Chapter 9

Sequences and the Binomial Theorem

9.1 Sequences

When we first introduced a function as a special type of relation in Section 1.3, we did not put any restrictions on the domain of the function. All we said was that the set of x-coordinates of the points in the function F is called the domain, and it turns out that any subset of the real numbers, regardless of how weird that subset may be, can be the domain of a function. As our exploration of functions continued beyond Section 1.3, we saw fewer and fewer functions with 'weird' domains. It is worth your time to go back through the text to see that the domains of the polynomial, rational, exponential, logarithmic and algebraic functions discussed thus far have fairly predictable domains which almost always consist of just a collection of intervals on the real line. This may lead some readers to believe that the only important functions in a College Algebra text have domains which consist of intervals and everything else was just introductory nonsense. In this section, we introduce **sequences** which are an important class of functions whose domains are the set of natural numbers.[1] Before we get to far ahead of ourselves, let's look at what the term 'sequence' means mathematically. Informally, we can think of a sequence as an infinite list of numbers. For example, consider the sequence

$$\frac{1}{2}, -\frac{3}{4}, \frac{9}{8}, -\frac{27}{16}, \ldots \tag{1}$$

As usual, the periods of ellipsis, ..., indicate that the proposed pattern continues forever. Each of the numbers in the list is called a **term**, and we call $\frac{1}{2}$ the 'first term', $-\frac{3}{4}$ the 'second term', $\frac{9}{8}$ the 'third term' and so forth. In numbering them this way, we are setting up a function, which we'll call a per tradition, between the natural numbers and the terms in the sequence.

[1] Recall that this is the set $\{1, 2, 3, \ldots\}$.

n	$a(n)$
1	$\frac{1}{2}$
2	$-\frac{3}{4}$
3	$\frac{9}{8}$
4	$-\frac{27}{16}$
$\vdots$	$\vdots$

In other words, $a(n)$ is the n^{th} term in the sequence. We formalize these ideas in our definition of a sequence and introduce some accompanying notation.

Definition 9.1. A **sequence** is a function a whose domain is the natural numbers. The value $a(n)$ is often written as a_n and is called the $\boldsymbol{n^{\text{th}}}$ **term** of the sequence. The sequence itself is usually denoted using the notation: a_n, $n \geq 1$ or the notation: $\{a_n\}_{n=1}^{\infty}$.

Applying the notation provided in Definition 9.1 to the sequence given (1), we have $a_1 = \frac{1}{2}$, $a_2 = -\frac{3}{4}$, $a_3 = \frac{9}{8}$ and so forth. Now suppose we wanted to know a_{117}, that is, the 117^{th} term in the sequence. While the pattern of the sequence is apparent, it would benefit us greatly to have an explicit formula for a_n. Unfortunately, there is no general algorithm that will produce a formula for every sequence, so any formulas we do develop will come from that greatest of teachers, experience. In other words, it is time for an example.

Example 9.1.1. Write the first four terms of the following sequences.

1. $a_n = \dfrac{5^{n-1}}{3^n}$, $n \geq 1$
2. $b_k = \dfrac{(-1)^k}{2k+1}$, $k \geq 0$
3. $\{2n-1\}_{n=1}^{\infty}$
4. $\left\{\dfrac{1+(-1)^i}{i}\right\}_{i=2}^{\infty}$
5. $a_1 = 7$, $a_{n+1} = 2 - a_n$, $n \geq 1$
6. $f_0 = 1$, $f_n = n \cdot f_{n-1}$, $n \geq 1$

Solution.

1. Since we are given $n \geq 1$, the first four terms of the sequence are a_1, a_2, a_3 and a_4. Since the notation a_1 means the same thing as $a(1)$, we obtain our first term by replacing every occurrence of n in the formula for a_n with $n = 1$ to get $a_1 = \frac{5^{1-1}}{3^1} = \frac{1}{3}$. Proceeding similarly, we get $a_2 = \frac{5^{2-1}}{3^2} = \frac{5}{9}$, $a_3 = \frac{5^{3-1}}{3^3} = \frac{25}{27}$ and $a_4 = \frac{5^{4-1}}{3^4} = \frac{125}{81}$.

2. For this sequence we have $k \geq 0$, so the first four terms are b_0, b_1, b_2 and b_3. Proceeding as before, replacing in this case the variable k with the appropriate whole number, beginning with 0, we get $b_0 = \frac{(-1)^0}{2(0)+1} = 1$, $b_1 = \frac{(-1)^1}{2(1)+1} = -\frac{1}{3}$, $b_2 = \frac{(-1)^2}{2(2)+1} = \frac{1}{5}$ and $b_3 = \frac{(-1)^3}{2(3)+1} = -\frac{1}{7}$. (This sequence is called an **alternating** sequence since the signs alternate between $+$ and $-$. The reader is encouraged to think what component of the formula is producing this effect.)

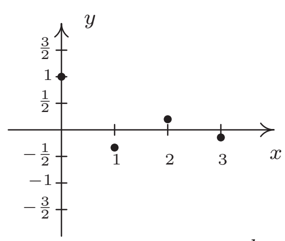
3. From $\{2n-1\}_{n=1}^{\infty}$, we have that $a_n = 2n-1$, $n \geq 1$. We get $a_1 = 1$, $a_2 = 3$, $a_3 = 5$ and $a_4 = 7$. (The first four terms are the first four odd natural numbers. The reader is encouraged to examine whether or not this pattern continues indefinitely.)

4. Here, we are using the letter i as a counter, not as the imaginary unit we saw in Section 3.4. Proceeding as before, we set $a_i = \frac{1+(-1)^i}{i}$, $i \geq 2$. We find $a_2 = 1$, $a_3 = 0$, $a_4 = \frac{1}{2}$ and $a_5 = 0$.

5. To obtain the terms of this sequence, we start with $a_1 = 7$ and use the equation $a_{n+1} = 2 - a_n$ for $n \geq 1$ to generate successive terms. When $n = 1$, this equation becomes $a_{1+1} = 2 - a_1$ which simplifies to $a_2 = 2 - a_1 = 2 - 7 = -5$. When $n = 2$, the equation becomes $a_{2+1} = 2 - a_2$ so we get $a_3 = 2 - a_2 = 2 - (-5) = 7$. Finally, when $n = 3$, we get $a_{3+1} = 2 - a_3$ so $a_4 = 2 - a_3 = 2 - 7 = -5$.

6. As with the problem above, we are given a place to start with $f_0 = 1$ and given a formula to build other terms of the sequence. Substituting $n = 1$ into the equation $f_n = n \cdot f_{n-1}$, we get $f_1 = 1 \cdot f_0 = 1 \cdot 1 = 1$. Advancing to $n = 2$, we get $f_2 = 2 \cdot f_1 = 2 \cdot 1 = 2$. Finally, $f_3 = 3 \cdot f_2 = 3 \cdot 2 = 6$. □

Some remarks about Example 9.1.1 are in order. We first note that since sequences are functions, we can graph them in the same way we graph functions. For example, if we wish to graph the sequence $\{b_k\}_{k=0}^{\infty}$ from Example 9.1.1, we graph the equation $y = b(k)$ for the values $k \geq 0$. That is, we plot the points $(k, b(k))$ for the values of k in the domain, $k = 0, 1, 2, \ldots$. The resulting collection of points is the graph of the sequence. Note that we do not connect the dots in a pleasing fashion as we are used to doing, because the domain is just the whole numbers in this case, not a collection of intervals of real numbers. If you feel a sense of nostalgia, you should see Section 1.2.



Graphing $y = b_k = \dfrac{(-1)^k}{2k+1}$, $k \geq 0$

Speaking of $\{b_k\}_{k=0}^{\infty}$, the astute and mathematically minded reader will correctly note that this technically isn't a sequence, since according to Definition 9.1, sequences are functions whose domains are the *natural* numbers, not the *whole* numbers, as is the case with $\{b_k\}_{k=0}^{\infty}$. In other words, to satisfy Definition 9.1, we need to shift the variable k so it starts at $k = 1$ instead of $k = 0$. To see how we can do this, it helps to think of the problem graphically. What we want is to shift the graph of $y = b(k)$ to the right one unit, and thinking back to Section 1.7, we can accomplish this by replacing k with $k - 1$ in the definition of $\{b_k\}_{k=0}^{\infty}$. Specifically, let $c_k = b_{k-1}$ where $k - 1 \geq 0$. We get $c_k = \frac{(-1)^{k-1}}{2(k-1)+1} = \frac{(-1)^{k-1}}{2k-1}$, where now $k \geq 1$. We leave to the reader to verify that $\{c_k\}_{k=1}^{\infty}$ generates the same list of numbers as does $\{b_k\}_{k=0}^{\infty}$, but the former satisfies Definition

9.1, while the latter does not. Like so many things in this text, we acknowledge that this point is pedantic and join the vast majority of authors who adopt a more relaxed view of Definition 9.1 to include any function which generates a list of numbers which can then be matched up with the natural numbers.[2] Finally, we wish to note the sequences in parts 5 and 6 are examples of sequences described **recursively**. In each instance, an initial value of the sequence is given which is then followed by a **recursion equation** − a formula which enables us to use known terms of the sequence to determine other terms. The terms of the sequence in part 6 are given a special name: $f_n = n!$ is called ***n*-factorial**. Using the '!' notation, we can describe the factorial sequence as: $0! = 1$ and $n! = n(n-1)!$ for $n \geq 1$. After $0! = 1$ the next four terms, written out in detail, are $1! = 1 \cdot 0! = 1 \cdot 1 = 1$, $2! = 2 \cdot 1! = 2 \cdot 1 = 2$, $3! = 3 \cdot 2! = 3 \cdot 2 \cdot 1 = 6$ and $4! = 4 \cdot 3! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. From this, we see a more informal way of computing $n!$, which is $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$ with $0! = 1$ as a special case. (We will study factorials in greater detail in Section 9.4.) The world famous Fibonacci Numbers are defined recursively and are explored in the exercises. While none of the sequences worked out to be the sequence in (1), they do give us some insight into what kinds of patterns to look for. Two patterns in particular are given in the next definition.

