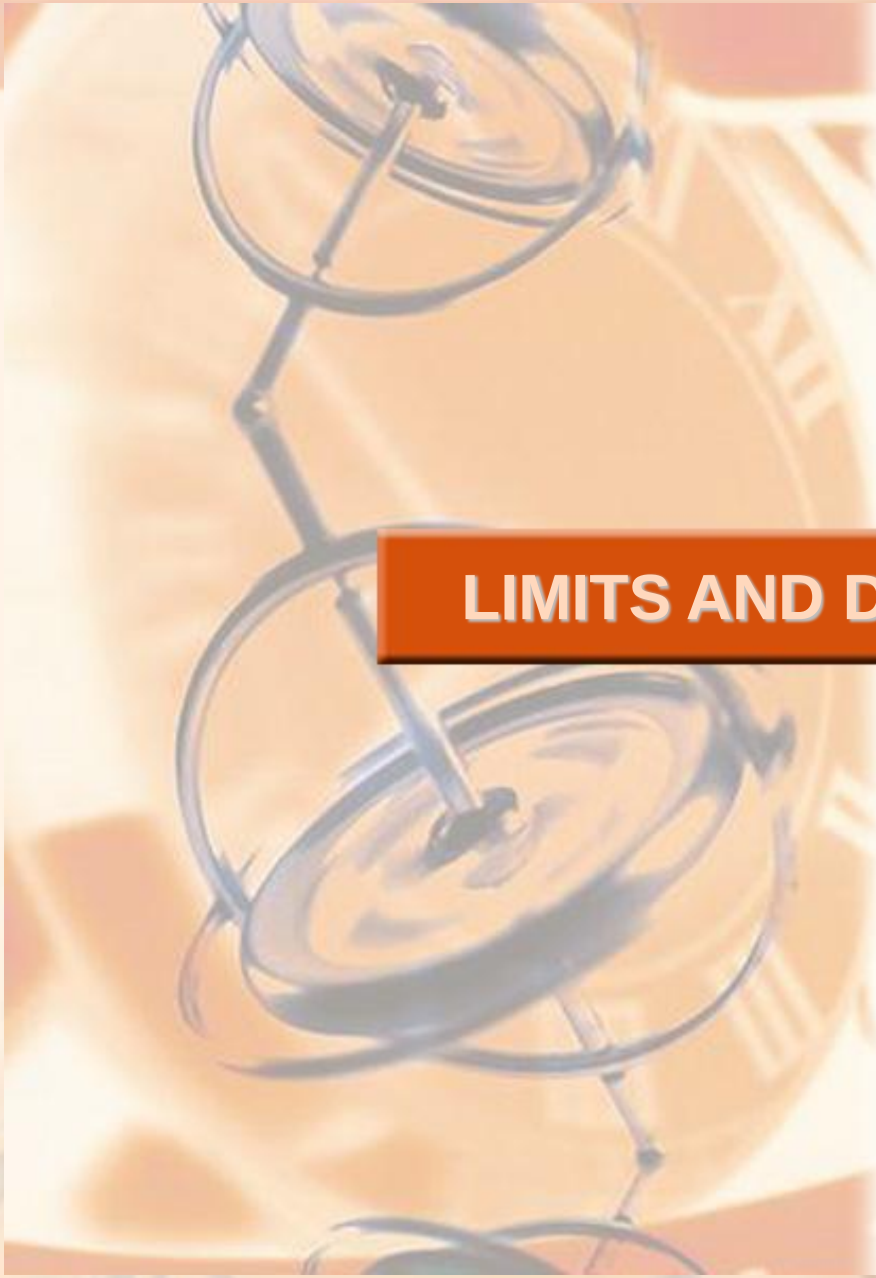


2

LIMITS AND DERIVATIVES

The idea of a limit underlies the various branches of calculus.

- It is therefore appropriate to begin our study of calculus by investigating limits and their properties.
- The special type of limit used to find tangents and velocities gives rise to the central idea in differential calculus—the derivative.

2.1

The Tangent and Velocity Problems

In this section, we will learn:

How limits arise when we attempt to find the tangent to a curve or the velocity of an object.

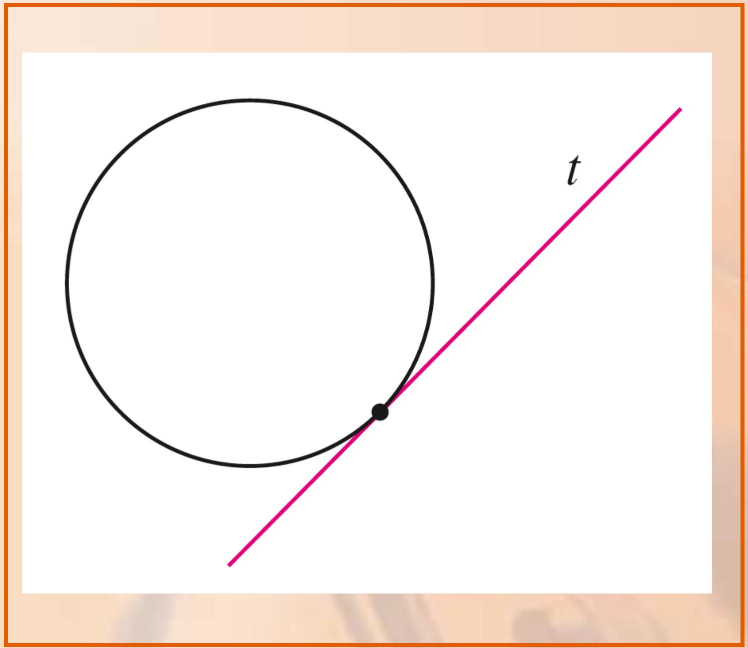
THE TANGENT PROBLEM

The word tangent is derived from the Latin word tangens, which means ‘touching.’ Thus, a tangent to a curve is a line that touches the curve.

