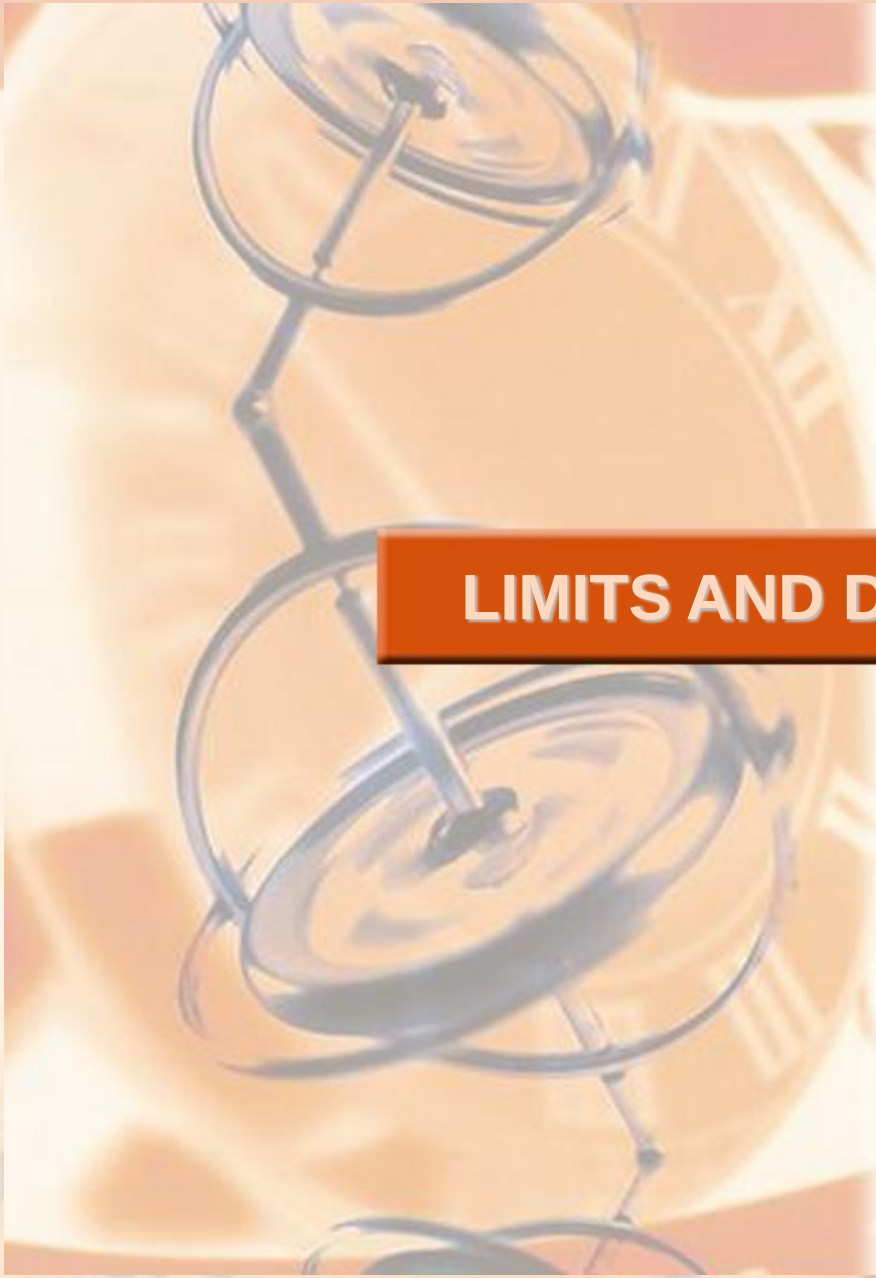


2

LIMITS AND DERIVATIVES

The intuitive definition of a limit given in Section 2.2 is inadequate for some purposes.

- This is because such phrases as '*x* is close to 2' and '*f*(*x*) gets closer and closer to *L*' are vague.
- In order to be able to prove conclusively that

$$\lim_{x \to 0}\left(x^3 + \frac{\cos 5x}{10{,}000}\right) = 0.0001 \text{ or } \lim_{x \to 0}\frac{\sin x}{x} = 1$$

we must make the definition of a limit precise.

2.4

The Precise Definition of a Limit

In this section, we will:

Define a limit precisely.

To motivate the precise definition of a limit, consider the function

$$f(x) = \begin{cases} 2x - 1 & if \;\; x \neq 3 \\ 6 & if \;\; x = 3 \end{cases}$$

- Intuitively, it is clear that when *x* is close to 3 but $x \neq 3$, then *f*(*x*) is close to 5—and so $\lim_{x \to 3} f(x) = 5$.

PRECISE DEFINITION OF LIMIT

To obtain more detailed information about how $f(x)$ varies when x is close to 3, we ask the following question.

- How close to 3 does x have to be so that $f(x)$ differs from 5 by less than 0.l?

PRECISE DEFINITION OF LIMIT

The distance from x to 3 is $|x - 3|$ and the distance from $f(x)$ to 5 is $|f(x) - 5|$.

So, our problem is to find a number δ such that

$$|f(x) - 5| < 0.1 \quad \text{if} \quad |x - 3| < \delta \quad \text{but} \quad x \neq 3$$

If $|x-3|>0$, then $x \neq 3$.

So, an equivalent formulation of our problem is to find a number δ such that

$$|f(x)-5|<0.1 \quad \text{if} \quad 0<|x-3|<\delta$$

Notice that, if $0 < |x - 3| < (0.1)/2 = 0.05$,

then

$$|f(x) - 5| = |(2x - 1) - 5| = |2x - 6| = 2|x - 3| < 0.1$$

That is,

$$|f(x) - 5| < 0.1 \text{ if } 0 < |x - 3| < 0.05$$

- Thus, an answer to the problem is given by $\delta = 0.05$.
- That is, if *x* is within a distance of 0.05 from 3, then *f*(*x*) will be within a distance of 0.1 from 5.

PRECISE DEFINITION OF LIMIT

If we change the number 0.1 in our problem to the smaller number 0.01, then, by using the same method, we find that *f*(*x*) will differ from 5 by less than 0.01—provided that *x* differs from 3 by less than (0.01)/2 = 0.005

$$|f(x)-5|<0.01 \quad \text{if} \quad 0<|x-3|<0.005$$

Similarly,

$$|f(x)-5| < 0.001 \quad \text{if} \quad 0 < |x-3| < 0.0005$$

- The numbers 0.1, 0.01, and 0.001 that we have considered are ‘error tolerances’ that we might allow.

PRECISE DEFINITION OF LIMIT

For 5 to be the precise limit of *f*(x) as *x* approaches 3, we must not only be able to bring the difference between *f*(*x*) and 5 below each of these three numbers, we must be able to bring it below any positive number.

- By the same reasoning, we can.

 Definition 1

If we write ε for an arbitrary positive number, then we find as before that:

$$|f(x)-5|<\varepsilon \quad \text{if} \quad 0<|x-3|<\delta=\frac{\varepsilon}{2}$$

- This is a precise way of saying that *f*(*x*) is close to 5 when *x* is close to 3.
- This is because (1) states that we can make the values of *f*(*x*) within an arbitrary distance ε from 5 by taking the values of *x* within a distance $\varepsilon/2$ from 3 (but $x \neq 3$).

