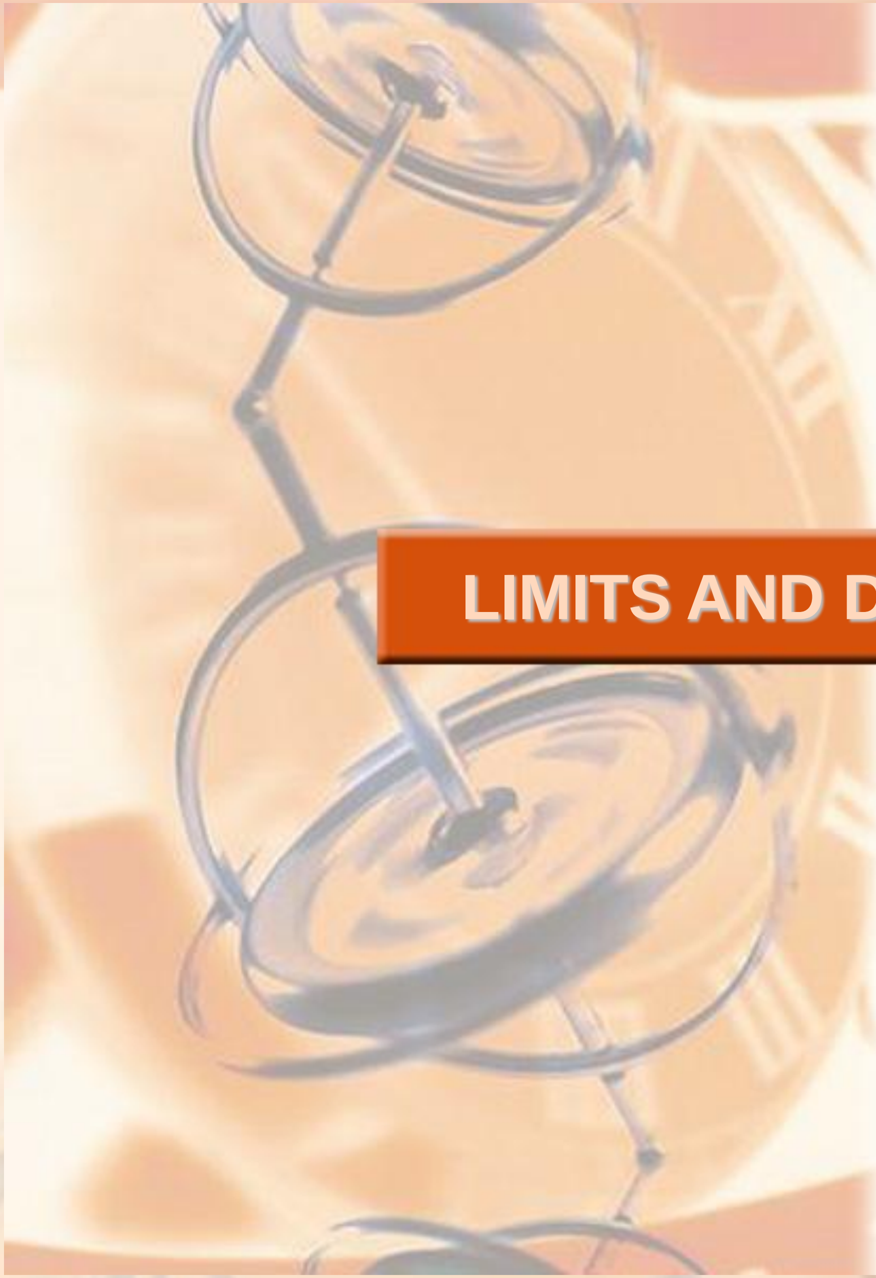


2

LIMITS AND DERIVATIVES

We have used calculators and graphs to guess the values of limits.

- However, we have learned that such methods don’t always lead to the correct answer.

2.3

Calculating Limits Using the Limit Laws

In this section, we will:

Use the Limit Laws to calculate limits.

Suppose that c is a constant and the limits $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist.

THE LIMIT LAWS

Then,

$$1. \lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

$$2. \lim_{x \to a}[f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

$$3. \lim_{x \to a}[cf(x)] = c \lim_{x \to a} f(x)$$

$$4. \lim_{x \to a}[f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

$$5. \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad \text{if } \lim_{x \to a} g(x) \neq 0$$

These laws can be stated verbally.

THE SUM LAW

The limit of a sum is the sum of the limits.

$$\lim_{x\to a}\left[f(x)+g(x)\right]=\lim_{x\to a}f(x)+\lim_{x\to a}g(x)$$

THE DIFFERENCE LAW

The limit of a difference is the difference of the limits.

$$\lim_{x \to a}\left[f(x) - g(x) \right] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

THE CONSTANT MULTIPLE LAW

The limit of a constant times a function is the constant times the limit of the function.

$$\lim_{x\to a}\left[cf(x)\right] = c\lim_{x\to a} f(x)$$

THE PRODUCT LAW

The limit of a product is the product of the limits.

$$\lim_{x \to a}\left[f(x)g(x) \right] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

THE QUOTIENT LAW

The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad if \lim_{x \to a} g(x) \neq 0$$

It is easy to believe that these properties are true.

- For instance, if $f(x)$ is close to L and $g(x)$ is close to M, it is reasonable to conclude that $f(x) + g(x)$ is close to $L + M$.
- This gives us an intuitive basis for believing that the Sum Law is true.