- In other words, a tangent line should have the same direction as the curve at the point of contact.

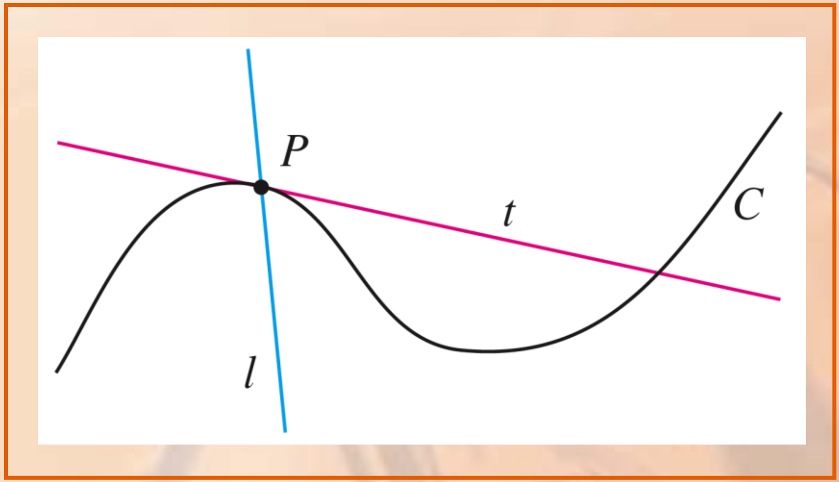
THE TANGENT PROBLEM

For a circle, we could simply follow Euclid and say that a tangent is a line that intersects the circle once and only once.



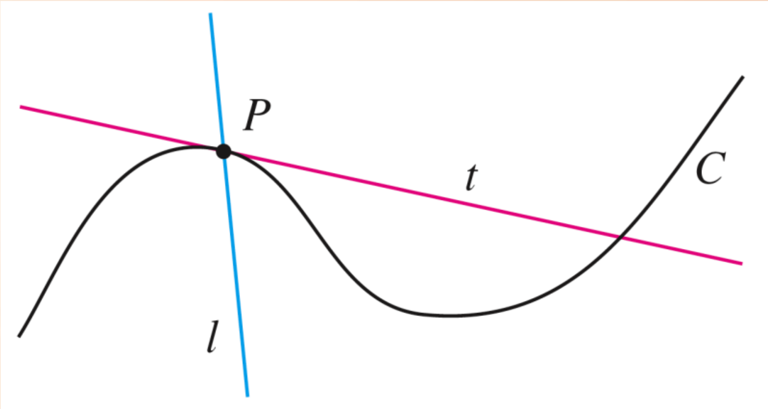
For more complicated curves, that definition is inadequate.

- The figure displays two lines *l* and *t* passing through a point *P* on a curve.
- The line *l* intersects only once, but it certainly does not look like what is thought of as a tangent.



THE TANGENT PROBLEM

- In contrast, the line *t* looks like a tangent, but it intersects twice.



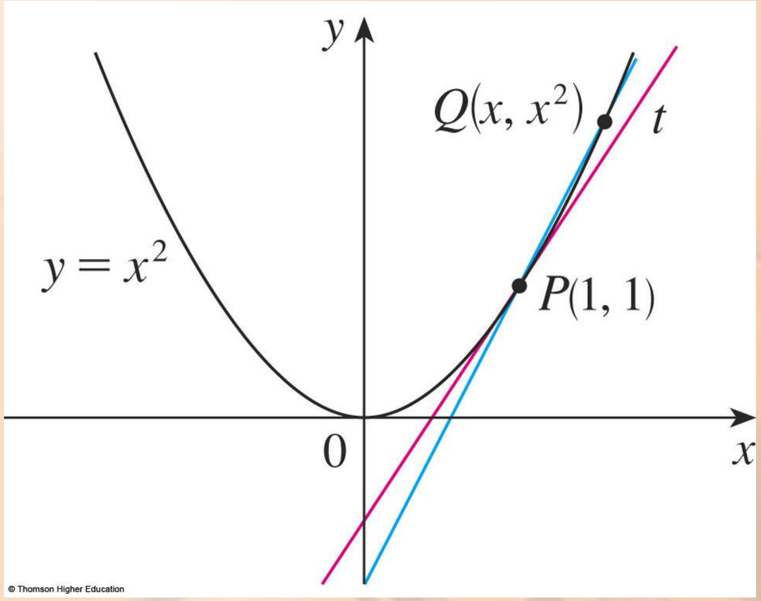
Example 1

Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1,1)$.

- We will be able to find an equation of the tangent line as soon as we know its slope *m*.
- The difficulty is that we know only one point, *P*, on *t*, whereas we need two points to compute the slope.