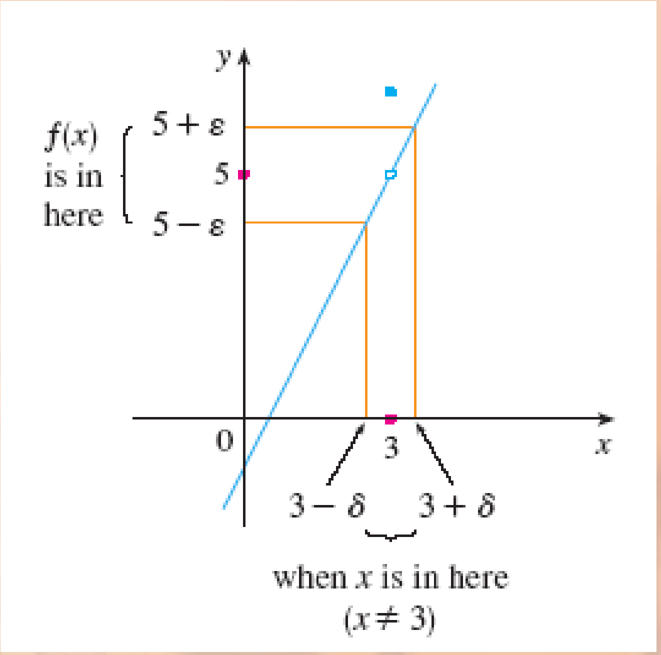
PRECISE DEFINITION OF LIMIT

Note that Definition 1 can be rewritten as follows.

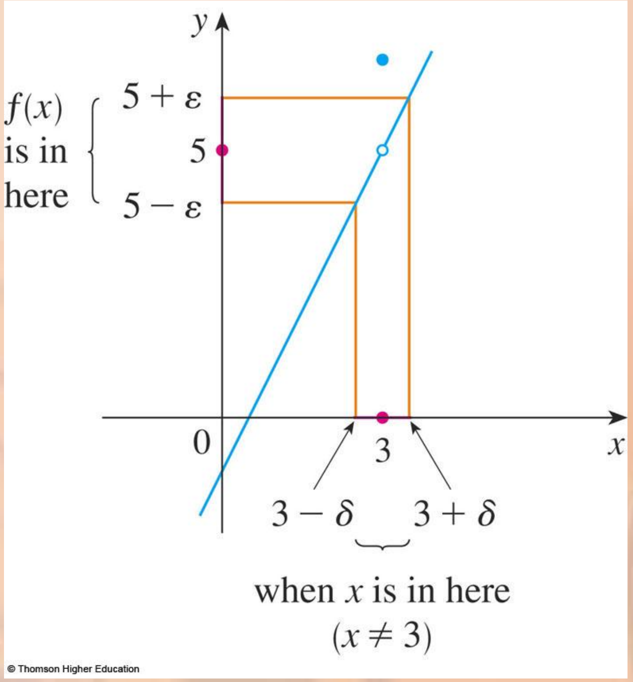
If $3-\delta < x < 3+\delta \quad (x \neq 3)$

then $5-\varepsilon < f(x) < 5+\varepsilon$

This is illustrated in the figure.



By taking the values of *x* $(\neq 3)$ to lie in the interval $(3-\delta, 3+\delta)$, we can make the values of *f*(*x*) lie in the interval $(5-\varepsilon, 5+\varepsilon)$.



Using (1) as a model,
we give a precise definition
of a limit.

Definition 2

Let *f* be a function defined on some open interval that contains the number *a*, except possibly at *a* itself.

Then, we say that the limit of *f*(*x*) as *x* approaches *a* is *L*, and we write $\lim_{x \to a} f(x) = L$ if, for every number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$\text{if } \; 0 < |x - a| < \delta \;\; \text{ then } \; |f(x) - L| < \varepsilon$$

PRECISE DEFINITION OF LIMIT

Since |*x* - *a*| is the distance from *x* to *a* and |*f*(*x*) - *L*| is the distance from *f*(*x*) to *L*, and since ε can be arbitrarily small, the definition can be expressed in words as follows.

- $\lim_{x \to a} f(x) = L$ means that the distance between *f*(*x*) and *L* can be made arbitrarily small by taking the distance from *x* to *a* sufficiently small (but not 0).
- Alternatively, $\lim_{x \to a} f(x) = L$ means that the values of *f*(*x*) can be made as close as we please to *L* by taking *x* close enough to *a* (but not equal to *a*).

PRECISE DEFINITION OF LIMIT

We can also reformulate Definition 2 in terms of intervals by observing that the inequality $|x-a|<\delta$ is equivalent to $-\delta<x-a<\delta$, which in turn can be written as $a-\delta<x<a+\delta$.

Also, $0 < |x - a|$ is true if and only if $x - a \neq 0$, that is, $x \neq a$.

Similarly, the inequality $|f(x) - L| < \varepsilon$ is equivalent to the pair of inequalities $L - \varepsilon < f(x) < L + \varepsilon$.

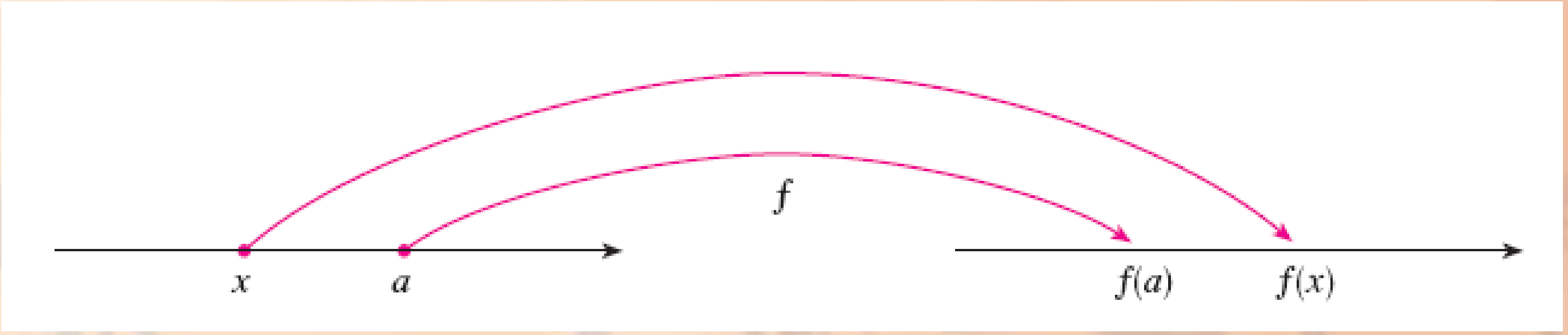
PRECISE DEFINITION OF LIMIT

Therefore, in terms of intervals, Definition 2 can be stated as follows.

$\lim_{x \to a} f(x) = L$ means that, for every $\varepsilon > 0$ (no matter how small ε is), we can find $\delta > 0$ such that, if *x* lies in the open interval $(a-\delta, a+\delta)$ and $x \neq a$, then *f*(*x*) lies in the open interval $(L-\varepsilon,\ L+\varepsilon)$.