Definition 9.2. Arithmetic and Geometric Sequences: Suppose $\{a_n\}_{n=k}^{\infty}$ is a sequence[a]

- If there is a number d so that $a_{n+1} = a_n + d$ for all $n \geq k$, then $\{a_n\}_{n=k}^{\infty}$ is called an **arithmetic sequence**. The number d is called the **common difference**.
- If there is a number r so that $a_{n+1} = ra_n$ for all $n \geq k$, then $\{a_n\}_{n=k}^{\infty}$ is called a **geometric sequence**. The number r is called the **common ratio**.

[a]Note that we have adjusted for the fact that not all 'sequences' begin at $n = 1$.

Both arithmetic and geometric sequences are defined in terms of recursion equations. In English, an arithmetic sequence is one in which we proceed from one term to the next by always *adding* the fixed number d. The name 'common difference' comes from a slight rewrite of the recursion equation from $a_{n+1} = a_n + d$ to $a_{n+1} - a_n = d$. Analogously, a geometric sequence is one in which we proceed from one term to the next by always *multiplying* by the same fixed number r. If $r \neq 0$, we can rearrange the recursion equation to get $\frac{a_{n+1}}{a_n} = r$, hence the name 'common ratio.' Some sequences are arithmetic, some are geometric and some are neither as the next example illustrates.[3]

Example 9.1.2. Determine if the following sequences are arithmetic, geometric or neither. If arithmetic, find the common difference d; if geometric, find the common ratio r.

1. $a_n = \dfrac{5^{n-1}}{3^n}$, $n \geq 1$
2. $b_k = \dfrac{(-1)^k}{2k+1}$, $k \geq 0$
3. $\{2n-1\}_{n=1}^{\infty}$
4. $\dfrac{1}{2}, -\dfrac{3}{4}, \dfrac{9}{8}, -\dfrac{27}{16}, \ldots$

[2]We're basically talking about the 'countably infinite' subsets of the real number line when we do this.
[3]Sequences which are both arithmetic and geometric are discussed in the Exercises.

Solution. A good rule of thumb to keep in mind when working with sequences is "When in doubt, write it out!" Writing out the first several terms can help you identify the pattern of the sequence should one exist.

1. From Example 9.1.1, we know that the first four terms of this sequence are $\frac{1}{3}$, $\frac{5}{9}$, $\frac{25}{27}$ and $\frac{125}{81}$. To see if this is an arithmetic sequence, we look at the successive differences of terms. We find that $a_2 - a_1 = \frac{5}{9} - \frac{1}{3} = \frac{2}{9}$ and $a_3 - a_2 = \frac{25}{27} - \frac{5}{9} = \frac{10}{27}$. Since we get different numbers, there is no 'common difference' and we have established that the sequence is *not* arithmetic. To investigate whether or not it is geometric, we compute the ratios of successive terms. The first three ratios
$$\frac{a_2}{a_1} = \frac{\frac{5}{9}}{\frac{1}{3}} = \frac{5}{3}, \quad \frac{a_3}{a_2} = \frac{\frac{25}{27}}{\frac{5}{9}} = \frac{5}{3} \quad \text{and} \quad \frac{a_4}{a_3} = \frac{\frac{125}{81}}{\frac{25}{27}} = \frac{5}{3}$$
suggest that the sequence is geometric. To prove it, we must show that $\frac{a_{n+1}}{a_n} = r$ for all n.
$$\frac{a_{n+1}}{a_n} = \frac{\frac{5^{(n+1)-1}}{3^{n+1}}}{\frac{5^{n-1}}{3^n}} = \frac{5^n}{3^{n+1}} \cdot \frac{3^n}{5^{n-1}} = \frac{5}{3}$$
This sequence is geometric with common ratio $r = \frac{5}{3}$.

2. Again, we have Example 9.1.1 to thank for providing the first four terms of this sequence: $1, -\frac{1}{3}, \frac{1}{5}$ and $-\frac{1}{7}$. We find $b_1 - b_0 = -\frac{4}{3}$ and $b_2 - b_1 = \frac{8}{15}$. Hence, the sequence is not arithmetic. To see if it is geometric, we compute $\frac{b_1}{b_0} = -\frac{1}{3}$ and $\frac{b_2}{b_1} = -\frac{3}{5}$. Since there is no 'common ratio,' we conclude the sequence is not geometric, either.

3. As we saw in Example 9.1.1, the sequence $\{2n-1\}_{n=1}^{\infty}$ generates the odd numbers: $1, 3, 5, 7, \ldots$. Computing the first few differences, we find $a_2 - a_1 = 2$, $a_3 - a_2 = 2$, and $a_4 - a_3 = 2$. This suggests that the sequence is arithmetic. To verify this, we find
$$a_{n+1} - a_n = (2(n+1) - 1) - (2n - 1) = 2n + 2 - 1 - 2n + 1 = 2$$
This establishes that the sequence is arithmetic with common difference $d = 2$. To see if it is geometric, we compute $\frac{a_2}{a_1} = 3$ and $\frac{a_3}{a_2} = \frac{5}{3}$. Since these ratios are different, we conclude the sequence is not geometric.

4. We met our last sequence at the beginning of the section. Given that $a_2 - a_1 = -\frac{5}{4}$ and $a_3 - a_2 = \frac{15}{8}$, the sequence is not arithmetic. Computing the first few ratios, however, gives us $\frac{a_2}{a_1} = -\frac{3}{2}$, $\frac{a_3}{a_2} = -\frac{3}{2}$ and $\frac{a_4}{a_3} = -\frac{3}{2}$. Since these are the only terms given to us, we assume that the pattern of ratios continue in this fashion and conclude that the sequence is geometric. □

We are now one step away from determining an explicit formula for the sequence given in (1). We know that it is a geometric sequence and our next result gives us the explicit formula we require.

Equation 9.1. Formulas for Arithmetic and Geometric Sequences:

- An arithmetic sequence with first term a and common difference d is given by

$$a_n = a + (n-1)d, \quad n \geq 1$$

- A geometric sequence with first term a and common ratio $r \neq 0$ is given by

$$a_n = ar^{n-1}, \quad n \geq 1$$

While the formal proofs of the formulas in Equation 9.1 require the techniques set forth in Section 9.3, we attempt to motivate them here. According to Definition 9.2, given an arithmetic sequence with first term a and common difference d, the way we get from one term to the next is by adding d. Hence, the terms of the sequence are: a, $a+d$, $a+2d$, $a+3d$, ... We see that to reach the nth term, we add d to a exactly $(n-1)$ times, which is what the formula says. The derivation of the formula for geometric series follows similarly. Here, we start with a and go from one term to the next by multiplying by r. We get a, ar, ar^2, ar^3 and so forth. The nth term results from multiplying a by r exactly $(n-1)$ times. We note here that the reason $r = 0$ is excluded from Equation 9.1 is to avoid an instance of 0^0 which is an indeterminant form.[4] With Equation 9.1 in place, we finally have the tools required to find an explicit formula for the nth term of the sequence given in (1). We know from Example 9.1.2 that it is geometric with common ratio $r = -\frac{3}{2}$. The first term is $a = \frac{1}{2}$ so by Equation 9.1 we get $a_n = ar^{n-1} = \frac{1}{2}\left(-\frac{3}{2}\right)^{n-1}$ for $n \geq 1$. After a touch of simplifying, we get $a_n = \frac{(-3)^{n-1}}{2^n}$ for $n \geq 1$. Note that we can easily check our answer by substituting in values of n and seeing that the formula generates the sequence given in (1). We leave this to the reader. Our next example gives us more practice finding patterns.

Example 9.1.3. Find an explicit formula for the n^{th} term of the following sequences.

1. $0.9, 0.09, 0.009, 0.0009, \ldots$
2. $\dfrac{2}{5}, 2, -\dfrac{2}{3}, -\dfrac{2}{7}, \ldots$
3. $1, -\dfrac{2}{7}, \dfrac{4}{13}, -\dfrac{8}{19}, \ldots$

Solution.

1. Although this sequence may seem strange, the reader can verify it is actually a geometric sequence with common ratio $r = 0.1 = \frac{1}{10}$. With $a = 0.9 = \frac{9}{10}$, we get $a_n = \frac{9}{10}\left(\frac{1}{10}\right)^{n-1}$ for $n \geq 0$. Simplifying, we get $a_n = \frac{9}{10^n}$, $n \geq 1$. There is more to this sequence than meets the eye and we shall return to this example in the next section.

2. As the reader can verify, this sequence is neither arithmetic nor geometric. In an attempt to find a pattern, we rewrite the second term with a denominator to make all the terms appear as fractions. We have $\frac{2}{5}, \frac{2}{1}, -\frac{2}{3}, -\frac{2}{7}, \ldots$. If we associate the negative '$-$' of the last two terms with the denominators we get $\frac{2}{5}, \frac{2}{1}, \frac{2}{-3}, \frac{2}{-7}, \ldots$. This tells us that we can tentatively sketch out the formula for the sequence as $a_n = \frac{2}{d_n}$ where d_n is the sequence of denominators.

[4]See the footnotes on page 237 in Section 3.1 and page 418 of Section 6.1.

Looking at the denominators $5, 1, -3, -7, \ldots$, we find that they go from one term to the next by subtracting 4 which is the same as adding -4. This means we have an arithmetic sequence on our hands. Using Equation 9.1 with $a = 5$ and $d = -4$, we get the nth denominator by the formula $d_n = 5 + (n-1)(-4) = 9 - 4n$ for $n \geq 1$. Our final answer is $a_n = \frac{2}{9-4n}$, $n \geq 1$.