Example 1

However, we can compute an approximation to m by choosing a nearby point *Q*(x, x^2) on the parabola and computing the slope m_{PQ} of the secant line *PQ*.



We choose $x \neq 1$ so that $Q \neq P$.

- Then,

$$m_{PQ} = \frac{x^2 - 1}{x - 1}$$

- For instance, for the point $Q(1.5, 2.25)$, we have:

$$m_{PQ} = \frac{2.25 - 1}{1.5 - 1} = \frac{1.25}{0.5} = 2.5$$

Example 1

The tables below the values of m_{PQ} for several values of x close to 1. The closer Q is to P, the closer x is to 1 and, it appears from the tables, the closer m_{PQ} is to 2.

- This suggests that the slope of the tangent line t should be $m = 2$.

x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

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Example 1

The slope of the tangent line is said to be the limit of the slopes of the secant lines. This is expressed symbolically as follows.

$$\lim_{Q \to P} m_{PQ} = m$$

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

Example 1

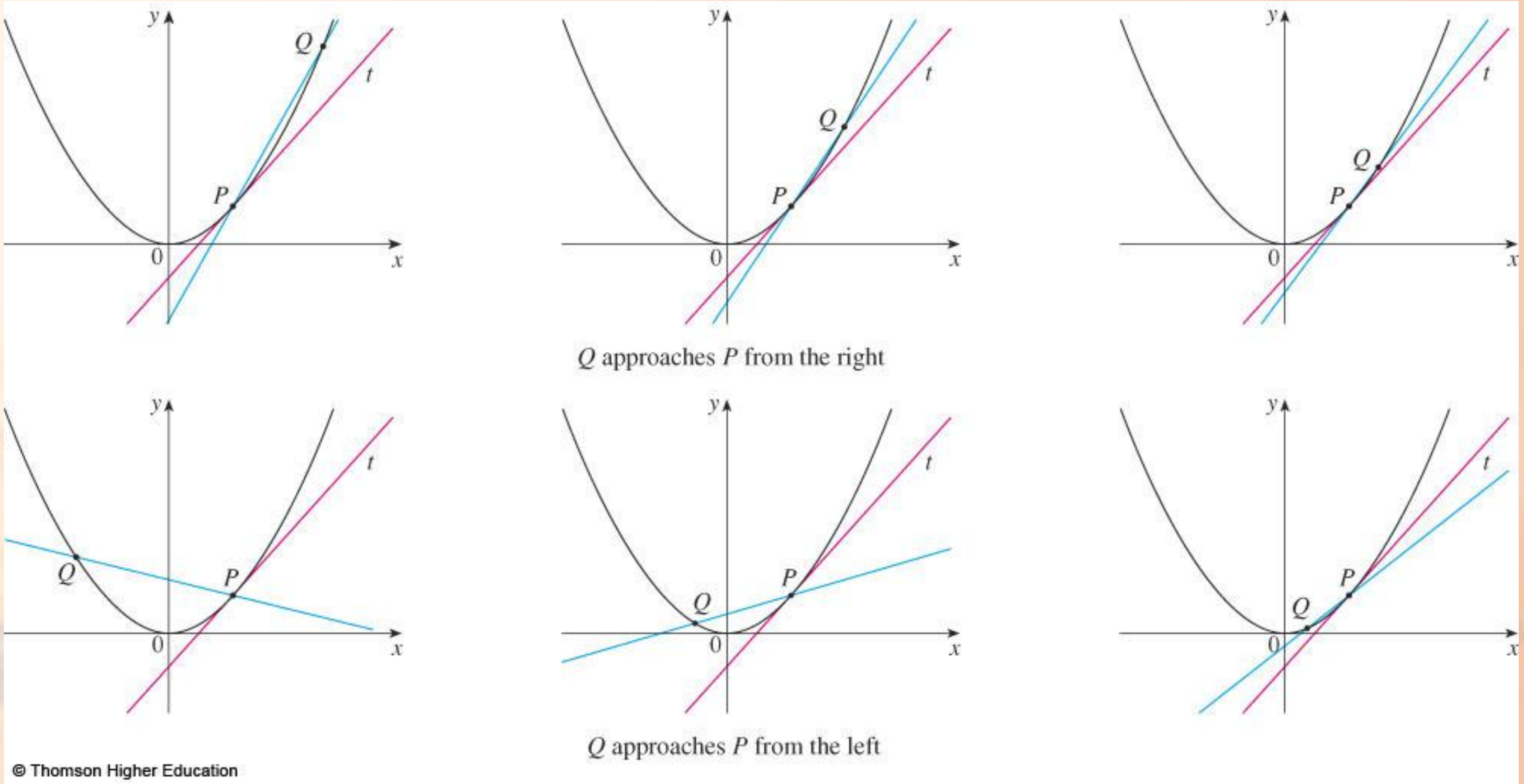
Assuming that the slope of the tangent line is indeed 2, we can use the point-slope form of the equation of a line to write the equation of the tangent line through (1, 1) as:

$y - 1 = 2(x - 1)$ or $y = 2x - 1$

THE TANGENT PROBLEM

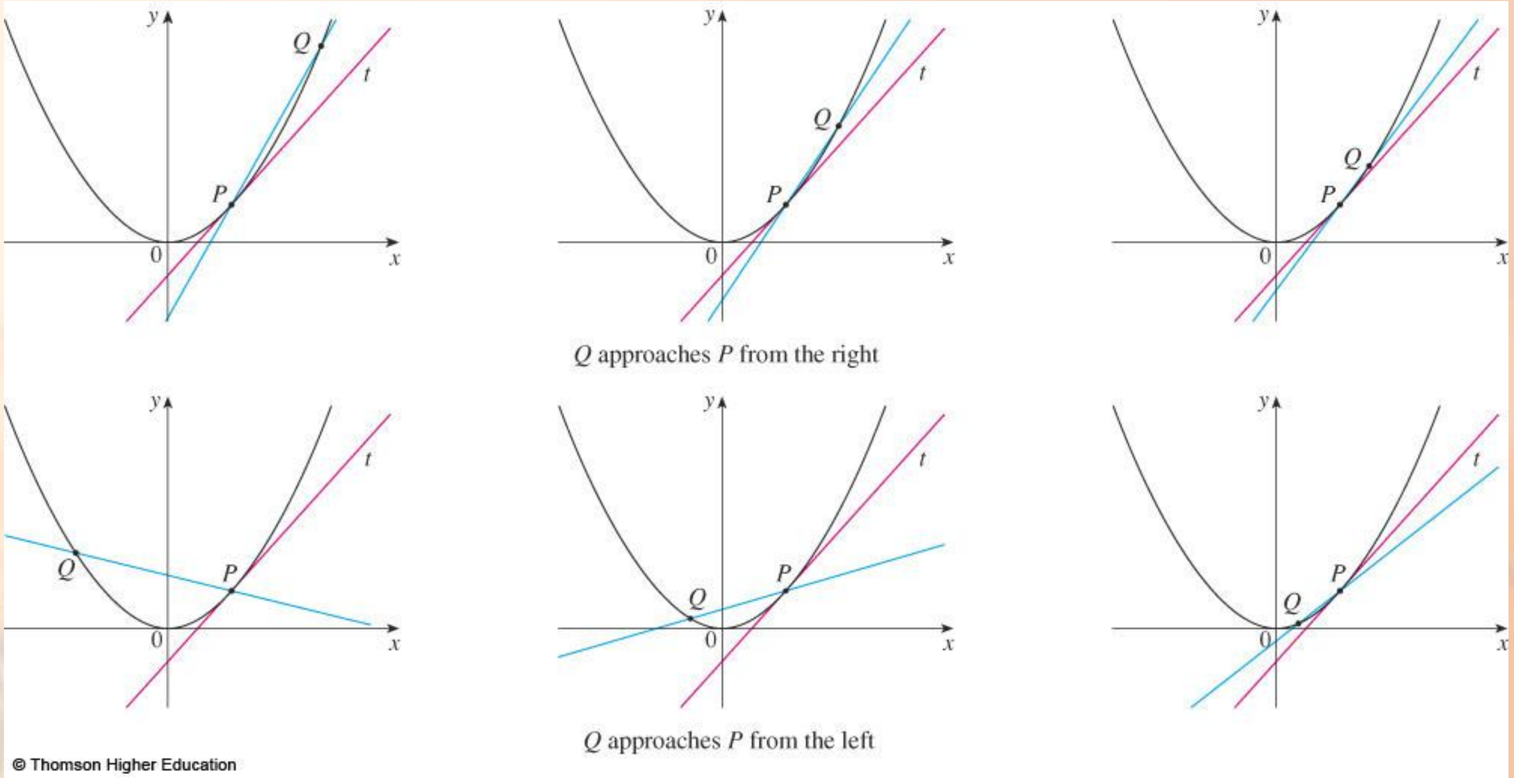
Example 1

The figure illustrates the limiting process that occurs in this example.



Example 1

As *Q* approaches *P* along the parabola, the corresponding secant lines rotate about *P* and approach the tangent line *t*.



THE TANGENT PROBLEM

Many functions that occur in science are not described by explicit equations, but by experimental data.

- The next example shows how to estimate the slope of the tangent line to the graph of such a function.

Example 2

The flash unit on a camera operates by storing charge on a capacitor and releasing it suddenly when the flash is set off.

- The data in the table describe the charge *Q* remaining on the capacitor (measured in microcoulombs) at time *t* (measured in seconds after the flash goes off).

t	Q
0.00	100.00
0.02	81.87
0.04	67.03
0.06	54.88
0.08	44.93
0.10	36.76

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THE TANGENT PROBLEM

Example 2

Using the data, you can draw the graph of this function and estimate the slope of the tangent line at the point where $t = 0.04$.