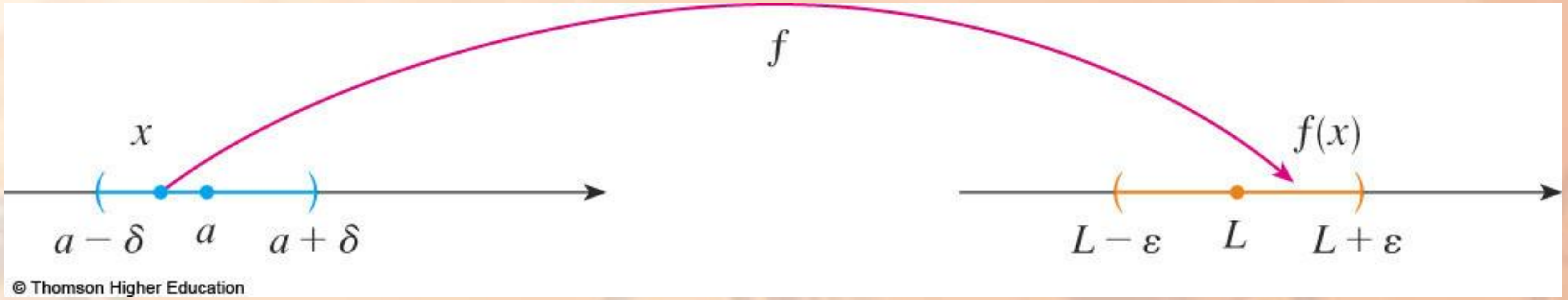
PRECISE DEFINITION OF LIMIT

We interpret this statement geometrically by representing a function by an arrow diagram as in the figure, where *f* maps a subset of $\mathbb{R}$ onto another subset of $\mathbb{R}$.



PRECISE DEFINITION OF LIMIT

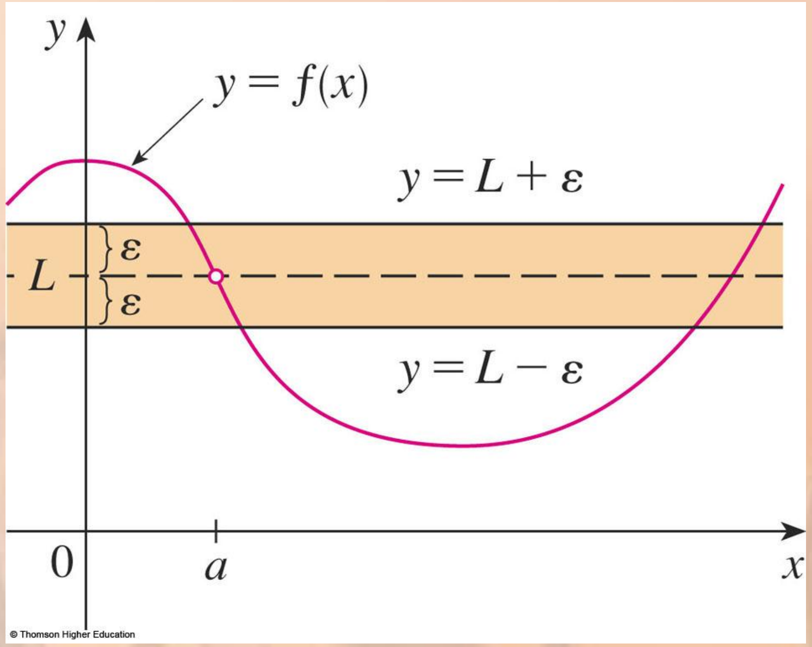
The definition of limit states that, if any small interval $(L-\varepsilon,\ L+\varepsilon)$ is given around *L*, then we can find an interval $(a-\delta, a+\delta)$ around *a* such that *f* maps all the points in $(a-\delta, a+\delta)$ (except possibly *a*) into the interval $(L-\varepsilon,\ L+\varepsilon)$.



Another geometric interpretation of limits can be given in terms of the graph of a function.

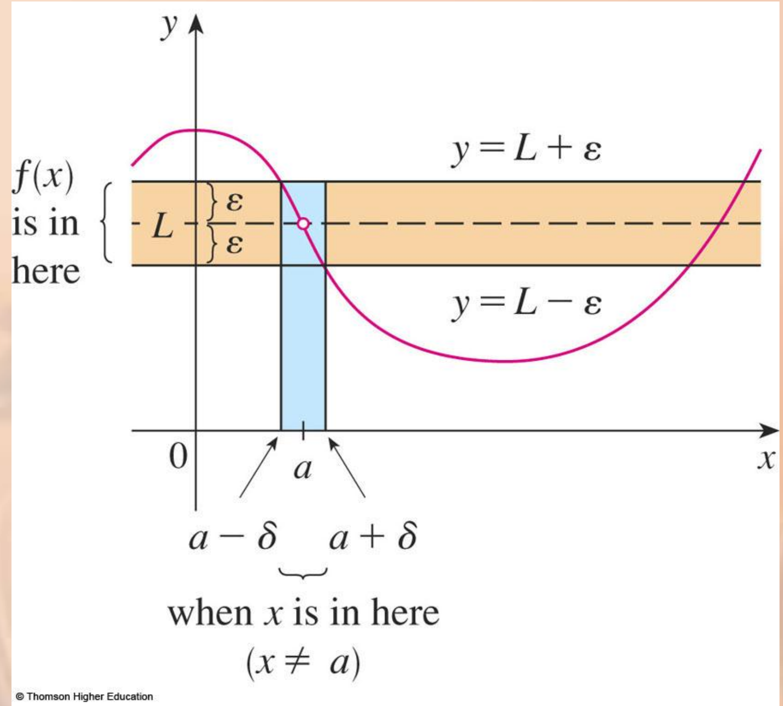
PRECISE DEFINITION OF LIMIT

If $\varepsilon > 0$ is given, then we draw the horizontal lines $y = L + \varepsilon$ and $y = L - \varepsilon$ and the graph of *f*.

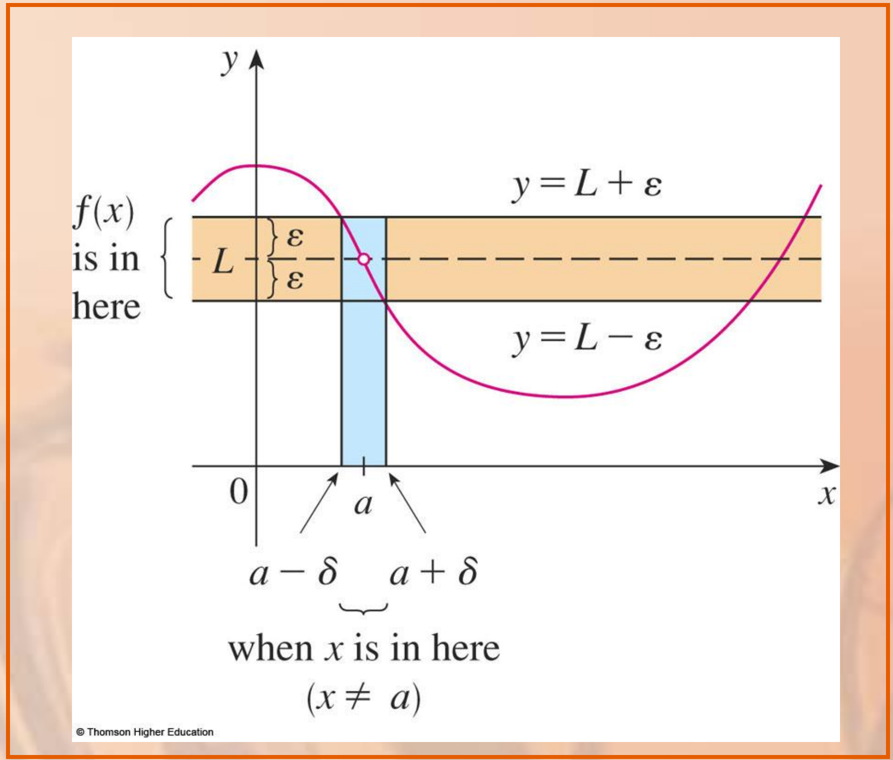


PRECISE DEFINITION OF LIMIT

If $\lim_{x\to a} f(x) = L$, then we can find a number $\delta > 0$ such that, if we restrict *x* to lie in the interval $(a-\delta, a+\delta)$ and take $x \neq a$, then the curve *y* = *f*(*x*) lies between the lines $y = L - \varepsilon$ and $y = L + \varepsilon$.