USING THE LIMIT LAWS

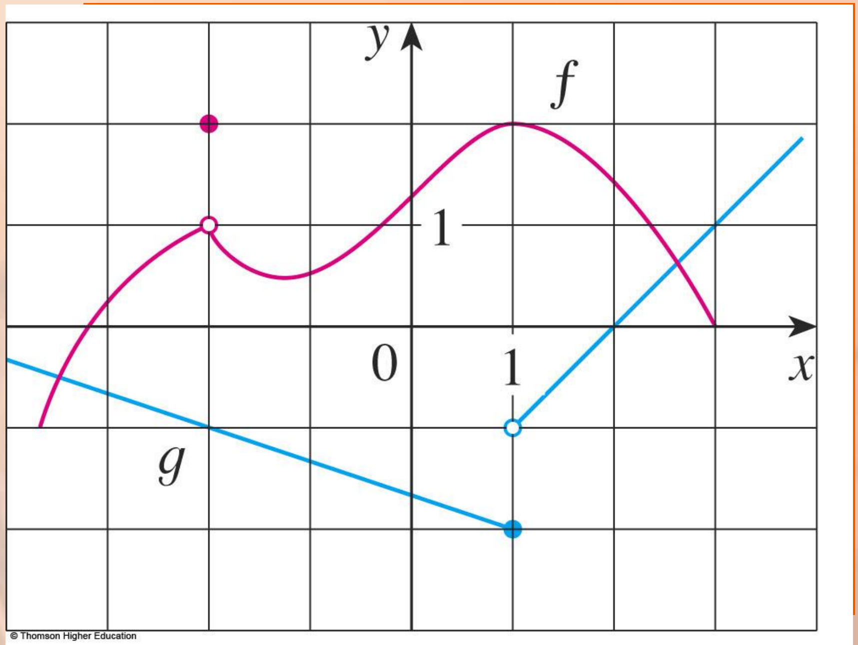
Example 1

Use the Limit Laws and the graphs of *f* and *g* in the figure to evaluate the following limits, if they exist.

a. $\lim_{x \to -2} [f(x) + 5g(x)]$

b. $\lim_{x \to 1} [f(x)g(x)]$

c. $\lim_{x \to 2} \frac{f(x)}{g(x)}$



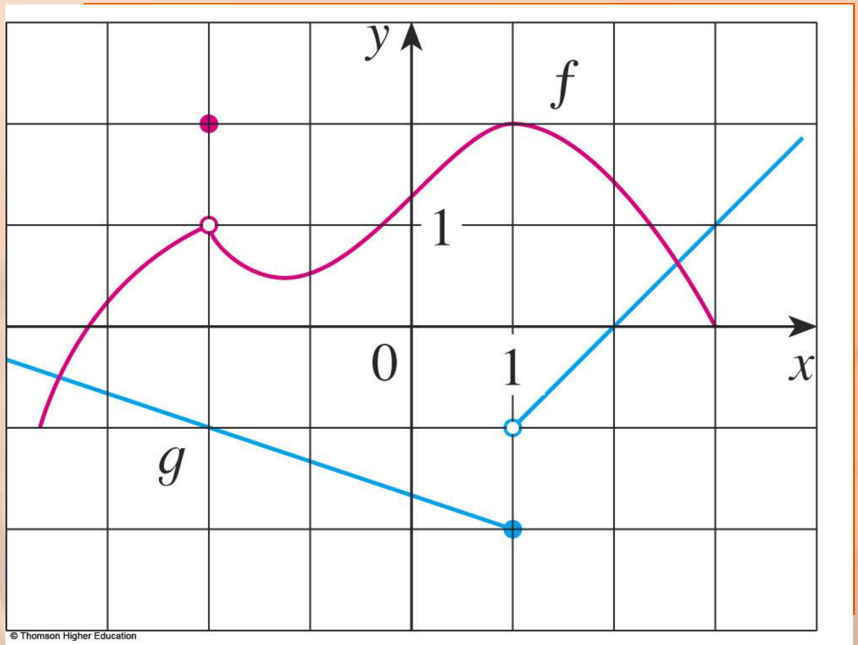
Example 1 a

From the graphs, we see that

$$\lim_{x\to -2} f(x) = 1 \text{ and } \lim_{x\to -2} g(x) = -1.$$

- Therefore, we have:

$$\lim_{x\to -2}\left[f(x) + 5g(x)\right]$$
$$= \lim_{x\to -2} f(x) + \lim_{x\to -2}\left[5g(x)\right]$$
$$= \lim_{x\to -2} f(x) + 5\lim_{x\to -2}\left[g(x)\right]$$
$$= 1 + 5(-1) = -4$$