- Remember, the slope of the tangent line represents the electric current flowing from the capacitor to the flash bulb measured in microamperes.

t	Q
0.00	100.00
0.02	81.87
0.04	67.03
0.06	54.88
0.08	44.93
0.10	36.76

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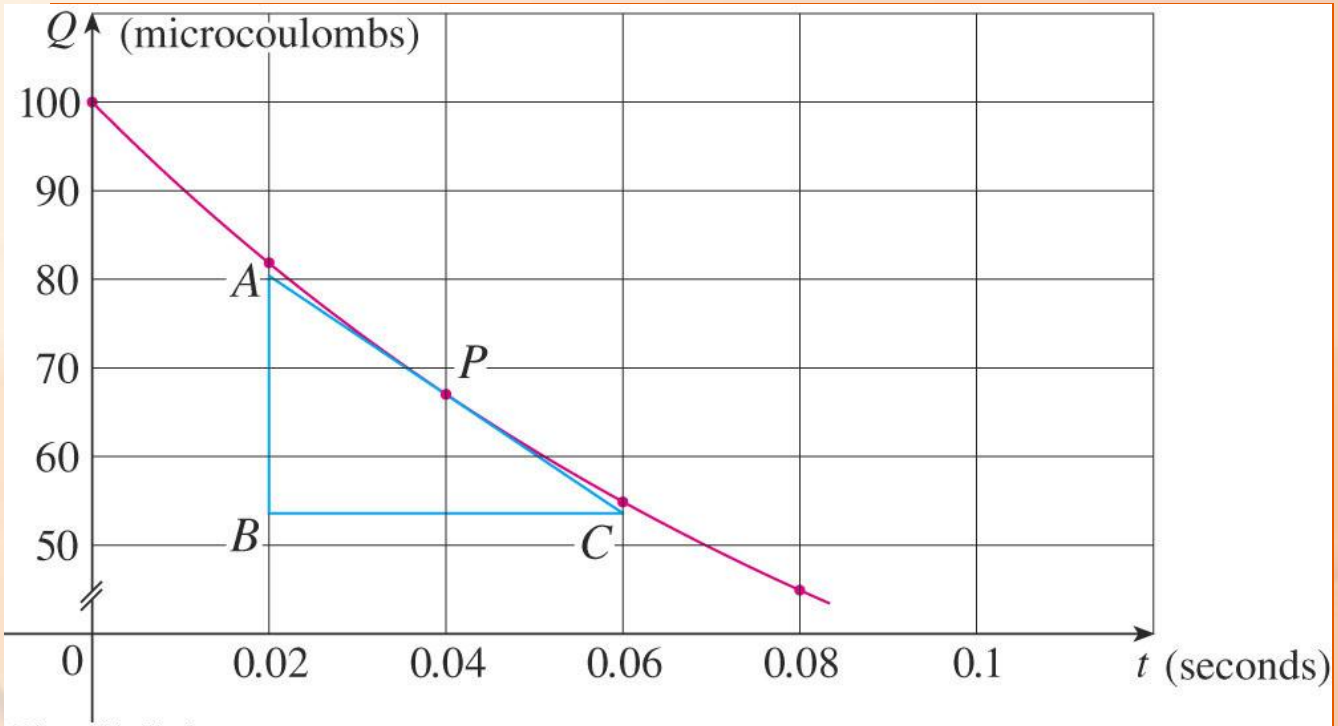
Example 2

In the figure, the given data are plotted and used to sketch a curve that approximates the graph of the function.



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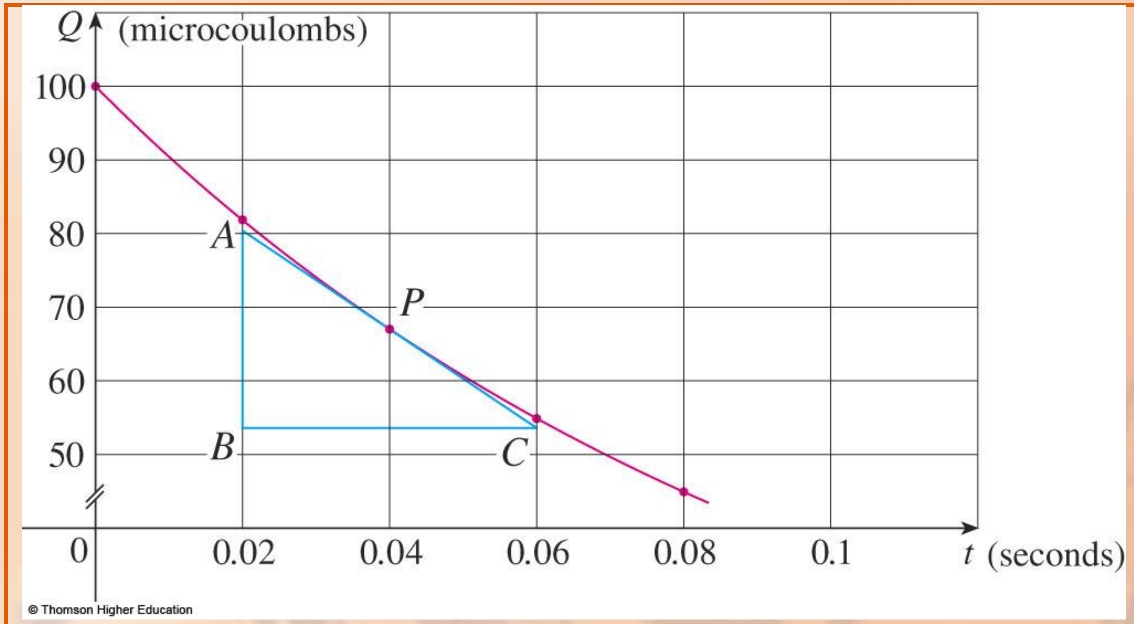
Example 2

Given the points *P*(0.04, 67.03) and *R*(0.00, 100.00), we find that the slope of the secant line *PR* is:

$$m_{PR} = \frac{100.00 - 67.03}{0.00 - 0.04} = -824.25$$



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Example 2

The table shows the results of similar calculations for the slopes of other secant lines.