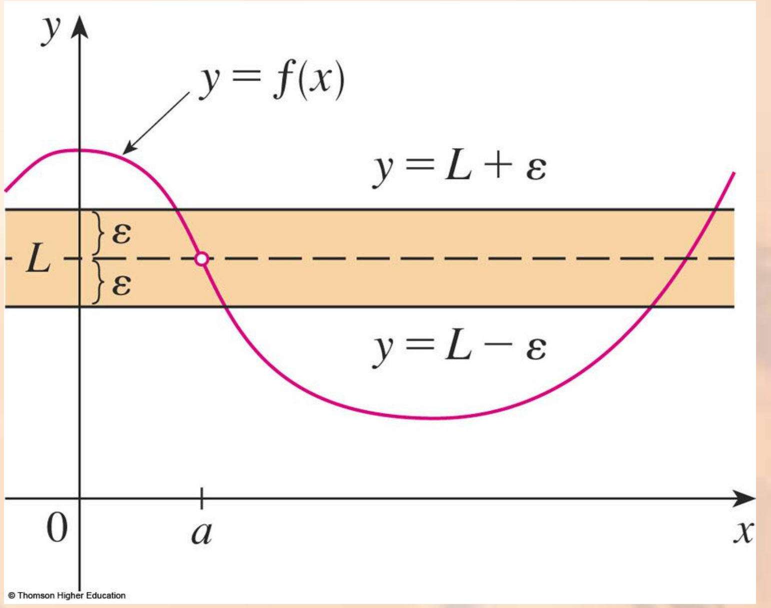


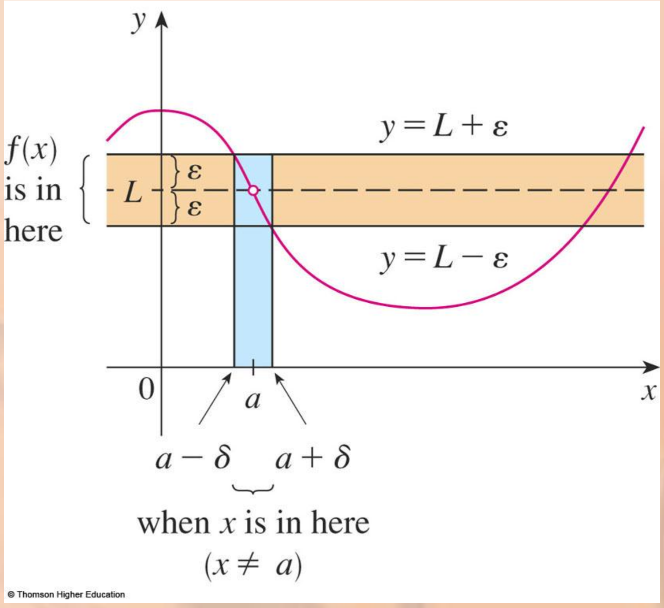
You can see that, if such a δ has been found, then any smaller δ will also work.



PRECISE DEFINITION OF LIMIT

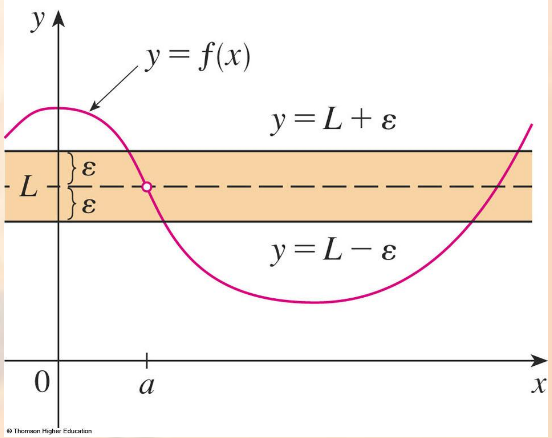
It is important to realize that the process illustrated in the figures must work for every positive number ε, no matter how small it is chosen.

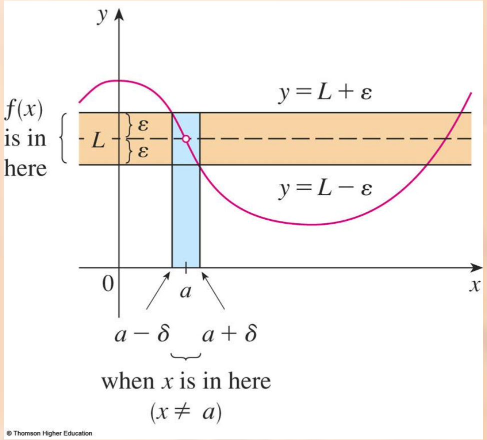


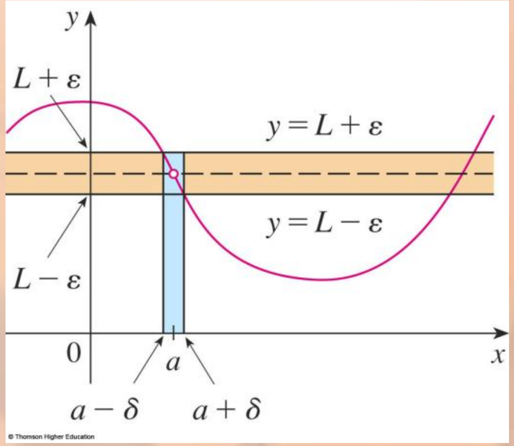


PRECISE DEFINITION OF LIMIT

The third figure shows that, if a smaller ε is chosen, then a smaller δ may be required.







 Example 1

Use a graph to find a number δ such that

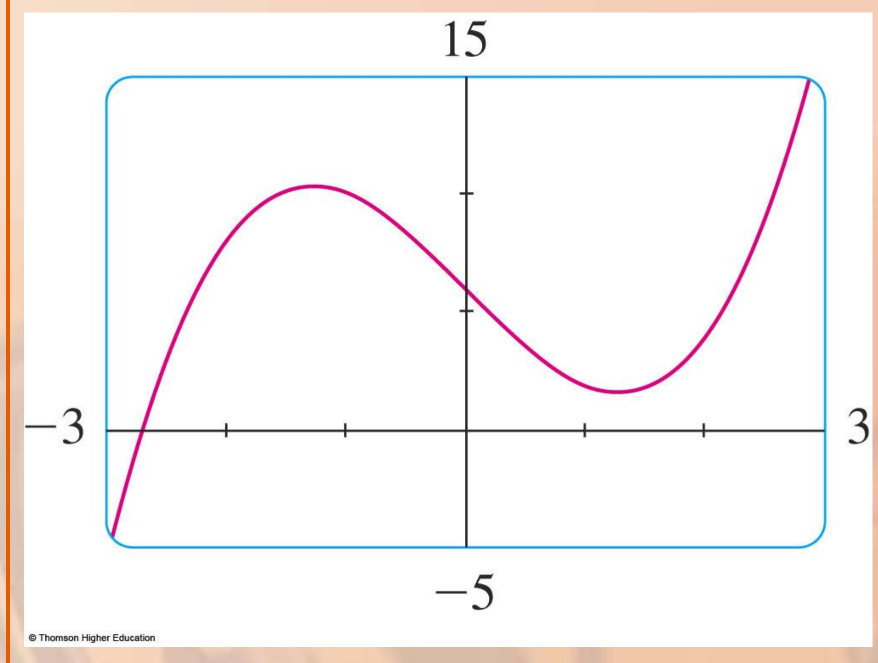
$$\text{if} \quad |x-1| < \delta \quad \text{then} \quad \left|\left(x^3 - 5x + 6\right) - 2\right| < 0.2$$

- In other words, find a number δ that corresponds to $\varepsilon = 0.2$ in the definition of a limit for the function $f(x) = x^3 - 5x + 6$ with *a* = 1 and *L* = 2.