USING THE LIMIT LAWS

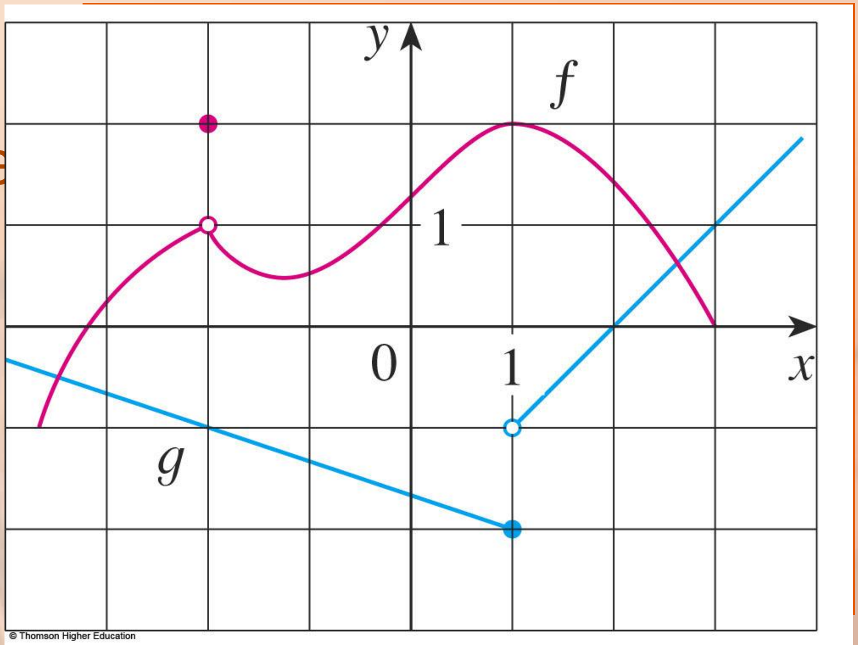
Example 1 b

We see that $\lim_{x \to 1} f(x) = 2$.

However, $\lim_{x \to 1} g(x)$ does not exist—because the left and right limits are different:

$\lim_{x \to 1^-} g(x) = -2$ and $\lim_{x \to 1^+} g(x) = -1$

- So, we can't use the Product Law for the desired limit.



USING THE LIMIT LAWS

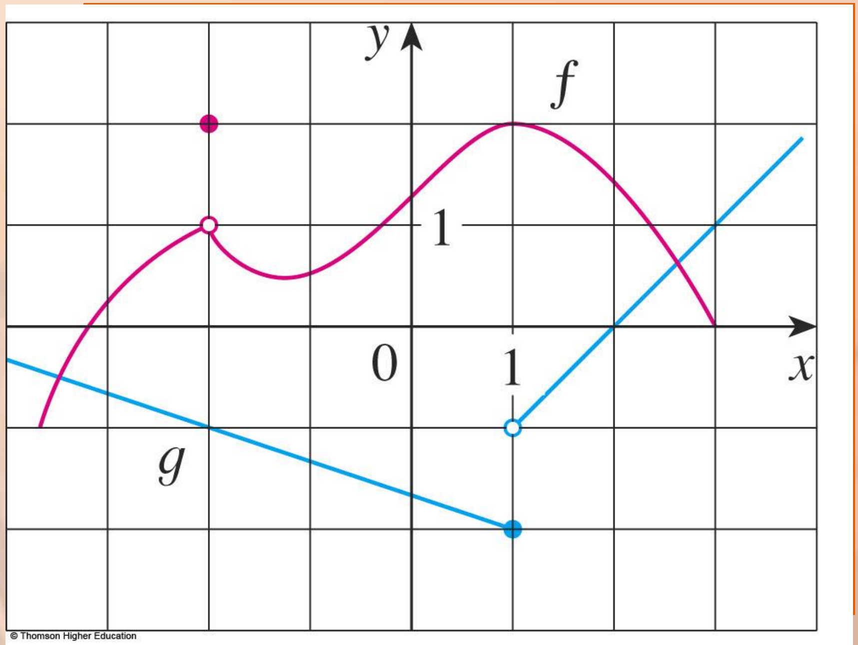
Example 1 b

However, we can use the Product Law for the one-sided limits:

$$\lim_{x \to 1^-}[f(x)g(x)] = 2 \cdot (-2) = -4 \text{ and}$$

$$\lim_{x \to 1^+}[f(x)g(x)] = 2 \cdot (-1) = -2$$

- The left and right limits aren't equal.
- So, $\lim_{x \to 1}[f(x)g(x)]$ does not exist.

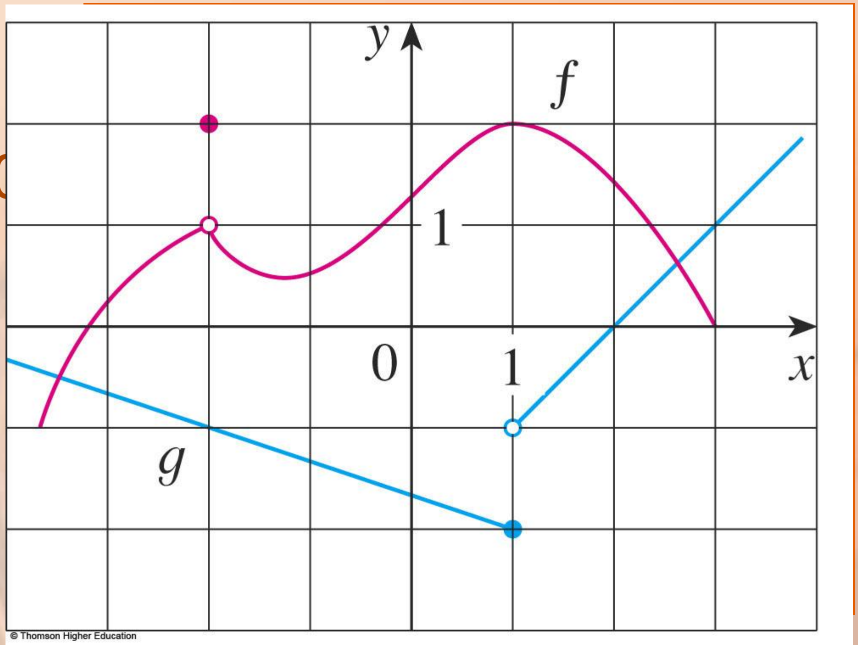


Example 1 c

The graphs show that $\lim_{x \to 2} f(x) \approx 1.4$ and $\lim_{x \to 2} g(x) = 0$.