- From this, we would expect the slope of the tangent line at $t = 0.04$ to lie somewhere between –742 and –607.5.

R	m_{PR}
(0.00, 100.00)	−824.25
(0.02, 81.87)	−742.00
(0.06, 54.88)	−607.50
(0.08, 44.93)	−552.50
(0.10, 36.76)	−504.50

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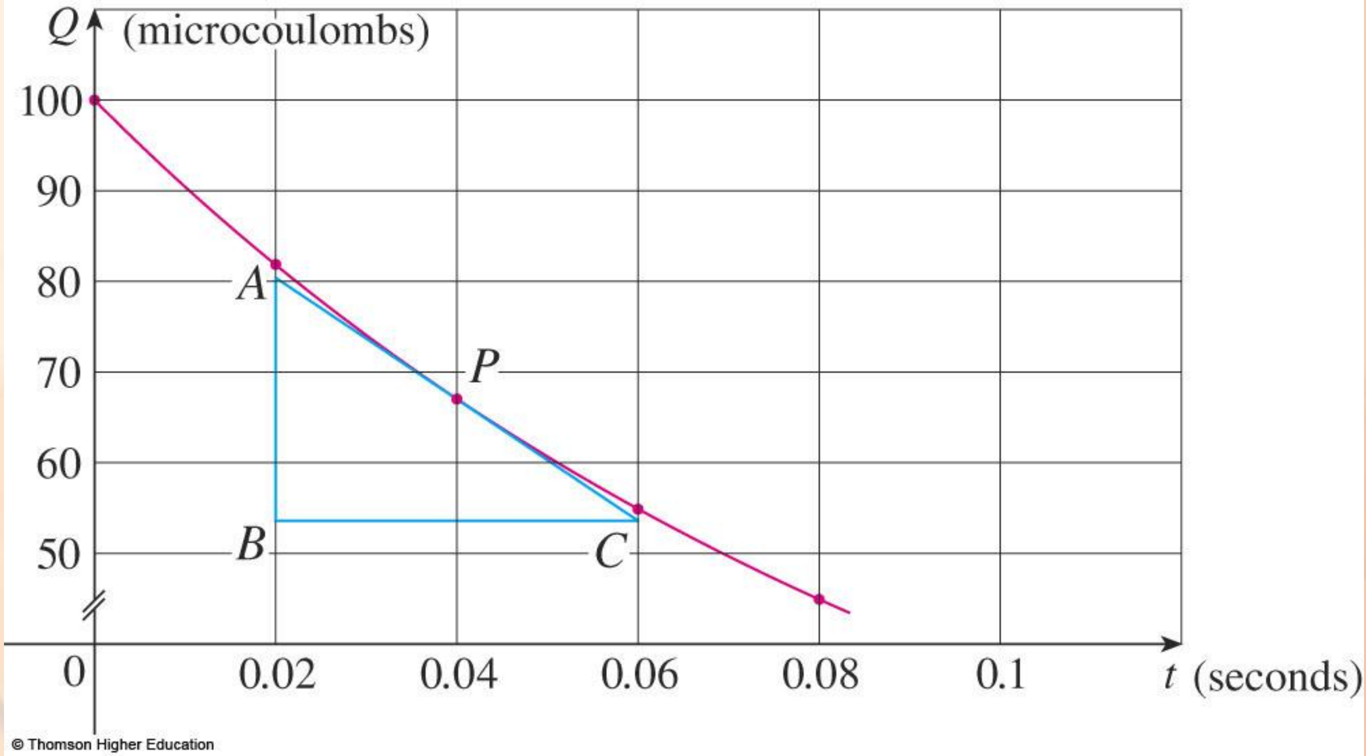
In fact, the average of the slopes of the two closest secant lines is:

$$\frac{1}{2}\left(-742-607.5\right)=-674.5$$

So, by this method, we estimate the slope of the tangent line to be –675.

Example 2

Another method is to draw an approximation to the tangent line at *P* and measure the sides of the triangle *ABC*.