3. The sequence as given is neither arithmetic nor geometric, so we proceed as in the last problem to try to get patterns individually for the numerator and denominator. Letting c_n and d_n denote the sequence of numerators and denominators, respectively, we have $a_n = \frac{c_n}{d_n}$. After some experimentation,[5] we choose to write the first term as a fraction and associate the negatives '$-$' with the numerators. This yields $\frac{1}{1}, \frac{-2}{7}, \frac{4}{13}, \frac{-8}{19}, \ldots$. The numerators form the sequence $1, -2, 4, -8, \ldots$ which is geometric with $a = 1$ and $r = -2$, so we get $c_n = (-2)^{n-1}$, for $n \geq 1$. The denominators $1, 7, 13, 19, \ldots$ form an arithmetic sequence with $a = 1$ and $d = 6$. Hence, we get $d_n = 1 + 6(n-1) = 6n - 5$, for $n \geq 1$. We obtain our formula for $a_n = \frac{c_n}{d_n} = \frac{(-2)^{n-1}}{6n-5}$, for $n \geq 1$. We leave it to the reader to show that this checks out. □

While the last problem in Example 9.1.3 was neither geometric nor arithmetic, it did resolve into a combination of these two kinds of sequences. If handed the sequence $2, 5, 10, 17, \ldots$, we would be hard-pressed to find a formula for a_n if we restrict our attention to these two archetypes. We said before that there is no general algorithm for finding the explicit formula for the nth term of a given sequence, and it is only through experience gained from evaluating sequences from explicit formulas that we learn to begin to recognize number patterns. The pattern $1, 4, 9, 16, \ldots$ is rather recognizable as the squares, so the formula $a_n = n^2$, $n \geq 1$ may not be too hard to determine. With this in mind, it's possible to see $2, 5, 10, 17, \ldots$ as the sequence $1+1, 4+1, 9+1, 16+1, \ldots$, so that $a_n = n^2 + 1$, $n \geq 1$. Of course, since we are given only a small *sample* of the sequence, we shouldn't be too disappointed to find out this isn't the *only* formula which generates this sequence. For example, consider the sequence defined by $b_n = -\frac{1}{4}n^4 + \frac{5}{2}n^3 - \frac{31}{4}n^2 + \frac{25}{2}n - 5$, $n \geq 1$. The reader is encouraged to verify that it also produces the terms $2, 5, 10, 17$. In fact, it can be shown that given any finite sample of a sequence, there are infinitely many explicit formulas all of which generate those same finite points. This means that there will be infinitely many correct answers to some of the exercises in this section.[6] Just because your answer doesn't match ours doesn't mean it's wrong. As always, when in doubt, write your answer out. As long as it produces the same terms in the same order as what the problem wants, your answer is correct.

Sequences play a major role in the Mathematics of Finance, as we have already seen with Equation 6.2 in Section 6.5. Recall that if we invest P dollars at an annual percentage rate r and compound the interest n times per year, the formula for A_k, the amount in the account after k compounding periods, is $A_k = P\left(1 + \frac{r}{n}\right)^k = \left[P\left(1 + \frac{r}{n}\right)\right]\left(1 + \frac{r}{n}\right)^{k-1}$, $k \geq 1$. We now spot this as a geometric sequence with first term $P\left(1 + \frac{r}{n}\right)$ and common ratio $\left(1 + \frac{r}{n}\right)$. In retirement planning, it is seldom the case that an investor deposits a set amount of money into an account and waits for it to grow. Usually, additional payments of principal are made at regular intervals and the value of the investment grows accordingly. This kind of investment is called an **annuity** and will be discussed in the next section once we have developed more mathematical machinery.

[5] Here we take 'experimentation' to mean a frustrating guess-and-check session.

[6] For more on this, see When Every Answer is Correct: Why Sequences and Number Patterns Fail the Test.

9.1.1 Exercises

In Exercises 1 - 13, write out the first four terms of the given sequence.

1. $a_n = 2^n - 1,\ n \geq 0$

2. $d_j = (-1)^{\frac{j(j+1)}{2}},\ j \geq 1$

3. $\{5k - 2\}_{k=1}^{\infty}$

4. $\left\{\dfrac{n^2+1}{n+1}\right\}_{n=0}^{\infty}$

5. $\left\{\dfrac{x^n}{n^2}\right\}_{n=1}^{\infty}$

6. $\left\{\dfrac{\ln(n)}{n}\right\}_{n=1}^{\infty}$

7. $a_1 = 3,\ a_{n+1} = a_n - 1,\ n \geq 1$

8. $d_0 = 12,\ d_m = \dfrac{d_{m-1}}{100},\ m \geq 1$

9. $b_1 = 2,\ b_{k+1} = 3b_k + 1,\ k \geq 1$

10. $c_0 = -2,\ c_j = \dfrac{c_{j-1}}{(j+1)(j+2)},\ j \geq 1$

11. $a_1 = 117,\ a_{n+1} = \dfrac{1}{a_n},\ n \geq 1$

12. $s_0 = 1,\ s_{n+1} = x^{n+1} + s_n,\ n \geq 0$

13. $F_0 = 1,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2},\ n \geq 2$ (This is the famous Fibonacci Sequence)

In Exercises 14 - 21 determine if the given sequence is arithmetic, geometric or neither. If it is arithmetic, find the common difference d; if it is geometric, find the common ratio r.

14. $\{3n - 5\}_{n=1}^{\infty}$

15. $a_n = n^2 + 3n + 2,\ n \geq 1$

16. $\dfrac{1}{3}, \dfrac{1}{6}, \dfrac{1}{12}, \dfrac{1}{24}, \ldots$

17. $\left\{3\left(\dfrac{1}{5}\right)^{n-1}\right\}_{n=1}^{\infty}$

18. $17, 5, -7, -19, \ldots$

19. $2, 22, 222, 2222, \ldots$

20. $0.9, 9, 90, 900, \ldots$

21. $a_n = \dfrac{n!}{2},\ n \geq 0.$

In Exercises 22 - 30, find an explicit formula for the n^{th} term of the given sequence. Use the formulas in Equation 9.1 as needed.

22. $3, 5, 7, 9, \ldots$

23. $1, -\dfrac{1}{2}, \dfrac{1}{4}, -\dfrac{1}{8}, \ldots$

24. $1, \dfrac{2}{3}, \dfrac{4}{5}, \dfrac{8}{7}, \ldots$

25. $1, \dfrac{2}{3}, \dfrac{1}{3}, \dfrac{4}{27}, \ldots$

26. $1, \dfrac{1}{4}, \dfrac{1}{9}, \dfrac{1}{16}, \ldots$

27. $x, -\dfrac{x^3}{3}, \dfrac{x^5}{5}, -\dfrac{x^7}{7}, \ldots$

28. $0.9, 0.99, 0.999, 0.9999, \ldots$ 29. $27, 64, 125, 216, \ldots$ 30. $1, 0, 1, 0, \ldots$

31. Find a sequence which is both arithmetic and geometric. (Hint: Start with $a_n = c$ for all n.)

32. Show that a geometric sequence can be transformed into an arithmetic sequence by taking the natural logarithm of the terms.

33. Thomas Robert Malthus is credited with saying, "The power of population is indefinitely greater than the power in the earth to produce subsistence for man. Population, when unchecked, increases in a geometrical ratio. Subsistence increases only in an arithmetical ratio. A slight acquaintance with numbers will show the immensity of the first power in comparison with the second." (See this webpage for more information.) Discuss this quote with your classmates from a sequences point of view.

34. This classic problem involving sequences shows the power of geometric sequences. Suppose that a wealthy benefactor agrees to give you one penny today and then double the amount she gives you each day for 30 days. So, for example, you get two pennies on the second day and four pennies on the third day. How many pennies do you get on the 30^{th} day? What is the total dollar value of the gift you have received?

35. Research the terms 'arithmetic mean' and 'geometric mean.' With the help of your classmates, show that a given term of a arithmetic sequence a_k, $k \geq 2$ is the arithmetic mean of the term immediately preceding, a_{k-1} it and immediately following it, a_{k+1}. State and prove an analogous result for geometric sequences.

36. Discuss with your classmates how the results of this section might change if we were to examine sequences of other mathematical things like complex numbers or matrices. Find an explicit formula for the n^{th} term of the sequence $i, -1, -i, 1, i, \ldots$. List out the first four terms of the matrix sequences we discussed in Exercise 8.3.1 in Section 8.3.

9.1.2 Answers

1. $0, 1, 3, 7$
2. $-1, -1, 1, 1$
3. $3, 8, 13, 18$
4. $1, 1, \frac{5}{3}, \frac{5}{2}$
5. $x, \frac{x^2}{4}, \frac{x^3}{9}, \frac{x^4}{16}$
6. $0, \frac{\ln(2)}{2}, \frac{\ln(3)}{3}, \frac{\ln(4)}{4}$
7. $3, 2, 1, 0$
8. $12, 0.12, 0.0012, 0.000012$
9. $2, 7, 22, 67$
10. $-2, -\frac{1}{3}, -\frac{1}{36}, -\frac{1}{720}$
11. $117, \frac{1}{117}, 117, \frac{1}{117}$
12. $1, x+1, x^2+x+1, x^3+x^2+x+1$
13. $1, 1, 2, 3$
14. arithmetic, $d = 3$
15. neither
16. geometric, $r = \frac{1}{2}$
17. geometric, $r = \frac{1}{5}$
18. arithmetic, $d = -12$
19. neither
20. geometric, $r = 10$
21. neither
22. $a_n = 1 + 2n,\ n \geq 1$
23. $a_n = \left(-\frac{1}{2}\right)^{n-1},\ n \geq 1$
24. $a_n = \frac{2^{n-1}}{2n-1},\ n \geq 1$
25. $a_n = \frac{n}{3^{n-1}},\ n \geq 1$
26. $a_n = \frac{1}{n^2},\ n \geq 1$
27. $\frac{(-1)^{n-1}x^{2n-1}}{2n-1},\ n \geq 1$
28. $a_n = \frac{10^n-1}{10^n},\ n \geq 1$
29. $a_n = (n+2)^3,\ n \geq 1$
30. $a_n = \frac{1+(-1)^{n-1}}{2},\ n \geq 1$

9.2 Summation Notation

In the previous section, we introduced sequences and now we shall present notation and theorems concerning the sum of terms of a sequence. We begin with a definition, which, while intimidating, is meant to make our lives easier.

Definition 9.3. Summation Notation: Given a sequence $\{a_n\}_{n=k}^{\infty}$ and numbers m and p satisfying $k \leq m \leq p$, the summation from m to p of the sequence $\{a_n\}$ is written

$$\sum_{n=m}^{p} a_n = a_m + a_{m+1} + \ldots + a_p$$

The variable n is called the **index of summation**. The number m is called the **lower limit of summation** while the number p is called the **upper limit of summation**.