Example 1

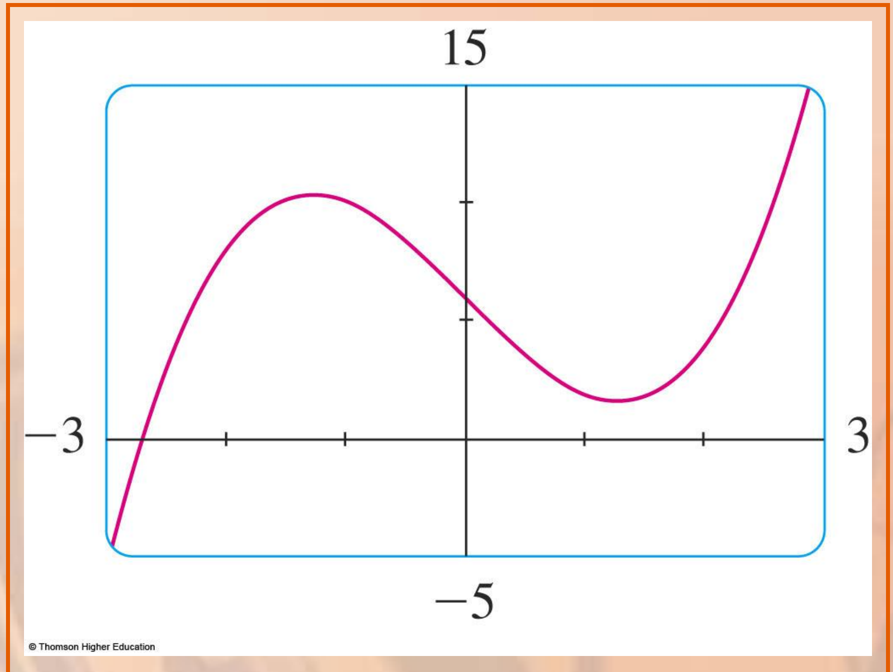
A graph of *f* is shown.

We are interested in the region near the point (1, 2).



Notice that we can rewrite the inequality $\left|\left(x^3-5x+6\right)-2\right|<0.2$ as $1.8<x^3-5x+6<2.2$

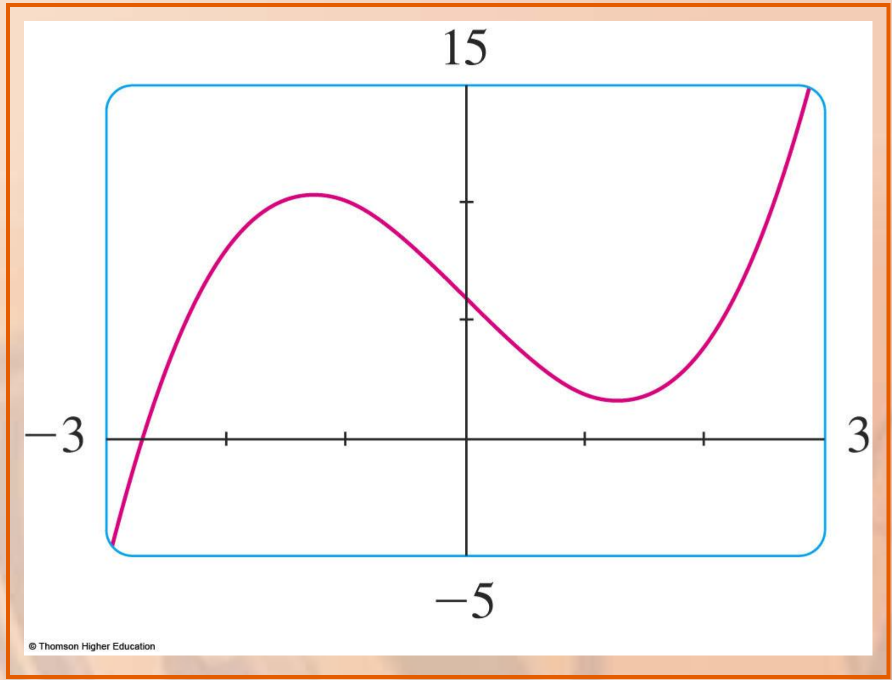
- So, we need to determine the values of *x* for which the curve $y=x^3-5x+6$ lies between the horizontal lines *y* = 1.8 and *y* = 2.2.



PRECISE DEFINITION OF LIMIT

Example 1

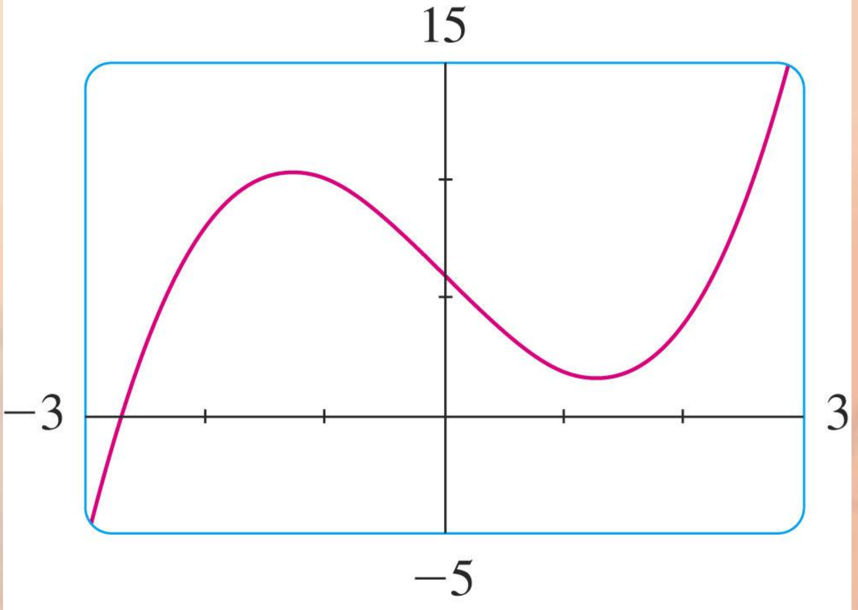
- Therefore, we graph the curves $y = x^3 - 5x + 6$, y = 1.8, and y = 2.2 near the point (1, 2).
- Then, we use the cursor to estimate that the x-coordinate of the point of intersection of the line y = 2.2 and the curve $y = x^3 - 5x + 6$ is about 0.911