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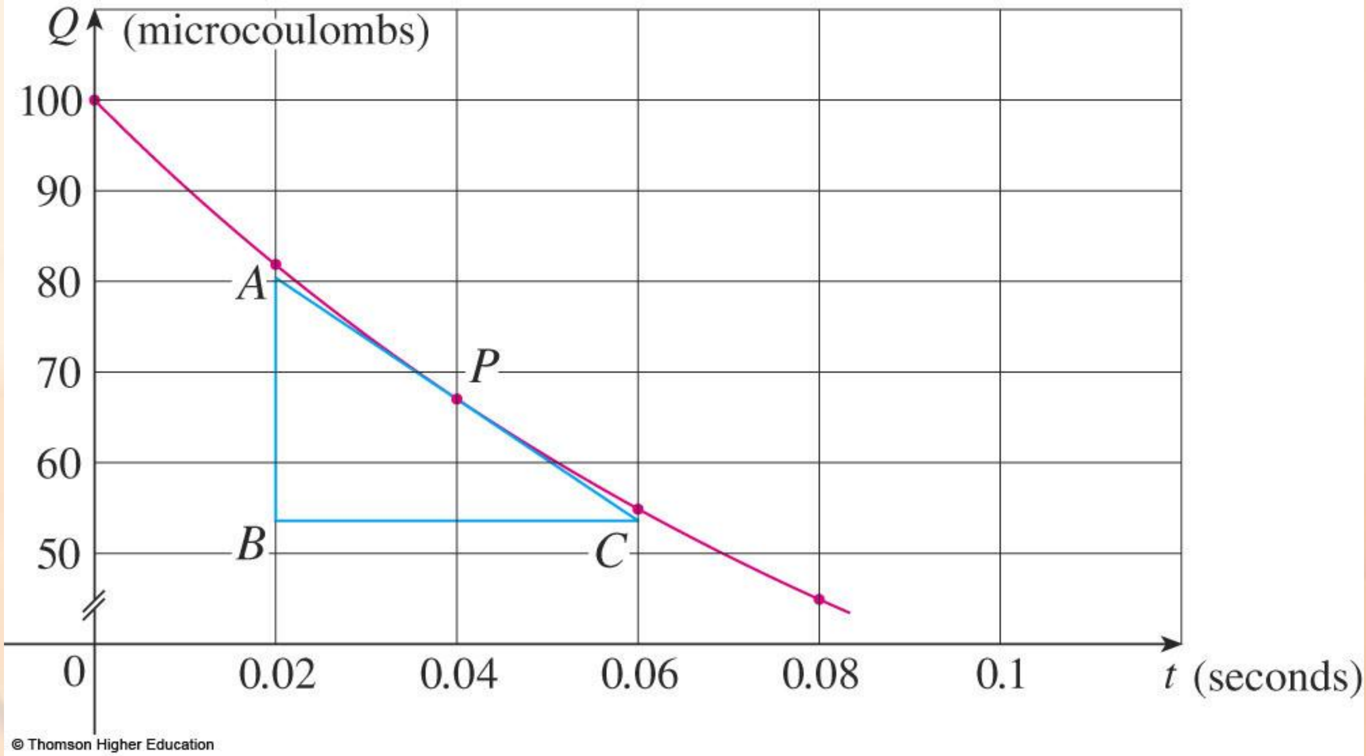
Example 2

This gives an estimate of the slope of the tangent line as:

$$-\frac{|AB|}{|BC|} \approx -\frac{80.4-53.6}{0.06-0.02} = -670$$



THE VELOCITY PROBLEM

If you watch the speedometer of a car as you travel in city traffic, you see that the needle doesn't stay still for very long. That is, the velocity of the car is not constant.

- We assume from watching the speedometer that the car has a definite velocity at each moment.
- How is the 'instantaneous' velocity defined?

Example 3

Investigate the example of a falling ball.

- Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground.
- Find the velocity of the ball after 5 seconds.

THE VELOCITY PROBLEM

Example 3

Through experiments carried out four centuries ago, Galileo discovered that the distance fallen by any freely falling body is proportional to the square of the time it has been falling.

- Remember, this model neglects air resistance.

Example 3

If the distance fallen after *t* seconds is denoted by *s(t)* and measured in meters, then Galileo's law is expressed by the following equation.

$$s(t) = 4.9t^2$$

Example 3

The difficulty in finding the velocity after 5 s is that you are dealing with a single instant of time ($t = 5$).

- No time interval is involved.

 Example 3

However, we can approximate the desired quantity by computing the average velocity over the brief time interval of a tenth of a second (from $t = 5$ to $t = 5.1$).

$$\text{average velocity} = \frac{\text{change in position}}{\text{time elapsed}}$$

$$= \frac{s(5.1) - s(5)}{0.1}$$

$$= \frac{4.9(5.1)^2 - 4.9(5)^2}{0.1}$$

$$= 49.49 \text{ m/s}$$

Example 3

The table shows the results of similar calculations of the average velocity over successively smaller time periods.

- It appears that, as we shorten the time period, the average velocity is becoming closer to 49 m/s.

Time interval	Average velocity (m/s)
$5 \leqslant t \leqslant 6$	53.9
$5 \leqslant t \leqslant 5.1$	49.49
$5 \leqslant t \leqslant 5.05$	49.245
$5 \leqslant t \leqslant 5.01$	49.049
$5 \leqslant t \leqslant 5.001$	49.0049

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THE VELOCITY PROBLEM

Example 3

- The instantaneous velocity when $t = 5$ is defined to be the limiting value of these average velocities over shorter and shorter time periods that start at $t = 5$.
- Thus, the (instantaneous) velocity after 5 s is:

$$v = 49 \text{ m/s}$$

Time interval	Average velocity (m/s)
$5 \leqslant t \leqslant 6$	53.9
$5 \leqslant t \leqslant 5.1$	49.49
$5 \leqslant t \leqslant 5.05$	49.245
$5 \leqslant t \leqslant 5.01$	49.049
$5 \leqslant t \leqslant 5.001$	49.0049