In English, Definition 9.3 is simply defining a short-hand notation for adding up the terms of the sequence $\{a_n\}_{n=k}^{\infty}$ from a_m through a_p. The symbol Σ is the capital Greek letter sigma and is shorthand for 'sum'. The lower and upper limits of the summation tells us which term to start with and which term to end with, respectively. For example, using the sequence $a_n = 2n - 1$ for $n \geq 1$, we can write the sum $a_3 + a_4 + a_5 + a_6$ as

$$\begin{aligned} \sum_{n=3}^{6}(2n-1) &= (2(3)-1)+(2(4)-1)+(2(5)-1)+(2(6)-1) \\ &= 5+7+9+11 \\ &= 32 \end{aligned}$$

The index variable is considered a 'dummy variable' in the sense that it may be changed to any letter without affecting the value of the summation. For instance,

$$\sum_{n=3}^{6}(2n-1) = \sum_{k=3}^{6}(2k-1) = \sum_{j=3}^{6}(2j-1)$$

One place you may encounter summation notation is in mathematical definitions. For example, summation notation allows us to define polynomials as functions of the form

$$f(x) = \sum_{k=0}^{n} a_k x^k$$

for real numbers a_k, $k = 0, 1, \ldots n$. The reader is invited to compare this with what is given in Definition 3.1. Summation notation is particularly useful when talking about matrix operations. For example, we can write the product of the ith row R_i of a matrix $A = [a_{ij}]_{m \times n}$ and the j^{th} column C_j of a matrix $B = [b_{ij}]_{n \times r}$ as

$$Ri \cdot Cj = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Again, the reader is encouraged to write out the sum and compare it to Definition 8.9. Our next example gives us practice with this new notation.

Example 9.2.1.

1. Find the following sums.

   (a) $\displaystyle\sum_{k=1}^{4} \frac{13}{100^k}$ (b) $\displaystyle\sum_{n=0}^{4} \frac{n!}{2}$ (c) $\displaystyle\sum_{n=1}^{5} \frac{(-1)^{n+1}}{n}(x-1)^n$

2. Write the following sums using summation notation.

   (a) $1 + 3 + 5 + \ldots + 117$

   (b) $1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + - \ldots + \dfrac{1}{117}$

   (c) $0.9 + 0.09 + 0.009 + \ldots 0.\underbrace{0\cdots0}_{n-1 \text{ zeros}}9$

Solution.

1. (a) We substitute $k = 1$ into the formula $\frac{13}{100^k}$ and add successive terms until we reach $k = 4$.

$$\begin{aligned}\sum_{k=1}^{4} \frac{13}{100^k} &= \frac{13}{100^1} + \frac{13}{100^2} + \frac{13}{100^3} + \frac{13}{100^4} \\ &= 0.13 + 0.0013 + 0.000013 + 0.00000013 \\ &= 0.13131313\end{aligned}$$

   (b) Proceeding as in (a), we replace every occurrence of n with the values 0 through 4. We recall the factorials, $n!$ as defined in number Example 9.1.1, number 6 and get:

$$\begin{aligned}\sum_{n=0}^{4} \frac{n!}{2} &= \frac{0!}{2} + \frac{1!}{2} + \frac{2!}{2} + \frac{3!}{2} = \frac{4!}{2} \\ &= \frac{1}{2} + \frac{1}{2} + \frac{2\cdot 1}{2} + \frac{3\cdot 2\cdot 1}{2} + \frac{4\cdot 3\cdot 2\cdot 1}{2} \\ &= \frac{1}{2} + \frac{1}{2} + 1 + 3 + 12 \\ &= 17\end{aligned}$$

   (c) We proceed as before, replacing the index n, but *not* the variable x, with the values 1 through 5 and adding the resulting terms.

$$\begin{aligned}\sum_{n=1}^{5} \frac{(-1)^{n+1}}{n}(x-1)^n &= \frac{(-1)^{1+1}}{1}(x-1)^1 + \frac{(-1)^{2+1}}{2}(x-1)^2 + \frac{(-1)^{3+1}}{3}(x-1)^3 \\ &\quad + \frac{(-1)^{1+4}}{4}(x-1)^4 + \frac{(-1)^{1+5}}{5}(x-1)^5 \\ &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5}\end{aligned}$$

2. The key to writing these sums with summation notation is to find the pattern of the terms. To that end, we make good use of the techniques presented in Section 9.1.

(a) The terms of the sum 1, 3, 5, etc., form an arithmetic sequence with first term $a = 1$ and common difference $d = 2$. We get a formula for the nth term of the sequence using Equation 9.1 to get $a_n = 1 + (n-1)2 = 2n - 1$, $n \geq 1$. At this stage, we have the formula for the terms, namely $2n - 1$, and the lower limit of the summation, $n = 1$. To finish the problem, we need to determine the upper limit of the summation. In other words, we need to determine which value of n produces the term 117. Setting $a_n = 117$, we get $2n - 1 = 117$ or $n = 59$. Our final answer is

$$1 + 3 + 5 + \ldots + 117 = \sum_{n=1}^{59} (2n-1)$$

(b) We rewrite all of the terms as fractions, the subtraction as addition, and associate the negatives '$-$' with the numerators to get

$$\frac{1}{1} + \frac{-1}{2} + \frac{1}{3} + \frac{-1}{4} + \ldots + \frac{1}{117}$$

The numerators, $1, -1$, etc. can be described by the geometric sequence[1] $c_n = (-1)^{n-1}$ for $n \geq 1$, while the denominators are given by the arithmetic sequence[2] $d_n = n$ for $n \geq 1$. Hence, we get the formula $a_n = \frac{(-1)^{n-1}}{n}$ for our terms, and we find the lower and upper limits of summation to be $n = 1$ and $n = 117$, respectively. Thus

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \ldots + \frac{1}{117} = \sum_{n=1}^{117} \frac{(-1)^{n-1}}{n}$$

(c) Thanks to Example 9.1.3, we know that one formula for the n^{th} term is $a_n = \frac{9}{10^n}$ for $n \geq 1$. This gives us a formula for the summation as well as a lower limit of summation. To determine the upper limit of summation, we note that to produce the $n - 1$ zeros to the right of the decimal point before the 9, we need a denominator of 10^n. Hence, n is

[1] This is indeed a geometric sequence with first term $a = 1$ and common ratio $r = -1$.
[2] It is an arithmetic sequence with first term $a = 1$ and common difference $d = 1$.

the upper limit of summation. Since n is used in the limits of the summation, we need to choose a different letter for the index of summation.[3] We choose k and get

$$0.9 + 0.09 + 0.009 + \ldots 0.\underbrace{0 \cdots 0}_{n-1 \text{ zeros}}9 \quad = \quad \sum_{k=1}^{n} \frac{9}{10^k}$$

□

The following theorem presents some general properties of summation notation. While we shall not have much need of these properties in Algebra, they do play a great role in Calculus. Moreover, there is much to be learned by thinking about why the properties hold. We invite the reader to prove these results. To get started, remember, "When in doubt, write it out!"

Theorem 9.1. Properties of Summation Notation: Suppose $\{a_n\}$ and $\{b_n\}$ are sequences so that the following sums are defined.

- $\displaystyle\sum_{n=m}^{p} (a_n \pm b_n) = \sum_{n=m}^{p} a_n \pm \sum_{n=m}^{p} b_n$
- $\displaystyle\sum_{n=m}^{p} c\, a_n = c \sum_{n=m}^{p} a_n$, for any real number c.
- $\displaystyle\sum_{n=m}^{p} a_n = \sum_{n=m}^{j} a_n + \sum_{n=j+1}^{p} a_n$, for any natural number $m \leq j < j+1 \leq p$.
- $\displaystyle\sum_{n=m}^{p} a_n = \sum_{n=m+r}^{p+r} a_{n-r}$, for any whole number r.

We now turn our attention to the sums involving arithmetic and geometric sequences. Given an arithmetic sequence $a_k = a + (k-1)d$ for $k \geq 1$, we let S denote the sum of the first n terms. To derive a formula for S, we write it out in two different ways

$$\begin{array}{ccccccccccc} S & = & a & + & (a+d) & + & \ldots & + & (a+(n-2)d) & + & (a+(n-1)d) \\ S & = & (a+(n-1)d) & + & (a+(n-2)d) & + & \ldots & + & (a+d) & + & a \end{array}$$

If we add these two equations and combine the terms which are aligned vertically, we get

$$2S = (2a + (n-1)d) + (2a + (n-1)d) + \ldots + (2a + (n-1)d) + (2a + (n-1)d)$$

The right hand side of this equation contains n terms, all of which are equal to $(2a + (n-1)d)$ so we get $2S = n(2a + (n-1)d)$. Dividing both sides of this equation by 2, we obtain the formula

[3]To see why, try writing the summation using 'n' as the index.

$$S = \frac{n}{2}(2a + (n-1)d)$$

If we rewrite the quantity $2a + (n-1)d$ as $a + (a + (n-1)d) = a_1 + a_n$, we get the formula

$$S = n\left(\frac{a_1 + a_n}{2}\right)$$

A helpful way to remember this last formula is to recognize that we have expressed the sum as the product of the number of terms n and the *average* of the first and n^{th} terms.

To derive the formula for the geometric sum, we start with a geometric sequence $a_k = ar^{k-1}$, $k \geq 1$, and let S once again denote the sum of the first n terms. Comparing S and rS, we get

$$\begin{array}{rclcccccccccccc} S & = & a & + & ar & + & ar^2 & + & \ldots & + & ar^{n-2} & + & ar^{n-1} & & \\ rS & = & & & ar & + & ar^2 & + & \ldots & + & ar^{n-2} & + & ar^{n-1} & + & ar^n \end{array}$$

Subtracting the second equation from the first forces all of the terms except a and ar^n to cancel out and we get $S - rS = a - ar^n$. Factoring, we get $S(1-r) = a\left(1 - r^n\right)$. Assuming $r \neq 1$, we can divide both sides by the quantity $(1-r)$ to obtain

$$S = a\left(\frac{1-r^n}{1-r}\right)$$

If we distribute a through the numerator, we get $a - ar^n = a_1 - a_{n+1}$ which yields the formula

$$S = \frac{a_1 - a_{n+1}}{1-r}$$

In the case when $r = 1$, we get the formula

$$S = \underbrace{a + a + \ldots + a}_{n \text{ times}} = n\,a$$

Our results are summarized below.

Equation 9.2. Sums of Arithmetic and Geometric Sequences:

- The sum S of the first n terms of an arithmetic sequence $a_k = a + (k-1)d$ for $k \geq 1$ is

$$S = \sum_{k=1}^{n} a_k = n\left(\frac{a_1 + a_n}{2}\right) = \frac{n}{2}(2a + (n-1)d)$$

- The sum S of the first n terms of a geometric sequence $a_k = ar^{k-1}$ for $k \geq 1$ is

1. $S = \sum_{k=1}^{n} a_k = \frac{a_1 - a_{n+1}}{1-r} = a\left(\frac{1-r^n}{1-r}\right)$, if $r \neq 1$.
2. $S = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} a = na$, if $r = 1$.