PRECISE DEFINITION OF LIMIT **Example 1**

- Similarly, $y = x^3 - 5x + 6$ intersects the line $y = 1.8$ when $x \approx 1.124$
- So, rounding to be safe, we can say that

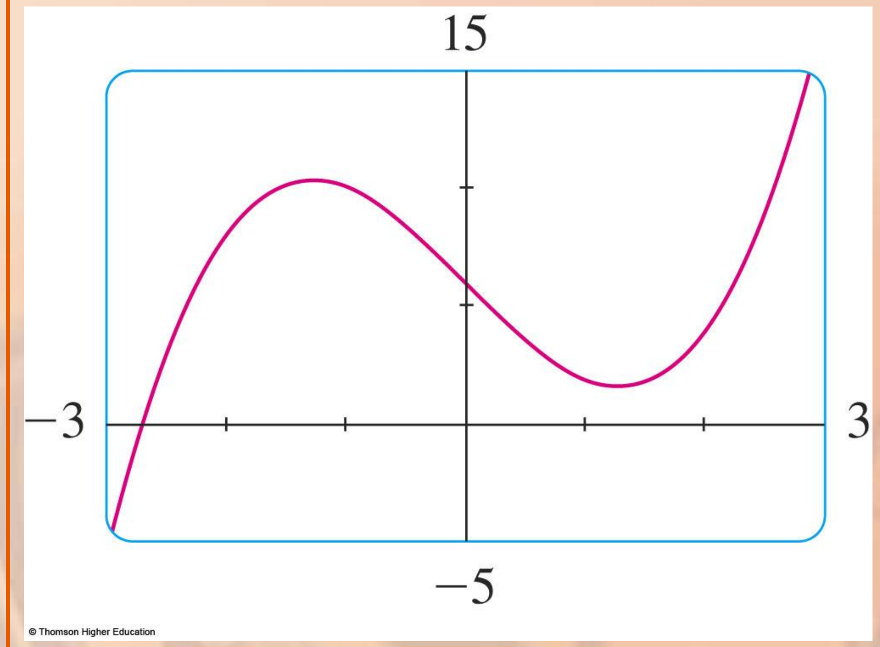
$$\text{if} \quad 0.92 < x < 1.12 \quad \text{then} \quad 1.8 < x^3 - 5x + 6 < 2.2$$



 Example 1

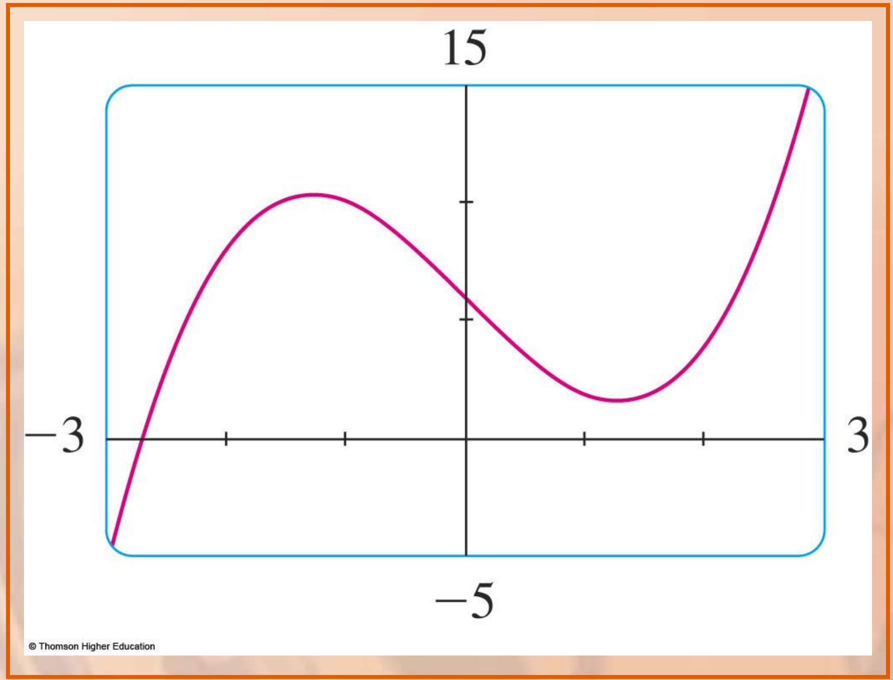
This interval (0.92, 1.12) is not symmetric about $x = 1$.

- The distance from $x = 1$ to the left endpoint is 1 - 0.92 = 0.08 and the distance to the right endpoint is 0.12



We can choose δ to be the smaller of these numbers—that is, $\delta = 0.08$

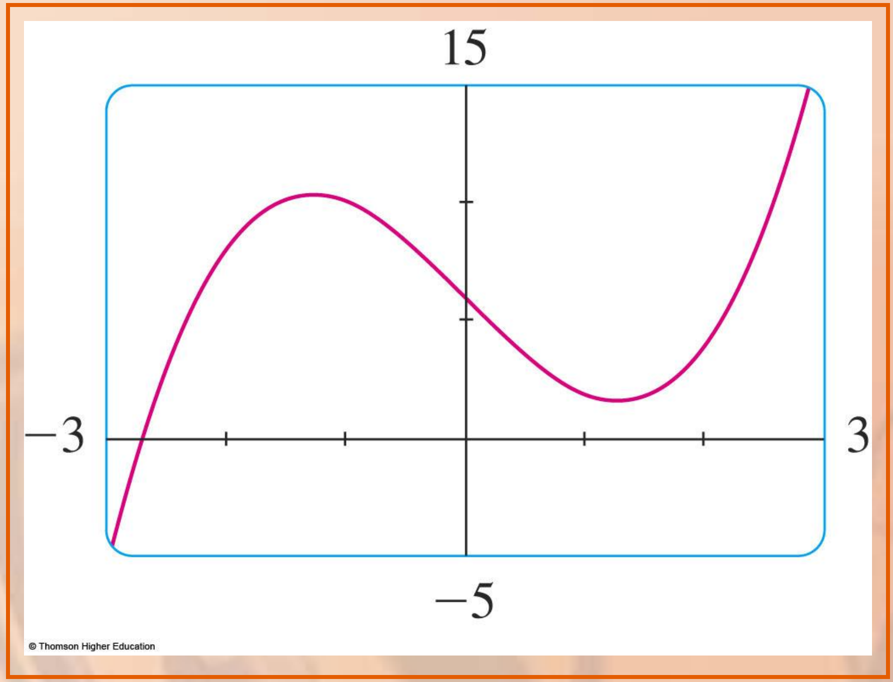
- Then, we can rewrite our inequalities in terms of distances: if $|x-1| < 0.08$ then $|(x^3-5x+6)-2| < 0.2$



 Example 1

This just says that, by keeping x within 0.08 of 1, we are able to keep $f(x)$ within 0.2 of 2.

- Though we chose $\delta = 0.08$, any smaller, positive value of δ would also have worked.



PRECISE DEFINITION OF LIMIT

The graphical procedure in the example gives an illustration of the definition for $\varepsilon = 0.2$

However, it does not prove that the limit is equal to 2.

- A proof has to provide a δ for every ε.

PRECISE DEFINITION OF LIMIT

In proving limit statements, it may be helpful to think of the definition of limit as a challenge.

- First, it challenges you with a number ε.
- Then, you must be able to produce a suitable δ.
- You have to be able to do this for every $\varepsilon > 0$, not just a particular ε.