As the limit of the denominator is 0, we can't use the Quotient Law.

- $\lim_{x \to 2} \frac{f(x)}{g(x)}$ does not exist.
- This is because the denominator approaches 0 while the numerator approaches a nonzero number.



THE POWER LAW

If we use the Product Law repeatedly with *f*(*x*) = *g*(*x*), we obtain the Power Law.

$$6.\ \lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$$

where *n* is a positive integer

In applying these six limit laws, we need to use two special limits.

$$7.\ \lim_{x \to a} c = c$$

$$8.\ \lim_{x \to a} x = a$$

- These limits are obvious from an intuitive point of view.
- State them in words or draw graphs of $y = c$ and $y = x$.

If we now put $f(x) = x$ in the Power Law and use Law 8, we get another useful special limit.

$$9. \lim_{x \to a} x^n = a^n$$

where n is a positive integer.

A similar limit holds for roots.

$$10. \lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$$

- If n is even, we assume that $a > 0$.

More generally, we have the Root Law.

$$11. \lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$$

where *n* is a positive integer.

- If *n* is even, we assume that $\lim_{x \to a} f(x) > 0$.

Example 2

Evaluate the following limits and justify each step.

a. $\lim_{x \to 5} (2x^2 - 3x + 4)$

b. $\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$

Example 2 a

$$\lim_{x \to 5}(2x^2 - 3x + 4)$$

$$= \lim_{x \to 5}(2x^2) - \lim_{x \to 5} 3x + \lim_{x \to 5} 4$$ (by Laws 2 and 1)

$$= 2\lim_{x \to 5} x^2 - 3\lim_{x \to 5} x + \lim_{x \to 5} 4$$ (by Law 3)

$$= 2(5^2) - 3(5) + 4$$ (by Laws 9, 8, and 7)

$$= 39$$

Example 2 b

We start by using the Quotient Law. However, its use is fully justified only at the final stage.

- That is when we see that the limits of the numerator and denominator exist and the limit of the denominator is not 0.

USING THE LIMIT LAWS

Example 2 b

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$$

$$= \frac{\lim_{x \to -2} (x^3 + 2x^2 - 1)}{\lim_{x \to -2} (5 - 3x)}$$

(by Law 5)

$$= \frac{\lim_{x \to -2} x^3 + 2 \lim_{x \to -2} x^2 - \lim_{x \to -2} 1}{\lim_{x \to -2} 5 - 3 \lim_{x \to -2} x}$$

(by Laws 1, 2, and 3)

$$= \frac{(-2)^3 - 2(-2)^2 - 1}{5 - 3(-2)} = -\frac{1}{11}$$

(by Laws 9, 8, and 7)

Note

If we let $f(x) = 2x^2 - 3x + 4$,

then $f(5) = 39$.

- In other words, we would have gotten the correct answer in Example 2 a by substituting 5 for *x*.
- Similarly, direct substitution provides the correct answer in Example 2 b.

Note

The functions in the example are a polynomial and a rational function, respectively.

- Similar use of the Limit Laws proves that direct substitution always works for such functions.

DIRECT SUBSTITUTION PROPERTY

We state this fact as follows.

If *f* is a polynomial or a rational function and *a* is in the domain of *f*, then

$$\lim_{x \to a} f(x) = f(a)$$

DIRECT SUBSTITUTION PROPERTY

Functions with the Direct Substitution Property are called ‘continuous at *a*.’

However, not all limits can be evaluated by direct substitution—as the following examples show.

Example 3

Find $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$.

- Let $f(x) = (x^2 - 1)/(x - 1)$.
- We can't find the limit by substituting $x = 1$ because $f(1)$ isn't defined.
- We can't apply the Quotient Law because the limit of the denominator is 0.
- Instead, we need to do some preliminary algebra.

We factor the numerator as a difference of squares.

$$\frac{x^2-1}{x-1}=\frac{(x-1)(x+1)}{(x-1)}$$

- The numerator and denominator have a common factor of *x* - 1.
- When we take the limit as *x* approaches 1, we have $x \neq 1$ and so $x-1 \neq 0$.

USING THE LIMIT LAWS

Example 3

- Therefore, we can cancel the common factor and compute the limit as follows:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x-1)(x+1)}{(x-1)}$$