While we have made an honest effort to derive the formulas in Equation 9.2, formal proofs require the machinery in Section 9.3. An application of the arithmetic sum formula which proves useful in Calculus results in formula for the sum of the first n natural numbers. The natural numbers themselves are a sequence[4] 1, 2, 3, ... which is arithmetic with $a = d = 1$. Applying Equation 9.2,

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

So, for example, the sum of the first 100 natural numbers[5] is $\frac{100(101)}{2} = 5050$.

An important application of the geometric sum formula is the investment plan called an **annuity**. Annuities differ from the kind of investments we studied in Section 6.5 in that payments are deposited into the account on an on-going basis, and this complicates the mathematics a little.[6] Suppose you have an account with annual interest rate r which is compounded n times per year. We let $i = \frac{r}{n}$ denote the interest rate per period. Suppose we wish to make ongoing deposits of P dollars at the *end* of each compounding period. Let A_k denote the amount in the account after k compounding periods. Then $A_1 = P$, because we have made our first deposit at the *end* of the first compounding period and no interest has been earned. During the second compounding period, we earn interest on A_1 so that our initial investment has grown to $A_1(1+i) = P(1+i)$ in accordance with Equation 6.1. When we add our second payment at the end of the second period, we get

$$A_2 = A_1(1+i) + P = P(1+i) + P = P(1+i)\left(1 + \frac{1}{1+i}\right)$$

The reason for factoring out the $P(1+i)$ will become apparent in short order. During the third compounding period, we earn interest on A_2 which then grows to $A_2(1+i)$. We add our third

[4]This is the identity function on the natural numbers!

[5]There is an interesting anecdote which says that the famous mathematician Carl Friedrich Gauss was given this problem in primary school and devised a very clever solution.

[6]The reader may wish to re-read the discussion on compound interest in Section 6.5 before proceeding.

payment at the end of the third compounding period to obtain

$$A_3 = A_2(1+i) + P = P(1+i)\left(1+\frac{1}{1+i}\right)(1+i) + P = P(1+i)^2\left(1+\frac{1}{1+i}+\frac{1}{(1+i)^2}\right)$$

During the fourth compounding period, A_3 grows to $A_3(1+i)$, and when we add the fourth payment, we factor out $P(1+i)^3$ to get

$$A_4 = P(1+i)^3\left(1+\frac{1}{1+i}+\frac{1}{(1+i)^2}+\frac{1}{(1+i)^3}\right)$$

This pattern continues so that at the end of the kth compounding, we get

$$A_k = P(1+i)^{k-1}\left(1+\frac{1}{1+i}+\frac{1}{(1+i)^2}+\ldots+\frac{1}{(1+i)^{k-1}}\right)$$

The sum in the parentheses above is the sum of the first k terms of a geometric sequence with $a = 1$ and $r = \frac{1}{1+i}$. Using Equation 9.2, we get

$$1+\frac{1}{1+i}+\frac{1}{(1+i)^2}+\ldots+\frac{1}{(1+i)^{k-1}} = 1\left(\frac{1-\dfrac{1}{(1+i)^k}}{1-\dfrac{1}{1+i}}\right) = \frac{(1+i)\left(1-(1+i)^{-k}\right)}{i}$$

Hence, we get

$$A_k = P(1+i)^{k-1}\left(\frac{(1+i)\left(1-(1+i)^{-k}\right)}{i}\right) = \frac{P\left((1+i)^k-1\right)}{i}$$

If we let t be the number of years this investment strategy is followed, then $k = nt$, and we get the formula for the future value of an **ordinary annuity**.

Equation 9.3. Future Value of an Ordinary Annuity: Suppose an annuity offers an annual interest rate r compounded n times per year. Let $i = \frac{r}{n}$ be the interest rate per compounding period. If a deposit P is made at the end of each compounding period, the amount A in the account after t years is given by

$$A = \frac{P\left((1+i)^{nt}-1\right)}{i}$$

The reader is encouraged to substitute $i = \frac{r}{n}$ into Equation 9.3 and simplify. Some familiar equations arise which are cause for pause and meditation. One last note: if the deposit P is made a the *beginning* of the compounding period instead of at the end, the annuity is called an **annuity-due**. We leave the derivation of the formula for the future value of an annuity-due as an exercise for the reader.

Example 9.2.2. An ordinary annuity offers a 6% annual interest rate, compounded monthly.

1. If monthly payments of \$50 are made, find the value of the annuity in 30 years.
2. How many years will it take for the annuity to grow to \$100,000?

Solution.

1. We have $r = 0.06$ and $n = 12$ so that $i = \frac{r}{n} = \frac{0.06}{12} = 0.005$. With $P = 50$ and $t = 30$,

$$A = \frac{50\left((1+0.005)^{(12)(30)} - 1\right)}{0.005} \approx 50225.75$$

Our final answer is \$50,225.75.

2. To find how long it will take for the annuity to grow to \$100,000, we set $A = 100000$ and solve for t. We isolate the exponential and take natural logs of both sides of the equation.

$$\begin{aligned} 100000 &= \frac{50\left((1+0.005)^{12t} - 1\right)}{0.005} \\ 10 &= (1.005)^{12t} - 1 \\ (1.005)^{12t} &= 11 \\ \ln\left((1.005)^{12t}\right) &= \ln(11) \\ 12t\ln(1.005) &= \ln(11) \\ t &= \tfrac{\ln(11)}{12\ln(1.005)} \approx 40.06 \end{aligned}$$

This means that it takes just over 40 years for the investment to grow to \$100,000. Comparing this with our answer to part 1, we see that in just 10 additional years, the value of the annuity nearly doubles. This is a lesson worth remembering. □

We close this section with a peek into Calculus by considering *infinite* sums, called **series**. Consider the number $0.\overline{9}$. We can write this number as

$$0.\overline{9} = 0.9999... = 0.9 + 0.09 + 0.009 + 0.0009 + \ldots$$

From Example 9.2.1, we know we can write the sum of the first n of these terms as

$$0.\underbrace{9\cdots9}_{n \text{ nines}} = .9 + 0.09 + 0.009 + \ldots 0.\underbrace{0\cdots0}_{n-1 \text{ zeros}}9 = \sum_{k=1}^{n} \frac{9}{10^k}$$

Using Equation 9.2, we have

$$\sum_{k=1}^{n} \frac{9}{10^k} = \frac{9}{10}\left(\frac{1-\dfrac{1}{10^{n+1}}}{1-\dfrac{1}{10}}\right) = 1 - \frac{1}{10^{n+1}}$$

It stands to reason that $0.\overline{9}$ is the same value of $1 - \frac{1}{10^{n+1}}$ as $n \to \infty$. Our knowledge of exponential expressions from Section 6.1 tells us that $\frac{1}{10^{n+1}} \to 0$ as $n \to \infty$, so $1 - \frac{1}{10^{n+1}} \to 1$. We have just argued that $0.\overline{9} = 1$, which may cause some distress for some readers.[7] Any non-terminating decimal can be thought of as an infinite sum whose denominators are the powers of 10, so the phenomenon of adding up infinitely many terms and arriving at a finite number is not as foreign of a concept as it may appear. We end this section with a theorem concerning geometric series.

Theorem 9.2. Geometric Series: Given the sequence $a_k = ar^{k-1}$ for $k \geq 1$, where $|r| < 1$,

$$a + ar + ar^2 + \ldots = \sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$$

If $|r| \geq 1$, the sum $a + ar + ar^2 + \ldots$ is not defined.

The justification of the result in Theorem 9.2 comes from taking the formula in Equation 9.2 for the sum of the first n terms of a geometric sequence and examining the formula as $n \to \infty$. Assuming $|r| < 1$ means $-1 < r < 1$, so $r^n \to 0$ as $n \to \infty$. Hence as $n \to \infty$,

$$\sum_{k=1}^{n} ar^{k-1} = a\left(\frac{1-r^n}{1-r}\right) \to \frac{a}{1-r}$$

As to what goes wrong when $|r| \geq 1$, we leave that to Calculus as well, but will explore some cases in the exercises.

[7] To make this more palatable, it is usually accepted that $0.\overline{3} = \frac{1}{3}$ so that $0.\overline{9} = 3\left(0.\overline{3}\right) = 3\left(\frac{1}{3}\right) = 1$. Feel better?

9.2.1 Exercises

In Exercises 1 - 8, find the value of each sum using Definition 9.3.

1. $\displaystyle\sum_{g=4}^{9}(5g+3)$
2. $\displaystyle\sum_{k=3}^{8}\frac{1}{k}$
3. $\displaystyle\sum_{j=0}^{5}2^j$
4. $\displaystyle\sum_{k=0}^{2}(3k-5)x^k$
5. $\displaystyle\sum_{i=1}^{4}\frac{1}{4}(i^2+1)$
6. $\displaystyle\sum_{n=1}^{100}(-1)^n$
7. $\displaystyle\sum_{n=1}^{5}\frac{(n+1)!}{n!}$
8. $\displaystyle\sum_{j=1}^{3}\frac{5!}{j!\,(5-j)!}$

In Exercises 9 - 16, rewrite the sum using summation notation.

9. $8+11+14+17+20$
10. $1-2+3-4+5-6+7-8$
11. $x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\dfrac{x^7}{7}$
12. $1+2+4+\cdots+2^{29}$
13. $2+\frac{3}{2}+\frac{4}{3}+\frac{5}{4}+\frac{6}{5}$
14. $-\ln(3)+\ln(4)-\ln(5)+\cdots+\ln(20)$
15. $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\frac{1}{36}$
16. $\frac{1}{2}(x-5)+\frac{1}{4}(x-5)^2+\frac{1}{6}(x-5)^3+\frac{1}{8}(x-5)^4$

In Exercises 17 - 28, use the formulas in Equation 9.2 to find the sum.