PRECISE DEFINITION OF LIMIT

Imagine a contest between two people, A and B.

Imagine yourself to be B.

- A stipulates that the fixed number *L* should be approximated by the values of *f*(*x*) to within a degree of accuracy ε (say 0.01).
- Then, B responds by finding a number δ such that, if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

PRECISE DEFINITION OF LIMIT

- Then, A may become more exacting and challenge B with a smaller value of ε (say, 0.0001).
- Again, B has to respond by finding a corresponding δ.
- Usually, the smaller the value of ε, the smaller the corresponding value of δ must be.
- If B always wins, no matter how small A makes ε, then $\lim_{x \to a} f(x) = L$.

Example 2

Prove that:

$$\lim_{x \to 3} (4x - 5) = 7$$

STEP 1: GUESSING THE VALUE **Example 2**

The first step is the preliminary analysis—guessing a value for δ .

- Let ε be a given positive number.
- We want to find a number δ such that

$$\text{if } \; 0 < |x-3| < \delta \; \text{ then } \; |(4x-5)-7| < \varepsilon$$

- However, $|(4x-5)-7| = |4x-12| = |4(x-3)| = 4|x-3|$

STEP 1: GUESSING THE VALUE **Example 2**

- Therefore, we want

$$\text{if} \quad 0 < |x-3| < \delta \quad \text{then} \quad 4|x-3| < \varepsilon$$

- That is,

$$\text{if} \quad 0 < |x-3| < \delta \quad \text{then} \quad |x-3| < \frac{\varepsilon}{4}$$

- This suggests that we should choose $\delta = \varepsilon/4$.

STEP 2: PROOF

Example 2

The second step is the proof—showing that this δ works.

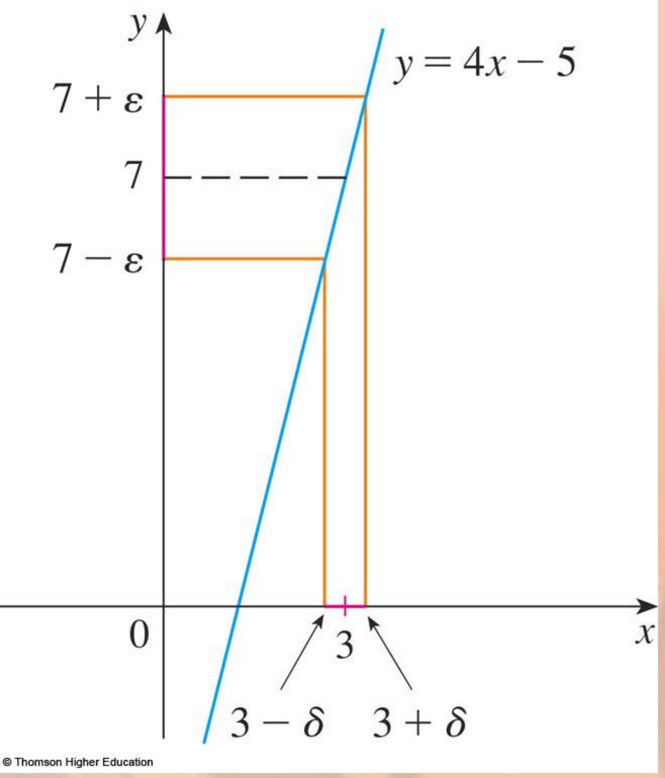
- Given $\varepsilon > 0$, choose $\delta = \varepsilon/4$.
- If $0 < |x-3| < \delta$, then

$$|(4x-5)-7| = |4x-12| = 4|x-3| < 4\delta = 4\left(\frac{\varepsilon}{4}\right) = \varepsilon$$

- Thus, if $0 < |x-3| < \delta$ then $|(4x-5)-7| < \varepsilon$
- Therefore, by the definition of a limit, $\lim_{x \to 3}(4x-5) = 7$

Example 2

The example is illustrated by the figure.



Note that, in the solution of the example, there were two stages—guessing and proving.

- We made a preliminary analysis that enabled us to guess a value for δ.
- In the second stage, though, we had to go back and prove in a careful, logical fashion that we had made a correct guess.

This procedure is typical of much of mathematics.

- Sometimes, it is necessary to first make an intelligent guess about the answer to a problem and then prove that the guess is correct.

PRECISE DEFINITION OF LIMIT

The intuitive definitions of one-sided limits given in Section 2.2 can be precisely reformulated as follows.

 Definition 3

Left-hand limit is defined as follows.

$$\lim_{x \to a^-} f(x) = L$$

if, for every number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$\text{if } \; a - \delta < x < a \; \text{ then } \left| f(x) - L \right| < \varepsilon$$

- Notice that Definition 3 is the same as Definition 2 except that *x* is restricted to lie in the left half $\left(a - \delta, a\right)$ of the interval $\left(a - \delta, a + \delta\right)$.

 Definition 4

Right-hand limit is defined as follows.

$$\lim_{x \to a^+} f(x) = L$$

if, for every number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$\text{if} \quad a < x < a + \delta \quad \text{then} \quad |f(x) - L| < \varepsilon$$

- In Definition 4, *x* is restricted to lie in the right half $(a, a+\delta)$ of the interval $(a-\delta, a+\delta)$.

Use Definition 4 to prove that:

$$\lim_{x \to 0^+} \sqrt{x} = 0$$

 Example 3

Let ε be a given positive number.

- Here, *a* = 0 and *L* = 0, so we want to find a number δ such that if $0 < x < \delta$ then $\left|\sqrt{x} - 0\right| < \varepsilon$.
- That is, if $0 < x < \delta$ then $\sqrt{x} < \varepsilon$.
- Squaring both sides of the inequality $\sqrt{x} < \varepsilon$, we get if $0 < x < \delta$ then $x < \varepsilon^2$.
- This suggests that we should choose $\delta = \varepsilon^2$.

STEP 2: PROOF

Example 3

Given $\varepsilon > 0$, let $\delta = \varepsilon^2$.

- If $0 < x < \delta$, then $\sqrt{x} < \sqrt{\delta} < \sqrt{\varepsilon^2} = \varepsilon$.
- So, $\left|\sqrt{x} - 0\right| < \varepsilon$.
- According to Definition 4, this shows that $\lim_{x \to 0^+} \sqrt{x} = 0$.

Prove that:

$$\lim_{x \to 3} x^2 = 9$$

STEP 1: GUESSING THE VALUE **Example 4**

Let $\varepsilon > 0$ be given.

- We have to find a number $\delta > 0$ such that

$$\text{if } \ 0 < |x-3| < \delta \ \text{ then } \ |x^2 - 9| < \varepsilon$$

- To connect $|x^2 - 9|$ with $|x-3|$ we write

$$|x^2 - 9| = |(x+3)(x-3)|$$

- Then, we want

$$\text{if } \ 0 < |x-3| < \delta \ \text{ then } \ |x+3||\,x-3| < \varepsilon$$

STEP 1: GUESSING THE VALUE **Example 4**

Notice that, if we can find a positive constant *C* such that $|x+3| < C$, then

$$|x+3||x-3| < C|x-3|$$

and we can make $C|x-3| < \varepsilon$ by taking $|x-3| < \varepsilon / C = \delta$.

STEP 1: GUESSING THE VALUE **Example 4**

We can find such a number *C* if we restrict *x* to lie in some interval centered at 3.

