 3| < 1 \Rightarrow 2 < x < 4 \Rightarrow |x + 3| < 7 \) (as in part I).

- We also have \( |x - 3| < \varepsilon / 7 \), so

\[
|x^2 - 9| = |x + 3||x - 3| < 7 \cdot \frac{\varepsilon}{7} = \varepsilon
\]

- This shows that \( \lim_{x \to 3} x^2 = 9 \).
As the example shows, it is not always easy to prove that limit statements are true using the $\varepsilon, \delta$ definition.

- In fact, if we had been given a more complicated function such as

$$f(x) = \frac{(6x^2 - 8x + 9)}{(2x^2 - 1)},$$

a proof would require a great deal of ingenuity.
Fortunately, this is unnecessary.

- This is because the Limit Laws stated in Section 2.3 can be proved using Definition 2.
- Then, the limits of complicated functions can be found rigorously from the Limit Laws—without resorting to the definition directly.
For instance, we prove the Sum Law.

- If \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} g(x) = M \) both exist, then

\[
\lim_{x \to a} [f(x) + g(x)] = L + M
\]
Let $\varepsilon > 0$ be given.

We must find $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta \text{ then } |f(x) + g(x) - (L + M)| < \varepsilon$$
Using the Triangle Inequality $|a + b| \leq |a| + |b|$
we can write:

$$|f(x) + g(x) - (L + M)| = |(f(x) - L) + (g(x) - M)|$$

$$\leq |f(x) - L| + |g(x) - M|$$
PROOF OF THE SUM LAW

We make $|f(x) + g(x) - (L + M)|$ less than $\varepsilon$ by making each of the terms $|f(x) - L|$ and $|g(x) - M|$ less than $\varepsilon/2$.

- Since $\varepsilon/2 > 0$ and $\lim_{x \to a} f(x) = L$, there exists a number $\delta_1 > 0$ such that
  
  if $0 < |x - a| < \delta_1$ then $|f(x) - L| < \frac{\varepsilon}{2}$

- Similarly, since $\lim_{x \to a} g(x) = M$, there exists a number $\delta_2 > 0$ such that
  
  if $0 < |x - a| < \delta_2$ then $|g(x) - M| < \frac{\varepsilon}{2}$
Let \( \delta = \min \{ \delta_1, \delta_2 \} \).

- Notice that
  
  if \( 0 < |x - a| < \delta \) then \( 0 < |x - a| < \delta_1 \) and \( 0 < |x - a| < \delta_2 \)

- So, \( |f(x) - L| < \frac{\varepsilon}{2} \) and \( |g(x) - M| < \frac{\varepsilon}{2} \)

- Therefore, by Definition 5,
  
  \[
  |f(x) + g(x) - (L + M)| \leq |f(x) - L| + |g(x) - M| \\
  < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
  \]
To summarize,

if \( 0 < |x - a| < \delta \) then \( |f(x) + g(x) - (L + M)| < \varepsilon \)

Thus, by the definition of a limit,

\[
\lim_{x \to a} [f(x) + g(x)] = L + M
\]
Infinite limits can also be defined in a precise way.

- The following is a precise version of Definition 4 in Section 2.2.
Let $f$ be a function defined on some open interval that contains the number $a$, except possibly at $a$ itself.

Then, $\lim_{x \to a} f(x) = \infty$ means that, for every positive number $M$, there is a positive number $\delta$ such that

$$\text{if } 0 < |x - a| < \delta \text{ then } f(x) > M$$
The definition states that the values of \( f(x) \) can be made arbitrarily large (larger than any given number \( M \)) by taking \( x \) close enough to \( a \) (within a distance \( \delta \), where \( \delta \) depends on \( M \), but with \( x \neq a \)).
A geometric illustration is shown in the figure.

- Given any horizontal line $y = M$, we can find a number $\delta > 0$ such that, if we restrict $x$ to lie in the interval $(a - \delta, a + \delta)$ but $x \neq a$, then the curve $y = f(x)$ lies above the line $y = M$.

- You can see that, if a larger $M$ is chosen, then a smaller $\delta$ may be required.
Use Definition 6 to prove that \( \lim_{x \to 0} \frac{1}{x^2} = \infty. \)

- Let \( M \) be a given positive number.
- We want to find a number \( \delta \) such that
  
  \[
  \text{if } 0 < |x| < \delta \text{ then } \frac{1}{x^2} > M
  \]

- However,
  
  \[
  \frac{1}{x^2} > M \iff x^2 < \frac{1}{M} \iff |x| < \frac{1}{\sqrt{M}}
  \]

- So, if we choose \( \delta = \frac{1}{\sqrt{M}} \) and \( 0 < |x| < \delta = \frac{1}{\sqrt{M}} \),
  then \( \frac{1}{x^2} > M \).

- This shows that \( \frac{1}{x^2} \to \infty \) as \( x \to 0 \).
Similarly, the following is a precise version of Definition 5 in Section 2.2.
Let $f$ be a function defined on some open interval that contains the number $a$, except possibly at $a$ itself. Then, $\lim_{x \to a} f(x) = -\infty$ means that, for every negative number $N$, there is a positive number $\delta$ such that

$$0 < |x - a| < \delta \text{ then } f(x) < N$$
This is illustrated by the figure.