17. $\displaystyle\sum_{n=1}^{10}5n+3$
18. $\displaystyle\sum_{n=1}^{20}2n-1$
19. $\displaystyle\sum_{k=0}^{15}3-k$
20. $\displaystyle\sum_{n=1}^{10}\left(\frac{1}{2}\right)^n$
21. $\displaystyle\sum_{n=1}^{5}\left(\frac{3}{2}\right)^n$
22. $\displaystyle\sum_{k=0}^{5}2\left(\frac{1}{4}\right)^k$
23. $1+4+7+\ldots+295$
24. $4+2+0-2-\ldots-146$
25. $1+3+9+\ldots+2187$
26. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{256}$
27. $3-\frac{3}{2}+\frac{3}{4}-\frac{3}{8}+-\cdots+\frac{3}{256}$
28. $\displaystyle\sum_{n=1}^{10}-2n+\left(\frac{5}{3}\right)^n$

In Exercises 29 - 32, use Theorem 9.2 to express each repeating decimal as a fraction of integers.

29. $0.\overline{7}$
30. $0.\overline{13}$
31. $10.\overline{159}$
32. $-5.8\overline{67}$

In Exercises 33 - 38, use Equation 9.3 to compute the future value of the annuity with the given terms. In all cases, assume the payment is made monthly, the interest rate given is the annual rate, and interest is compounded monthly.

33. payments are \$300, interest rate is 2.5%, term is 17 years.

34. payments are \$50, interest rate is 1.0%, term is 30 years.

35. payments are \$100, interest rate is 2.0%, term is 20 years

36. payments are \$100, interest rate is 2.0%, term is 25 years

37. payments are \$100, interest rate is 2.0%, term is 30 years

38. payments are \$100, interest rate is 2.0%, term is 35 years

39. Suppose an ordinary annuity offers an annual interest rate of 2%, compounded monthly, for 30 years. What should the monthly payment be to have \$100,000 at the end of the term?

40. Prove the properties listed in Theorem 9.1.

41. Show that the formula for the future value of an annuity due is

$$A = P(1+i)\left[\frac{(1+i)^{nt}-1}{i}\right]$$

42. Discuss with your classmates what goes wrong when trying to find the following sums.[8]

(a) $\displaystyle\sum_{k=1}^{\infty} 2^{k-1}$ (b) $\displaystyle\sum_{k=1}^{\infty} (1.0001)^{k-1}$ (c) $\displaystyle\sum_{k=1}^{\infty} (-1)^{k-1}$

[8]When in doubt, write them out!

9.2.2 Answers

1. 213
2. $\frac{341}{280}$
3. 63
4. $-5 - 2x + x^2$
5. $\frac{17}{2}$
6. 0
7. 20
8. 25
9. $\displaystyle\sum_{k=1}^{5}(3k+5)$
10. $\displaystyle\sum_{k=1}^{8}(-1)^{k-1}k$
11. $\displaystyle\sum_{k=1}^{4}(-1)^{k-1}\frac{x^{2k-1}}{2k-1}$
12. $\displaystyle\sum_{k=1}^{30}2^{k-1}$
13. $\displaystyle\sum_{k=1}^{5}\frac{k+1}{k}$
14. $\displaystyle\sum_{k=3}^{20}(-1)^{k}\ln(k)$
15. $\displaystyle\sum_{k=1}^{6}\frac{(-1)^{k-1}}{k^2}$
16. $\displaystyle\sum_{k=1}^{4}\frac{1}{2k}(x-5)^k$
17. 305
18. 400
19. -72
20. $\dfrac{1023}{1024}$
21. $\dfrac{633}{32}$
22. $\dfrac{1365}{512}$
23. 14652
24. -5396
25. 3280
26. $\dfrac{255}{256}$
27. $\dfrac{513}{256}$
28. $\dfrac{17771050}{59049}$
29. $\dfrac{7}{9}$
30. $\dfrac{13}{99}$
31. $\dfrac{3383}{333}$
32. $-\dfrac{5809}{990}$
33. \$76,163.67
34. \$20,981.40
35. \$29,479.69
36. \$38,882.12
37. 49,272.55
38. 60,754.80
39. For \$100,000, the monthly payment is $\approx$ \$202.95.

9.3 Mathematical Induction

The Chinese philosopher Confucius is credited with the saying, "A journey of a thousand miles begins with a single step." In many ways, this is the central theme of this section. Here we introduce a method of proof, Mathematical Induction, which allows us to *prove* many of the formulas we have merely *motivated* in Sections 9.1 and 9.2 by starting with just a single step. A good example is the formula for arithmetic sequences we touted in Equation 9.1. Arithmetic sequences are defined recursively, starting with $a_1 = a$ and then $a_{n+1} = a_n + d$ for $n \geq 1$. This tells us that we start the sequence with a and we go from one term to the next by successively adding d. In symbols,

$$a, a + d, a + 2d, a + 3d, a + 4d + \ldots$$

The pattern *suggested* here is that to reach the nth term, we start with a and add d to it exactly $n - 1$ times, which lead us to our formula $a_n = a + (n - 1)d$ for $n \geq 1$. But how do we *prove* this to be the case? We have the following.

The Principle of Mathematical Induction (PMI): Suppose $P(n)$ is a sentence involving the natural number n.

IF

1. $P(1)$ is true **and**
2. whenever $P(k)$ is true, it follows that $P(k + 1)$ is also true

THEN the sentence $P(n)$ is true for all natural numbers n.

The Principle of Mathematical Induction, or PMI for short, is exactly that - a principle.[1] It is a property of the natural numbers we either choose to accept or reject. In English, it says that if we want to prove that a formula works for all natural numbers n, we start by showing it is true for $n = 1$ (the '**base step**') and then show that if it is true for a generic natural number k, it must be true for the next natural number, $k + 1$ (the '**inductive step**'). The notation $P(n)$ acts just like function notation. For example, if $P(n)$ is the sentence (formula) '$n^2 + 1 = 3$', then $P(1)$ would be '$1^2 + 1 = 3$', which is false. The construction $P(k + 1)$ would be '$(k + 1)^2 + 1 = 3$'. As usual, this new concept is best illustrated with an example. Returning to our quest to prove the formula for an arithmetic sequence, we first identify $P(n)$ as the formula $a_n = a + (n - 1)d$. To prove this formula is valid for all natural numbers n, we need to do two things. First, we need to establish that $P(1)$ is true. In other words, is it true that $a_1 = a + (1 - 1)d$? The answer is yes, since this simplifies to $a_1 = a$, which is part of the definition of the arithmetic sequence. The second thing we need to show is that whenever $P(k)$ is true, it follows that $P(k + 1)$ is true. In other words, we *assume* $P(k)$ is true (this is called the '**induction hypothesis**') and *deduce* that $P(k + 1)$ is also true. Assuming $P(k)$ to be true seems to invite disaster - after all, isn't this essentially what we're trying to prove in the first place? To help explain this step a little better, we show how this works for specific values of n. We've already established $P(1)$ is true, and we now want to show that $P(2)$

[1] Another word for this you may have seen is 'axiom.'

is true. Thus we need to show that $a_2 = a+(2-1)d$. Since $P(1)$ is true, we have $a_1 = a$, and by the definition of an arithmetic sequence, $a_2 = a_1+d = a+d = a+(2-1)d$. So $P(2)$ is true. We now use the fact that $P(2)$ is true to show that $P(3)$ is true. Using the fact that $a_2 = a+(2-1)d$, we show $a_3 = a+(3-1)d$. Since $a_3 = a_2+d$, we get $a_3 = (a+(2-1)d)+d = a+2d = a+(3-1)d$, so we have shown $P(3)$ is true. Similarly, we can use the fact that $P(3)$ is true to show that $P(4)$ is true, and so forth. In general, if $P(k)$ is true (i.e., $a_k = a+(k-1)d$) we set out to show that $P(k+1)$ is true (i.e., $a_{k+1} = a+((k+1)-1)d$). Assuming $a_k = a+(k-1)d$, we have by the definition of an arithmetic sequence that $a_{k+1} = a_k + d$ so we get $a_{k+1} = (a+(k-1)d)+d = a+kd = a+((k+1)-1)d$. Hence, $P(k+1)$ is true.

In essence, by showing that $P(k+1)$ must always be true when $P(k)$ is true, we are showing that the formula $P(1)$ can be used to get the formula $P(2)$, which in turn can be used to derive the formula $P(3)$, which in turn can be used to establish the formula $P(4)$, and so on. Thus as long as $P(k)$ is true for some natural number k, $P(n)$ is true for all of the natural numbers n which follow k. Coupling this with the fact $P(1)$ is true, we have established $P(k)$ is true for all natural numbers which follow $n = 1$, in other words, all natural numbers n. One might liken Mathematical Induction to a repetitive process like climbing stairs.[2] If you are sure that (1) you can get on the stairs (the base case) and (2) you can climb from any one step to the next step (the inductive step), then presumably you can climb the entire staircase.[3] We get some more practice with induction in the following example.

Example 9.3.1. Prove the following assertions using the Principle of Mathematical Induction.

1. The sum formula for arithmetic sequences: $\displaystyle\sum_{j=1}^{n}(a+(j-1)d) = \frac{n}{2}(2a+(n-1)d)$.
2. For a complex number z, $(\overline{z})^n = \overline{z^n}$ for $n \geq 1$.
3. $3^n > 100n$ for $n > 5$.
4. Let A be an $n \times n$ matrix and let A' be the matrix obtained by replacing a row R of A with cR for some real number c. Use the definition of determinant to show $\det(A') = c\det(A)$.

Solution.

1. We set $P(n)$ to be the equation we are asked to prove. For $n = 1$, we compare both sides of the equation given in $P(n)$

$$\begin{aligned}\sum_{j=1}^{1}(a+(j-1)d) &\stackrel{?}{=} \frac{1}{2}(2a+(1-1)d)\\ a+(1-1)d &\stackrel{?}{=} \frac{1}{2}(2a)\\ a &= a \checkmark\end{aligned}$$

[2]Falling dominoes is the most widely used metaphor in the mainstream College Algebra books.
[3]This is how Carl climbed the stairs in the Cologne Cathedral. Well, that, and encouragement from Kai.