- In fact, since we are interested only in values of *x* that are close to 3, it is reasonable to assume that *x* is within a distance 1 from 3, that is, $|x-3|<1$.
- Then, $2<x<4$, so $5<x+3<7$.
- Thus, we have $|x+3|<7$, and so *C* = 7 is a suitable choice for the constant.

However, now, there are two restrictions on $|x-3|$, namely

$$|x-3|<1 \quad \text{and} \quad |x-3|<\frac{\varepsilon}{C}=\frac{\varepsilon}{7}$$

- To make sure that both inequalities are satisfied, we take δ to be the smaller of the two numbers 1 and $\varepsilon/7$.
- The notation for this is $\delta=\min\left\{1,\varepsilon/7\right\}$.

STEP 2: PROOF

Example 4

Given $\varepsilon > 0$, let $\delta = \min\left\{1, \varepsilon/7\right\}$.

- If $0 < |x-3| < \delta$, then $|x-3| < 1 \Rightarrow 2 < x < 4 \Rightarrow |x+3| < 7$ (as in part I).

- We also have $|x-3| < \varepsilon/7$, so

$$|x^2 - 9| = |x+3||x-3| < 7 \cdot \frac{\varepsilon}{7} = \varepsilon$$

- This shows that $\lim_{x \to 3} x^2 = 9$.

As the example shows, it is not always easy to prove that limit statements are true using the ε, δ definition.

- In fact, if we had been given a more complicated function such as

$$f(x) = \left(6x^2 - 8x + 9\right) / \left(2x^2 - 1\right),$$

a proof would require a great deal of ingenuity.

Fortunately, this is unnecessary.

- This is because the Limit Laws stated in Section 2.3 can be proved using Definition 2.
- Then, the limits of complicated functions can be found rigorously from the Limit Laws—without resorting to the definition directly.

For instance, we prove the Sum Law.

- If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ both exist, then

$$\lim_{x\to a}\left[f(x)+g(x)\right] = L + M$$

PROOF OF THE SUM LAW

Let $\varepsilon > 0$ be given.

- We must find $\delta > 0$ such that

  if $0 < |x - a| < \delta$ then $|f(x) + g(x) - (L + M)| < \varepsilon$

PROOF OF THE SUM LAW

Definition 5

Using the Triangle Inequality $|a+b| \leq |a|+|b|$ we can write:

$$|f(x)+g(x)-(L+M)| = |(f(x)-L)+(g(x)-M)|$$

$$\leq |f(x)-L| + |g(x)-M|$$

PROOF OF THE SUM LAW

We make $\left|f(x)+g(x)-(L+M)\right|$ less than ε by making each of the terms $\left|f(x)-L\right|$ and $\left|g(x)-M\right|$ less than $\varepsilon/2$.

- Since $\varepsilon/2>0$ and $\lim_{x\to a} f(x)=L$, there exists a number $\delta_1>0$ such that

$$\text{if } 0<|x-a|<\delta_1 \text{ then } |f(x)-L|<\frac{\varepsilon}{2}$$

- Similarly, since $\lim_{x\to a} g(x)=M$, there exists a number $\delta_2>0$ such that

$$\text{if } 0<|x-a|<\delta_2 \text{ then } |g(x)-M|<\frac{\varepsilon}{2}$$

PROOF OF THE SUM LAW

Let $\delta = \min\left\{\delta_1, \delta_2\right\}$.

- Notice that

  if $0 < |x - a| < \delta$ then $0 < |x - a| < \delta_1$ and $0 < |x - a| < \delta_2$

- So, $|f(x) - L| < \dfrac{\varepsilon}{2}$ and $|g(x) - M| < \dfrac{\varepsilon}{2}$

- Therefore, by Definition 5,

  $$|f(x) + g(x) - (L + M)| \le |f(x) - L| + |g(x) - M|$$

  $$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

PROOF OF THE SUM LAW

To summarize,

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |f(x) + g(x) - (L + M)| < \varepsilon$$

Thus, by the definition of a limit,

$$\lim_{x \to a} [f(x) + g(x)] = L + M$$

Infinite limits can also be defined in a precise way.

- The following is a precise version of Definition 4 in Section 2.2.

Definition 6

Let *f* be a function defined on some open interval that contains the number *a*, except possibly at *a* itself.

Then, $\lim_{x \to a} f(x) = \infty$ means that, for every positive number *M*, there is a positive number δ such that

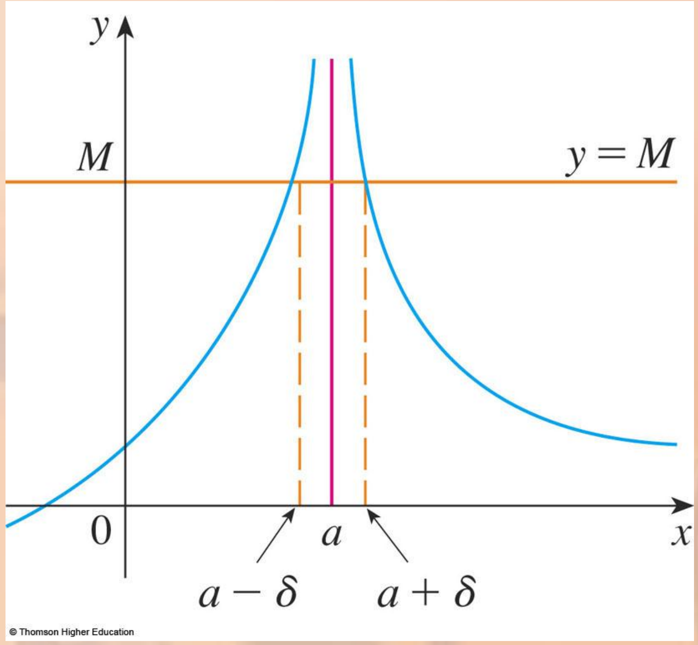
$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad f(x) > M$$

INFINITE LIMITS

The definition states that the values of *f*(*x*) can be made arbitrarily large (larger than any given number *M*) by taking *x* close enough to *a* (within a distance δ, where δ depends on *M*, but with $x \neq a$).

A geometric illustration is shown in the figure.

- Given any horizontal line $y = M$, we can find a number $\delta > 0$ such that, if we restrict x to lie in the interval $(a-\delta, a+\delta)$ but $x \neq a$, then the curve $y = f(x)$ lies above the line $y = M$.
- You can see that, if a larger M is chosen, then a smaller δ may be required.



Example 5

Use Definition 6 to prove that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

- Let *M* be a given positive number.
- We want to find a number δ such that

$$\text{if } 0 < |x| < \delta \text{ then } 1/x^2 > M$$

- However,

$$\frac{1}{x^2} > M \Leftrightarrow x^2 < \frac{1}{M} \Leftrightarrow |x| < \frac{1}{\sqrt{M}}$$

- So, if we choose $\delta = 1/\sqrt{M}$ and $0 < |x| < \delta = 1/\sqrt{M}$, then $1/x^2 > M$.
- This shows that $\frac{1}{x^2} \to \infty$ as $x \to 0$.

Similarly, the following is a precise version of Definition 5 in Section 2.2.

Definition 7

Let *f* be a function defined on some open interval that contains the number *a*, except possibly at *a* itself.

Then, $\lim_{x \to a} f(x) = -\infty$ means that, for every negative number *N*, there is a positive number δ such that

$$\text{if} \quad 0 < |x - a| < \delta \ \text{then} \ f(x) < N$$

This is illustrated by the figure.