$$= \lim_{x \to 1} (x+1)$$

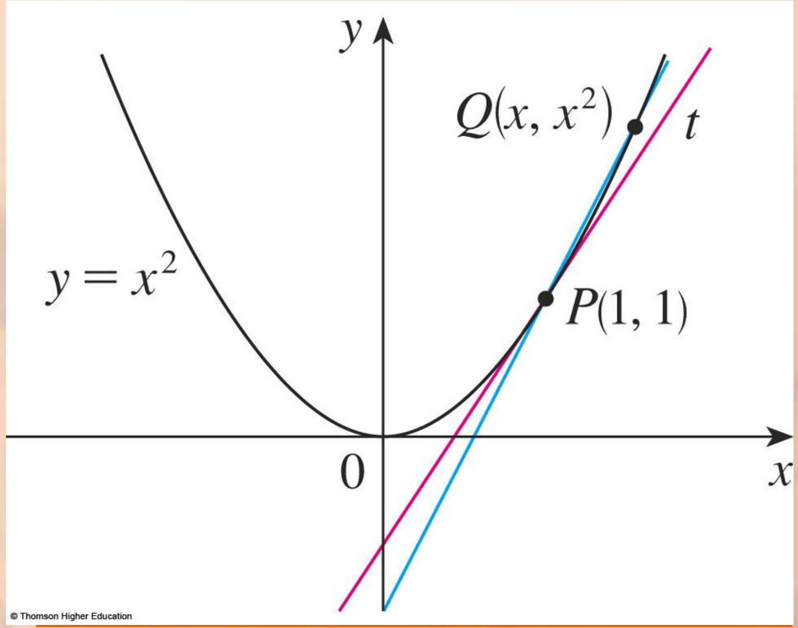
$$= 1 + 1$$

$$= 2$$

USING THE LIMIT LAWS

Example 3

The limit in the example arose in Section 2.1 when we were trying to find the tangent to the parabola $y = x^2$ at the point (1, 1).



Note

In the example, we were able to compute the limit by replacing the given function $f(x) = (x^2 - 1)/(x - 1)$ by a simpler function with the same limit, $g(x) = x + 1$.

- This is valid because $f(x) = g(x)$ except when $x = 1$ and, in computing a limit as x approaches 1, we don't consider what happens when x is actually equal to 1.

Note

In general, we have the following useful fact.

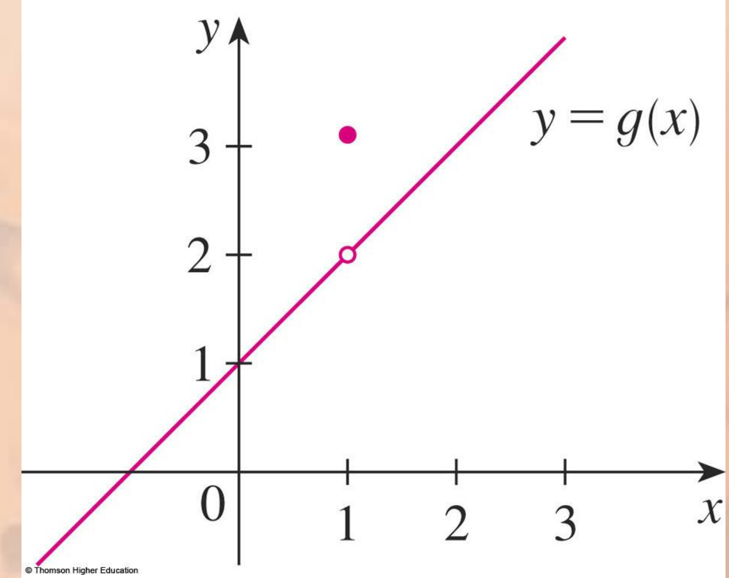
If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$, provided the limits exist.

USING THE LIMIT LAWS

Example 4

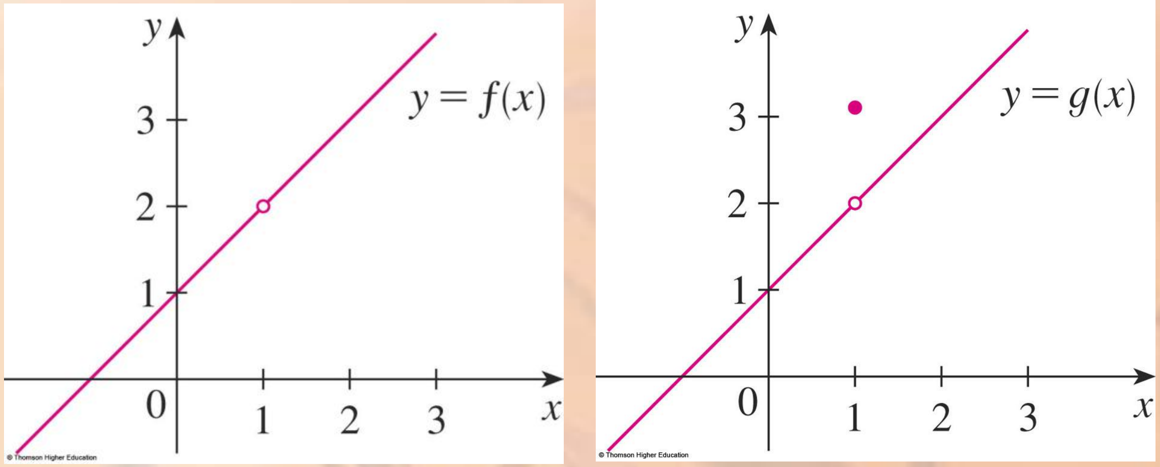
Find $\lim_{x \to 1} g(x)$ where $g(x) = \begin{cases} x+1 & if \quad x \neq 1 \\ \pi & if \quad x = 1 \end{cases}$.

- Here, *g* is defined at *x* = 1 and $g(1) = \pi$.
- However, the value of a limit as *x* approaches 1 does not depend on the value of the function at 1.
- Since *g*(*x*) = *x* + 1 for $x \neq 1$, we have $\lim_{x \to 1} g(x) = \lim_{x \to 1}(x+1) = 2$.



Note that the values of the functions in Examples 3 and 4 are identical except when $x = 1$.

So, they have the same limit as x approaches 1.



Example 5

Evaluate $\lim_{h \to 0} \dfrac{(3+h)^2 - 9}{h}$.