This shows the base case $P(1)$ is true. Next we assume $P(k)$ is true, that is, we assume

$$\sum_{j=1}^{k}(a+(j-1)d) = \frac{k}{2}(2a+(k-1)d)$$

and attempt to use this to show $P(k+1)$ is true. Namely, we must show

$$\sum_{j=1}^{k+1}(a+(j-1)d) = \frac{k+1}{2}(2a+(k+1-1)d)$$

To see how we can use $P(k)$ in this case to prove $P(k+1)$, we note that the sum in $P(k+1)$ is the sum of the first $k+1$ terms of the sequence $a_k = a+(k-1)d$ for $k \geq 1$ while the sum in $P(k)$ is the sum of the first k terms. We compare both side of the equation in $P(k+1)$.

$$\underbrace{\sum_{j=1}^{k+1}(a+(j-1)d)}_{\text{summing the first } k+1 \text{ terms}} \stackrel{?}{=} \frac{k+1}{2}(2a+(k+1-1)d)$$

$$\underbrace{\sum_{j=1}^{k}(a+(j-1)d)}_{\text{summing the first } k \text{ terms}} + \underbrace{(a+(k+1-1)d)}_{\text{adding the } (k+1)\text{st term}} \stackrel{?}{=} \frac{k+1}{2}(2a+kd)$$

$$\underbrace{\frac{k}{2}(2a+(k-1)d)}_{\text{Using } P(k)} + (a+kd) \stackrel{?}{=} \frac{(k+1)(2a+kd)}{2}$$

$$\frac{k(2a+(k-1)d)+2(a+kd)}{2} \stackrel{?}{=} \frac{2ka+k^2d+2a+kd}{2}$$

$$\frac{2ka+2a+k^2d+kd}{2} = \frac{2ka+2a+k^2d+kd}{2} \checkmark$$

Since all of our steps on both sides of the string of equations are reversible, we conclude that the two sides of the equation are equivalent and hence, $P(k+1)$ is true. By the Principle of Mathematical Induction, we have that $P(n)$ is true for all natural numbers n.

2. We let $P(n)$ be the formula $(\overline{z})^n = \overline{z^n}$. The base case $P(1)$ is $(\overline{z})^1 = \overline{z^1}$, which reduces to $\overline{z} = \overline{z}$ which is true. We now assume $P(k)$ is true, that is, we assume $(\overline{z})^k = \overline{z^k}$ and attempt to show that $P(k+1)$ is true. Since $(\overline{z})^{k+1} = (\overline{z})^k\,\overline{z}$, we can use the induction hypothesis and

write $(\overline{z})^k = \overline{z^k}$. Hence, $(\overline{z})^{k+1} = (\overline{z})^k\ \overline{z} = \overline{z^k}\,\overline{z}$. We now use the product rule for conjugates[4] to write $\overline{z^k}\,\overline{z} = \overline{z^k z} = \overline{z^{k+1}}$. This establishes $(\overline{z})^{k+1} = \overline{z^{k+1}}$, so that $P(k+1)$ is true. Hence, by the Principle of Mathematical Induction, $(\overline{z})^n = \overline{z^n}$ for all $n \geq 1$.

3. The first wrinkle we encounter in this problem is that we are asked to prove this formula for $n > 5$ instead of $n \geq 1$. Since n is a natural number, this means our base step occurs at $n = 6$. We can still use the PMI in this case, but our conclusion will be that the formula is valid for all $n \geq 6$. We let $P(n)$ be the inequality $3^n > 100n$, and check that $P(6)$ is true. Comparing $3^6 = 729$ and $100(6) = 600$, we see $3^6 > 100(6)$ as required. Next, we assume that $P(k)$ is true, that is we assume $3^k > 100k$. We need to show that $P(k+1)$ is true, that is, we need to show $3^{k+1} > 100(k+1)$. Since $3^{k+1} = 3 \cdot 3^k$, the induction hypothesis gives $3^{k+1} = 3 \cdot 3^k > 3(100k) = 300k$. We are done if we can show $300k > 100(k+1)$ for $k \geq 6$. Solving $300k > 100(k+1)$ we get $k > \frac{1}{2}$. Since $k \geq 6$, we know this is true. Putting all of this together, we have $3^{k+1} = 3 \cdot 3^k > 3(100k) = 300k > 100(k+1)$, and hence $P(k+1)$ is true. By induction, $3^n > 100n$ for all $n \geq 6$.

4. To prove this determinant property, we use induction on n, where we take $P(n)$ to be that the property we wish to prove is true for all $n \times n$ matrices. For the base case, we note that if A is a 1×1 matrix, then $A = [a]$ so $A' = [ca]$. By definition, $\det(A) = a$ and $\det(A') = ca$ so we have $\det(A') = c\det(A)$ as required. Now suppose that the property we wish to prove is true for all $k \times k$ matrices. Let A be a $(k+1) \times (k+1)$ matrix. We have two cases, depending on whether or not the row R being replaced is the first row of A.

Case 1: The row R being replaced is the first row of A. By definition,

$$\det(A') = \sum_{p=1}^{n} a'_{1p} C'_{1p}$$

where the $1p$ cofactor of A' is $C'_{1p} = (-1)^{(1+p)} \det\left(A'_{1p}\right)$ and A'_{1p} is the $k \times k$ matrix obtained by deleting the 1st row and pth column of A'.[5] Since the first row of A' is c times the first row of A, we have $a'_{1p} = c\, a_{1p}$. In addition, since the remaining rows of A' are identical to those of A, $A'_{1p} = A_{1p}$. (To obtain these matrices, the first row of A' is removed.) Hence $\det\left(A'_{1p}\right) = \det\left(A_{1p}\right)$, so that $C'_{1p} = C_{1p}$. As a result, we get

$$\det(A') = \sum_{p=1}^{n} a'_{1p} C'_{1p} = \sum_{p=1}^{n} c\, a_{1p} C_{1p} = c \sum_{p=1}^{n} a_{1p} C_{1p} = c\det(A),$$

as required. Hence, $P(k+1)$ is true in this case, which means the result is true in this case for all natural numbers $n \geq 1$. (You'll note that we did not use the induction hypothesis at all in this case. It is possible to restructure the proof so that induction is only used where

[4]See Exercise 54 in Section 3.4.
[5]See Section 8.5 for a review of this notation.

it is needed. While mathematically more elegant, it is less intuitive, and we stand by our approach because of its pedagogical value.)

Case 2: The row R being replaced is the not the first row of A. By definition,

$$\det(A') = \sum_{p=1}^{n} a'_{1p} C'_{1p},$$

where in this case, $a'_{1p} = a_{1p}$, since the first rows of A and A' are the same. The matrices A'_{1p} and A_{1p}, on the other hand, are different but in a very predictable way − the row in A'_{1p} which corresponds to the row cR in A' is exactly c times the row in A_{1p} which corresponds to the row R in A. In other words, A'_{1p} and A_{1p} are $k \times k$ matrices which satisfy the induction hypothesis. Hence, we know $\det\left(A'_{1p}\right) = c \det\left(A_{1p}\right)$ and $C'_{1p} = c\, C_{1p}$. We get

$$\det(A') = \sum_{p=1}^{n} a'_{1p} C'_{1p} = \sum_{p=1}^{n} a_{1p} c\, C_{1p} = c \sum_{p=1}^{n} a_{1p} C_{1p} = c \det(A),$$

which establishes $P(k + 1)$ to be true. Hence by induction, we have shown that the result holds in this case for $n \geq 1$ and we are done. □

While we have used the Principle of Mathematical Induction to prove some of the formulas we have merely motivated in the text, our main use of this result comes in Section 9.4 to prove the celebrated Binomial Theorem. The ardent Mathematics student will no doubt see the PMI in many courses yet to come. Sometimes it is explicitly stated and sometimes it remains hidden in the background. If ever you see a property stated as being true ‘for all natural numbers n’, it’s a solid bet that the formal proof requires the Principle of Mathematical Induction.

9.3.1 EXERCISES

In Exercises 1 - 7, prove each assertion using the Principle of Mathematical Induction.

1. $\displaystyle\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$

2. $\displaystyle\sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}$

3. $2^n > 500n$ for $n > 12$

4. $3^n \geq n^3$ for $n \geq 4$

5. Use the Product Rule for Absolute Value to show $|x^n| = |x|^n$ for all real numbers x and all natural numbers $n \geq 1$

6. Use the Product Rule for Logarithms to show $\log(x^n) = n\log(x)$ for all real numbers $x > 0$ and all natural numbers $n \geq 1$.

7. $\left[\begin{array}{cc} a & 0 \\ 0 & b \end{array}\right]^n = \left[\begin{array}{cc} a^n & 0 \\ 0 & b^n \end{array}\right]$ for $n \geq 1$.

8. Prove Equations 9.1 and 9.2 for the case of geometric sequences. That is:

   (a) For the sequence $a_1 = a$, $a_{n+1} = ra_n$, $n \geq 1$, prove $a_n = ar^{n-1}$, $n \geq 1$.

   (b) $\displaystyle\sum_{j=1}^{n} ar^{n-1} = a\left(\frac{1-r^n}{1-r}\right)$, if $r \neq 1$, $\displaystyle\sum_{j=1}^{n} ar^{n-1} = na$, if $r = 1$.

9. Prove that the determinant of a lower triangular matrix is the product of the entries on the main diagonal. (See Exercise 8.3.1 in Section 8.3.) Use this result to then show $\det(I_n) = 1$ where I_n is the $n \times n$ identity matrix.

10. Discuss the classic 'paradox' All Horses are the Same Color problem with your classmates.

9.3.2 Selected Answers

1. Let $P(n)$ be the sentence $\displaystyle\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$. For the base case, $n=1$, we get

$$\begin{aligned} \sum_{j=1}^{1} j^2 &\stackrel{?}{=} \frac{(1)(1+1)(2(1)+1)}{6} \\ 1^2 &= 1\,\checkmark \end{aligned}$$

We now assume $P(k)$ is true and use it to show $P(k+1)$ is true. We have

$$\begin{aligned} \sum_{j=1}^{k+1} j^2 &\stackrel{?}{=} \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \\ \sum_{j=1}^{k} j^2 + (k+1)^2 &\stackrel{?}{=} \frac{(k+1)(k+2)(2k+3)}{6} \\ \underbrace{\frac{k(k+1)(2k+1)}{6}}_{\text{Using } P(k)} + (k+1)^2 &\stackrel{?}{=} \frac{(k+1)(k+2)(2k+3)}{6} \\ \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} &\stackrel{?}{=} \frac{(k+1)(k+2)(2k+3)}{6} \\ \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} &\stackrel{?}{=} \frac{(k+1)(k+2)(2k+3)}{6} \\ \frac{(k+1)(k(2k+1) + 6(k+1))}{6} &\stackrel{?}{=} \frac{(k+1)(k+2)(2k+3)}{6} \\ \frac{(k+1)\left(2k^2+7k+6\right)}{6} &\stackrel{?}{=} \frac{(k+1)(k+2)(2k+3)}{6} \\ \frac{(k+1)(k+2)(2k+3)}{6} &= \frac{(k+1)(k+2)(2k+3)}{6}\,\checkmark \end{aligned}$$

By induction, $\displaystyle\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$ is true for all natural numbers $n \geq 1$.