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Example 3

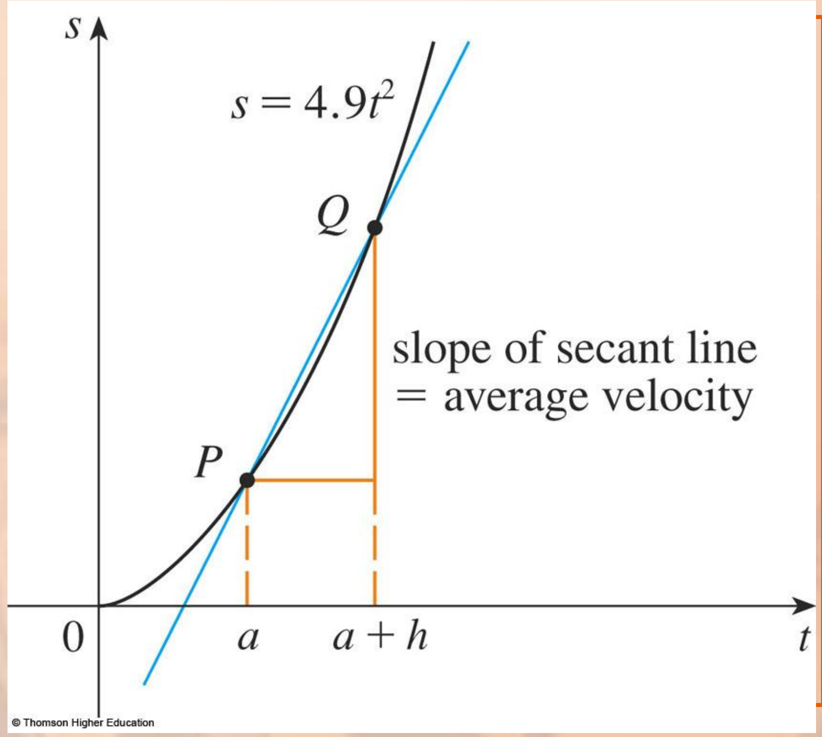
You may have the feeling that the calculations used in solving the problem are very similar to those used earlier to find tangents.

- There is a close connection between the tangent problem and the problem of finding velocities.

Example 3

If we draw the graph of the distance function of the ball and consider the points $P(a, 4.9a^2)$ and $Q(a + h, 4.9(a + h)^2)$, then the slope of the secant line PQ is:

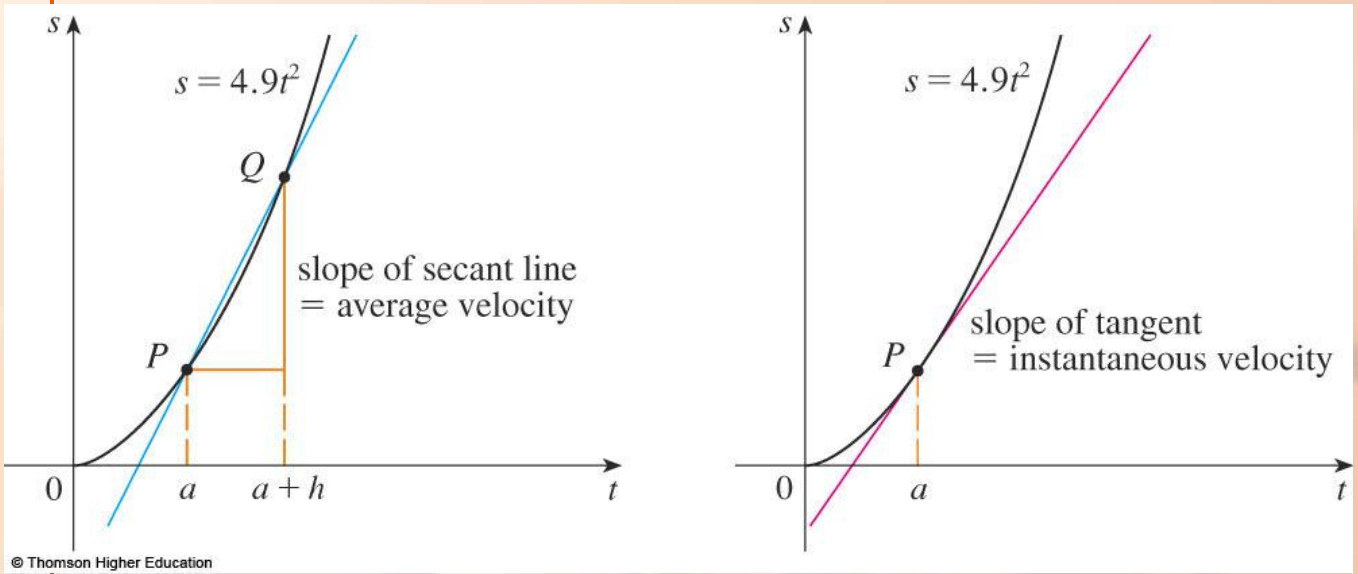
$$m_{PQ} = \frac{4.9(a+h)^2 - 4.9a^2}{(a+h)-a}$$



Example 3

That is the same as the average velocity over the time interval [*a*, *a* + *h*].

- Therefore, the velocity at time $t = a$ (the limit of these average velocities as *h* approaches 0) must be equal to the slope of the tangent line at *P* (the limit of the slopes of the secant lines).



THE VELOCITY PROBLEM

Examples 1 and 3 show that to solve tangent and velocity problems we must be able to find limits.

After studying methods for computing limits for the next five sections we will return to the problem of finding tangents and velocities in Section 2.7.