- If we define $F(h) = \dfrac{(3+h)^2 - 9}{h}$, then, we can't compute $\lim_{h \to 0} F(h)$ by letting *h* = 0 since *F*(0) is undefined.
- However, if we simplify *F*(h) algebraically, we find that:

$$F(h) = \frac{(9 + 6h + h^2) - 9}{h} = \frac{6h + h^2}{h} = 6 + h$$

USING THE LIMIT LAWS

Example 5

- Recall that we consider only $h \neq 0$ when letting *h* approach 0.
- Thus,

$$\lim_{h \to 0} \frac{(3+h)^2 - 9}{h}$$

$$= \lim_{h \to 0} (6+h)$$

$$= 6$$

Example 6

Find $\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2}$.

- We can't apply the Quotient Law immediately—since the limit of the denominator is 0.
- Here, the preliminary algebra consists of rationalizing the numerator.

USING THE LIMIT LAWS

Example 6

- Thus,

$$\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2}$$

$$= \lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2} \cdot \frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3}$$

$$= \lim_{t \to 0} \frac{(t^2+9)-9}{t^2(\sqrt{t^2+9}+3)}$$

$$= \lim_{t \to 0} \frac{t^2}{t^2(\sqrt{t^2+9}+3)}$$

$$= \lim_{t \to 0} \frac{1}{\sqrt{t^2+9}+3}$$

$$= \frac{1}{\sqrt{\lim_{t \to 0}(t^2+9)}+3} = \frac{1}{3+3} = \frac{1}{6}$$

Theorem 1

Some limits are best calculated by first finding the left- and right-hand limits.

The following theorem states that a two-sided limit exists if and only if both the one-sided limits exist and are equal.

$$\lim_{x \to a} f(x) = L \text{ if and only if } \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

- When computing one-sided limits, we use the fact that the Limit Laws also hold for one-sided limits.

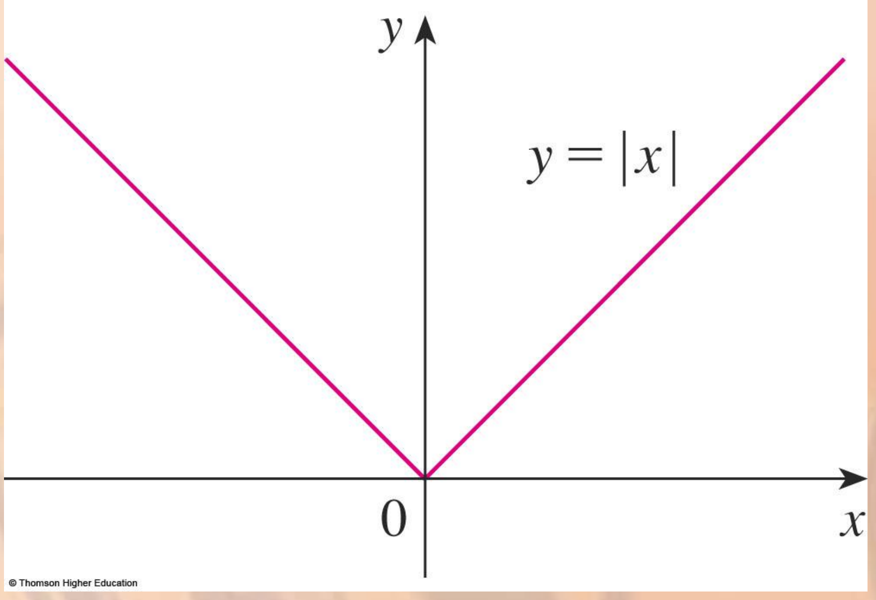
Example 7

Show that $\lim_{x \to 0} |x| = 0$.

- Recall that: $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$
- Since |*x*| = *x* for *x* > 0 , we have: $\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0$
- Since |*x*| = -*x* for *x* < 0, we have: $\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0$
- Therefore, by Theorem 1, $\lim_{x \to 0} |x| = 0$.

Example 7

The result looks plausible from the figure.



Example 8

Prove that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} 1 = 1$$

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = \lim_{x \to 0^-} (-1) = -1$$

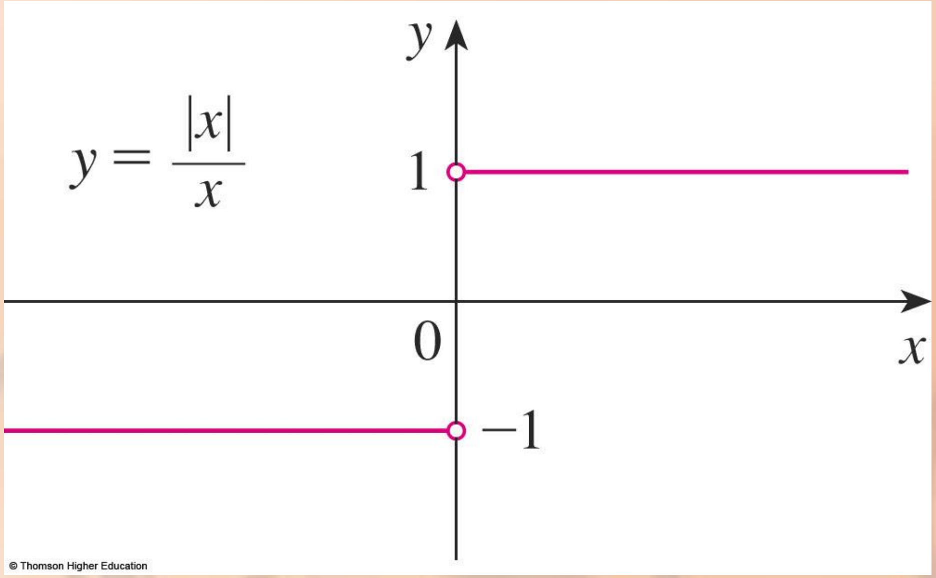
- Since the right- and left-hand limits are different, it follows from Theorem 1 that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

USING THE LIMIT LAWS

Example 8

The graph of the function $f(x) = |x|/x$ is shown in the figure.