4. Let $P(n)$ be the sentence $3^n > n^3$. Our base case is $n = 4$ and we check $3^4 = 81$ and $4^3 = 64$ so that $3^4 > 4^3$ as required. We now assume $P(k)$ is true, that is $3^k > k^3$, and try to show $P(k+1)$ is true. We note that $3^{k+1} = 3 \cdot 3^k > 3k^3$ and so we are done if we can show $3k^3 > (k+1)^3$ for $k \geq 4$. We can solve the inequality $3x^3 > (x+1)^3$ using the techniques of Section 5.3, and doing so gives us $x > \frac{1}{\sqrt[3]{3}-1} \approx 2.26$. Hence, for $k \geq 4$, $3^{k+1} = 3 \cdot 3^k > 3k^3 > (k+1)^3$ so that $3^{k+1} > (k+1)^3$. By induction, $3^n > n^3$ is true for all natural numbers $n \geq 4$.

6. Let $P(n)$ be the sentence $\log(x^n) = n\log(x)$. For the duration of this argument, we assume $x > 0$. The base case $P(1)$ amounts checking that $\log\left(x^1\right) = 1\log(x)$ which is clearly true. Next we assume $P(k)$ is true, that is $\log\left(x^k\right) = k\log(x)$ and try to show $P(k+1)$ is true. Using the Product Rule for Logarithms along with the induction hypothesis, we get

$$\log\left(x^{k+1}\right) = \log\left(x^k \cdot x\right) = \log\left(x^k\right) + \log(x) = k\log(x) + \log(x) = (k+1)\log(x)$$

Hence, $\log\left(x^{k+1}\right) = (k+1)\log(x)$. By induction $\log(x^n) = n\log(x)$ is true for all $x > 0$ and all natural numbers $n \geq 1$.

9. Let A be an $n \times n$ lower triangular matrix. We proceed to prove the $\det(A)$ is the product of the entries along the main diagonal by inducting on n. For $n = 1$, $A = [a]$ and $\det(A) = a$, so the result is (trivially) true. Next suppose the result is true for $k \times k$ lower triangular matrices. Let A be a $(k+1) \times (k+1)$ lower triangular matrix. Expanding $\det(A)$ along the first row, we have

$$\det(A) = \sum_{p=1}^{n} a_{1p} C_{1p}$$

Since $a_{1p} = 0$ for $2 \leq p \leq k+1$, this simplifies $\det(A) = a_{11}C_{11}$. By definition, we know that $C_{11} = (-1)^{1+1}\det\left(A_{11}\right) = \det\left(A_{11}\right)$ where A_{11} is $k \times k$ matrix obtained by deleting the first row and first column of A. Since A is lower triangular, so is A_{11} and, as such, the induction hypothesis applies to A_{11}. In other words, $\det\left(A_{11}\right)$ is the product of the entries along A_{11}'s main diagonal. Now, the entries on the main diagonal of A_{11} are the entries a_{22}, a_{33}, $\ldots$, $a_{(k+1)(k+1)}$ from the main diagonal of A. Hence,

$$\det(A) = a_{11}\det\left(A_{11}\right) = a_{11}\left(a_{22}a_{33}\cdots a_{(k+1)(k+1)}\right) = a_{11}a_{22}a_{33}\cdots a_{(k+1)(k+1)}$$

We have $\det(A)$ is the product of the entries along its main diagonal. This shows $P(k+1)$ is true, and, hence, by induction, the result holds for all $n \times n$ upper triangular matrices. The $n \times n$ identity matrix I_n is a lower triangular matrix whose main diagonal consists of all 1's. Hence, $\det\left(I_n\right) = 1$, as required.

9.4 The Binomial Theorem

In this section, we aim to prove the celebrated **Binomial Theorem**. Simply stated, the Binomial Theorem is a formula for the expansion of quantities $(a+b)^n$ for natural numbers n. In Elementary and Intermediate Algebra, you should have seen specific instances of the formula, namely

$$\begin{aligned}(a+b)^1 &= a+b \\ (a+b)^2 &= a^2+2ab+b^2 \\ (a+b)^3 &= a^3+3a^2b+3ab^2+b^3\end{aligned}$$

If we wanted the expansion for $(a+b)^4$ we would write $(a+b)^4 = (a+b)(a+b)^3$ and use the formula that we have for $(a+b)^3$ to get $(a+b)^4 = (a+b)\left(a^3+3a^2b+3ab^2+b^3\right) = a^4+4a^3b+6a^2b^2+4ab^3+b^4$. Generalizing this a bit, we see that if we have a formula for $(a+b)^k$, we can obtain a formula for $(a+b)^{k+1}$ by rewriting the latter as $(a+b)^{k+1} = (a+b)(a+b)^k$. Clearly this means Mathematical Induction plays a major role in the proof of the Binomial Theorem.[1] Before we can state the theorem we need to revisit the sequence of factorials which were introduced in Example 9.1.1 number 6 in Section 9.1.

Definition 9.4. Factorials: For a whole number n, ***n* factorial**, denoted $n!$, is the term f_n of the sequence $f_0 = 1$, $f_n = n \cdot f_{n-1}$, $n \geq 1$.

Recall this means $0! = 1$ and $n! = n(n-1)!$ for $n \geq 1$. Using the recursive definition, we get: $1! = 1 \cdot 0! = 1 \cdot 1 = 1$, $2! = 2 \cdot 1! = 2 \cdot 1 = 2$, $3! = 3 \cdot 2! = 3 \cdot 2 \cdot 1 = 6$ and $4! = 4 \cdot 3! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. Informally, $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$ with $0! = 1$ as our 'base case.' Our first example familiarizes us with some of the basic computations involving factorials.

Example 9.4.1.

1. Simplify the following expressions.

   (a) $\dfrac{3!\,2!}{0!}$ (b) $\dfrac{7!}{5!}$ (c) $\dfrac{1000!}{998!\,2!}$ (d) $\dfrac{(k+2)!}{(k-1)!}$, $k \geq 1$

2. Prove $n! > 3^n$ for all $n \geq 7$.

Solution.

1. We keep in mind the mantra, "When in doubt, write it out!" as we simplify the following.

   (a) We have been programmed to react with alarm to the presence of a 0 in the denominator, but in this case $0! = 1$, so the fraction is defined after all. As for the numerator, $3! = 3 \cdot 2 \cdot 1 = 6$ and $2! = 2 \cdot 1 = 2$, so we have $\frac{3!\,2!}{0!} = \frac{(6)(2)}{1} = 12$.

[1] It's pretty much the reason Section 9.3 is in the book.

(b) We have $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$ while $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. Dividing, we get $\frac{7!}{5!} = \frac{5040}{120} = 42$. While this is correct, we note that we could have saved ourselves some of time had we proceeded as follows

$$\frac{7!}{5!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{7 \cdot 6 \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 7 \cdot 6 = 42$$

In fact, should we want to fully exploit the recursive nature of the factorial, we can write

$$\frac{7!}{5!} = \frac{7 \cdot 6 \cdot 5!}{5!} = \frac{7 \cdot 6 \cdot \cancel{5!}}{\cancel{5!}} = 42$$

(c) Keeping in mind the lesson we learned from the previous problem, we have

$$\frac{1000!}{998!\,2!} = \frac{1000 \cdot 999 \cdot 998!}{998! \cdot 2!} = \frac{1000 \cdot 999 \cdot \cancel{998!}}{\cancel{998!} \cdot 2!} = \frac{999000}{2} = 499500$$

(d) This problem continues the theme which we have seen in the previous two problems. We first note that since $k+2$ is larger than $k-1$, $(k+2)!$ contains all of the factors of $(k-1)!$ and as a result we can get the $(k-1)!$ to cancel from the denominator. To see this, we begin by writing out $(k+2)!$ starting with $(k+2)$ and multiplying it by the numbers which precede it until we reach $(k-1)$: $(k+2)! = (k+2)(k+1)(k)(k-1)!$. As a result, we have

$$\frac{(k+2)!}{(k-1)!} = \frac{(k+2)(k+1)(k)(k-1)!}{(k-1)!} = \frac{(k+2)(k+1)(k)\cancel{(k-1)!}}{\cancel{(k-1)!}} = k(k+1)(k+2)$$

The stipulation $k \geq 1$ is there to ensure that all of the factorials involved are defined.

2. We proceed by induction and let $P(n)$ be the inequality $n! > 3^n$. The base case here is $n = 7$ and we see that $7! = 5040$ is larger than $3^7 = 2187$, so $P(7)$ is true. Next, we assume that $P(k)$ is true, that is, we assume $k! > 3^k$ and attempt to show $P(k+1)$ follows. Using the properties of the factorial, we have $(k+1)! = (k+1)k!$ and since $k! > 3^k$, we have $(k+1)! > (k+1)3^k$. Since $k \geq 7$, $k+1 \geq 8$, so $(k+1)3^k \geq 8 \cdot 3^k > 3 \cdot 3^k = 3^{k+1}$. Putting all of this together, we have $(k+1)! = (k+1)k! > (k+1)3^k > 3^{k+1}$ which shows $P(k+1)$ is true. By the Principle of Mathematical Induction, we have $n! > 3^n$ for all $n \geq 7$. □

Of all of the mathematical animals we have discussed in the text, factorials grow most quickly. In problem 2 of Example 9.4.1, we proved that $n!$ overtakes 3^n at $n = 7$. 'Overtakes' may be too polite a word, since $n!$ thoroughly trounces 3^n for $n \geq 7$, as any reasonable set of data will show. It can be shown that for any real number $x > 0$, not only does $n!$ eventually overtake x^n, but the ratio $\frac{x^n}{n!} \to 0$ as $n \to \infty$.[2]

Applications of factorials in the wild often involve counting arrangements. For example, if you have fifty songs on your mp3 player and wish arrange these songs in a playlist in which the order of the

[2]This fact is far more important than you could ever possibly imagine.