It supports the one-sided limits that we found.



USING THE LIMIT LAWS

Example 9

If $f(x) = \begin{cases} \sqrt{x-4} & \text{if } x > 4 \\ 8-2x & \text{if } x < 4 \end{cases}$

determine whether $\lim_{x \to 4} f(x)$ exists.

- Since $f(x) = \sqrt{x-4}$ for $x > 4$, we have:

$$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \sqrt{x-4}$$

$$= \sqrt{4-4} = 0$$

- Since $f(x) = 8 - 2x$ for $x < 4$, we have:

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} (8-2x)$$

$$= 8 - 2 \cdot 4 = 0$$

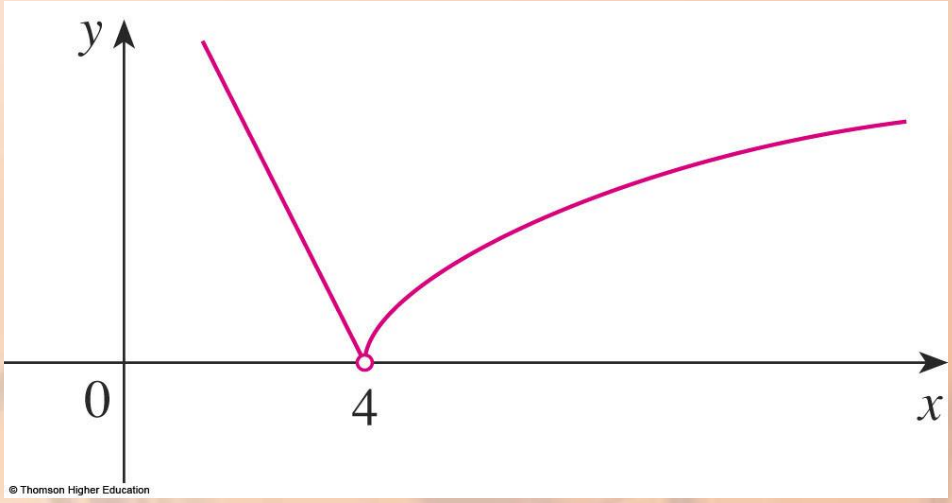
Example 9

- The right- and left-hand limits are equal.
- Thus, the limit exists and

$$\lim_{x \to 4} f(x) = 0.$$

Example 9

The graph of *f* is shown in the figure.



GREATEST INTEGER FUNCTION

The greatest integer function is defined by $[\![x]\!]$ = the largest integer that is less than or equal to *x*.

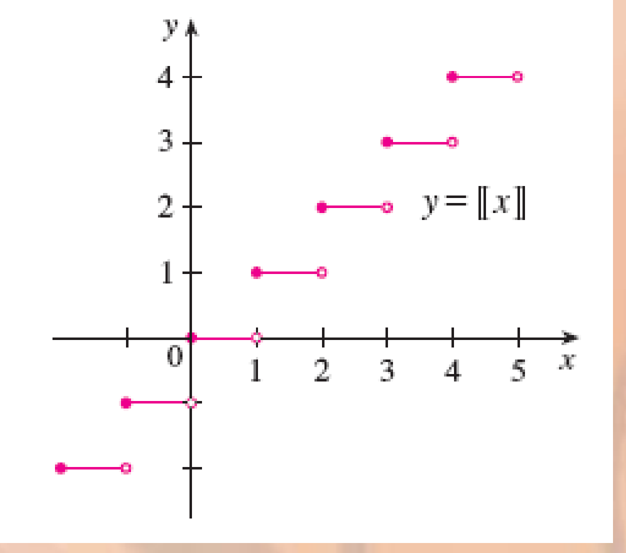
- For instance, $[\![4]\!] = 4$, $[\![4.8]\!] = 4$, $[\![\pi]\!] = 3$, $[\![\sqrt{2}]\!] = 1$, and $[\![-\frac{1}{2}]\!] = -1$.
- The greatest integer function is sometimes called the floor function.

USING THE LIMIT LAWS

Example 10

Show that $\lim_{x \to 3} [\![x]\!]$ does not exist.

- The graph of the greatest integer function is shown in the figure.



USING THE LIMIT LAWS

Example 10

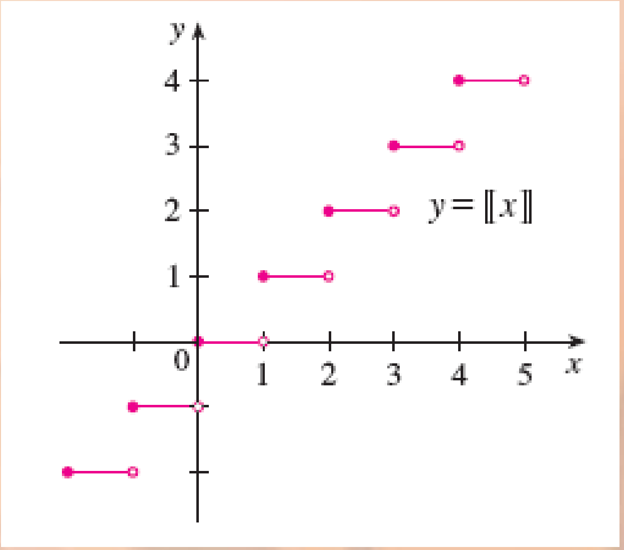
- Since $[\![x]\!] = 3$ for $3 \le x < 4$, we have:

$$\lim_{x \to 3^+} x = \lim_{x \to 3^+} 3 = 3$$

- Since $[\![x]\!] = 2$ for $2 \le x < 3$, we have:

$$\lim_{x \to 3^-} [\![x]\!] = \lim_{x \to 3^-} 2 = 2$$

- As these one-sided limits are not equal, $\lim_{x \to 3} [\![x]\!]$ does not exist by Theorem 1.



The next two theorems give two additional properties of limits.

Theorem 2

If $f(x) \le g(x)$ when *x* is near *a* (except possibly at *a*) and the limits of *f* and *g* both exist as *x* approaches *a*, then

$$\lim_{x \to a} f(x) \le \lim_{x \to a} g(x)$$

Theorem 3

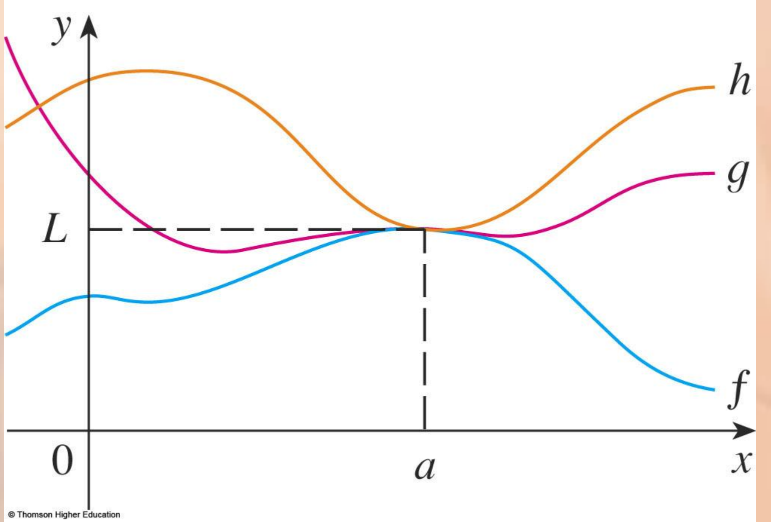
The Squeeze Theorem states that, if $f(x) \le g(x) \le h(x)$ when *x* is near (except possibly at *a*) and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then

$$\lim_{x \to a} g(x) = L$$

- The Squeeze Theorem is sometimes called the Sandwich Theorem or the Pinching Theorem.

The theorem is illustrated by the figure.

- It states that, if $g(x)$ is squeezed between $f(x)$ and $h(x)$ near a and if f and h have the same limit L at a, then g is forced to have the same limit L at a.



Example 11

Show that $\lim_{x \to 0} = x^2 \sin \frac{1}{x} = 0.$

- Note that we cannot use $\lim_{x \to 0} x^2 \sin \frac{1}{x} = \lim_{x \to 0} x^2 \cdot \lim_{x \to 0} \sin \frac{1}{x}$
- This is because $\lim_{x \to 0} \sin(1/x)$ does not exist.

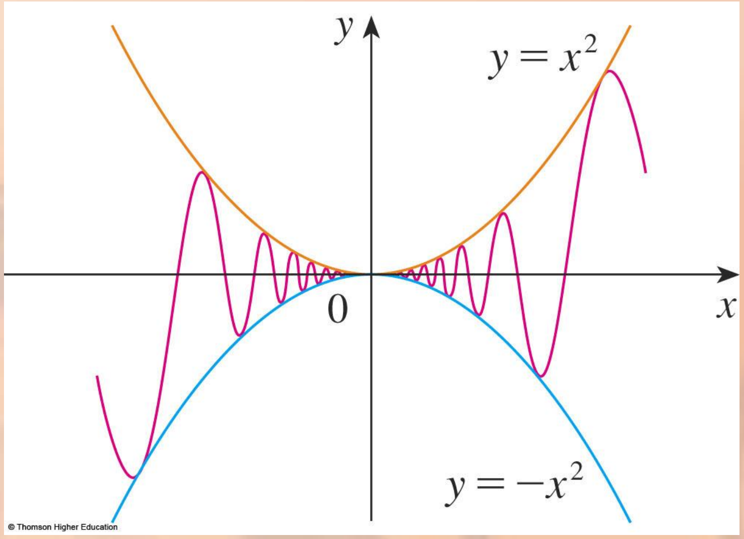
USING THE LIMIT LAWS

Example 11

- However, since $-1 \le \sin\frac{1}{x} \le 1$, we have:

$$-x^2 \le x^2 \sin\frac{1}{x} \le x^2$$

- This is illustrated by the figure.



USING THE LIMIT LAWS

Example 11

- We know that: $\lim_{x \to 0} x^2 = 0$ and $\lim_{x \to 0}(-x^2) = 0$

- Taking $f(x) = -x^2$, $g(x) = x^2 \sin(1/x)$, and $h(x) = x^2$ in the Squeeze Theorem, we obtain:

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$$