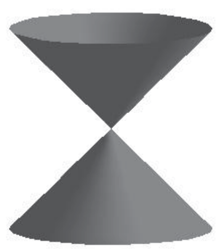
Chapter 7

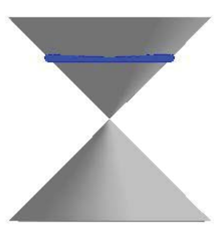
Hooked on Conics

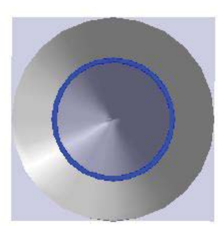
7.1 Introduction to Conics

In this chapter, we study the **Conic Sections** - literally 'sections of a cone'. Imagine a double-napped cone as seen below being 'sliced' by a plane.

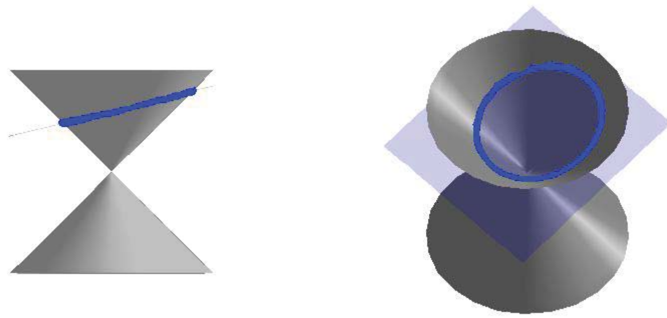


If we slice the cone with a horizontal plane the resulting curve is a **circle**.

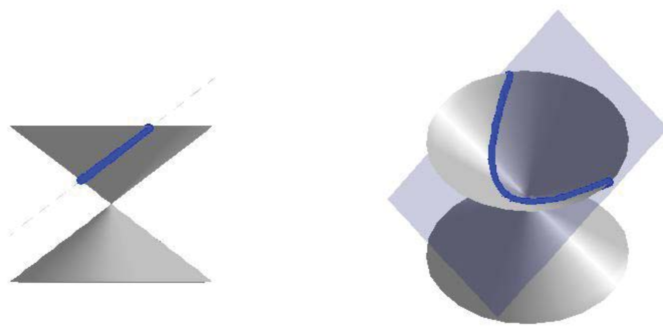




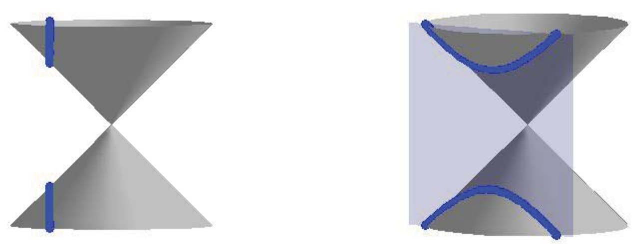
Tilting the plane ever so slightly produces an **ellipse**.



If the plane cuts parallel to the cone, we get a **parabola**.

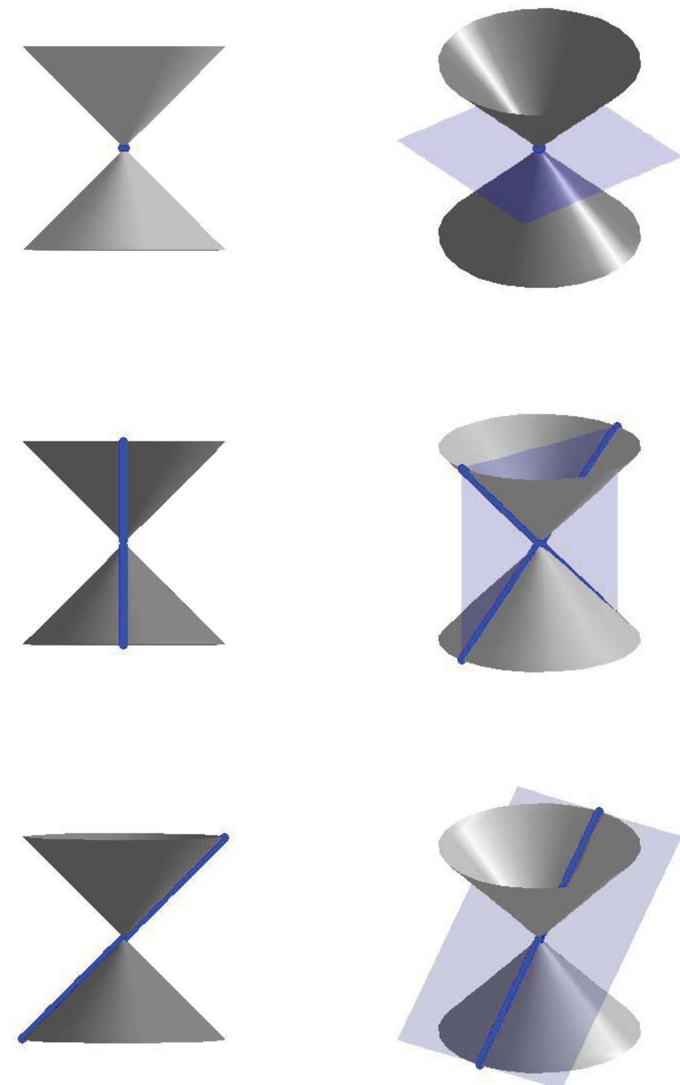


If we slice the cone with a vertical plane, we get a **hyperbola**.



For a wonderful animation describing the conics as intersections of planes and cones, see Dr. Louis Talman's Mathematics Animated Website.

If the slicing plane contains the vertex of the cone, we get the so-called 'degenerate' conics: a point, a line, or two intersecting lines.

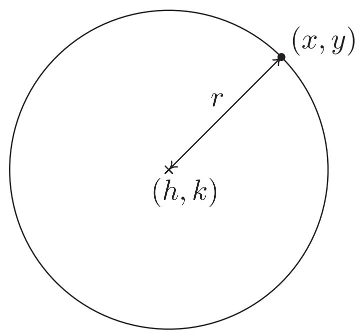


We will focus the discussion on the non-degenerate cases: circles, parabolas, ellipses, and hyperbolas, in that order. To determine equations which describe these curves, we will make use of their definitions in terms of distances.

7.2 CIRCLES

Recall from Geometry that a circle can be determined by fixing a point (called the center) and a positive number (called the radius) as follows.

Definition 7.1. A **circle** with center (h, k) and radius $r > 0$ is the set of all points (x, y) in the plane whose distance to (h, k) is r.



From the picture, we see that a point (x, y) is on the circle if and only if its distance to (h, k) is r. We express this relationship algebraically using the Distance Formula, Equation 1.1, as

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

By squaring both sides of this equation, we get an equivalent equation (since $r > 0$) which gives us the standard equation of a circle.

Equation 7.1. The Standard Equation of a Circle: The equation of a circle with center (h, k) and radius $r > 0$ is $(x-h)^2 + (y-k)^2 = r^2$.

Example 7.2.1. Write the standard equation of the circle with center $(-2, 3)$ and radius 5.

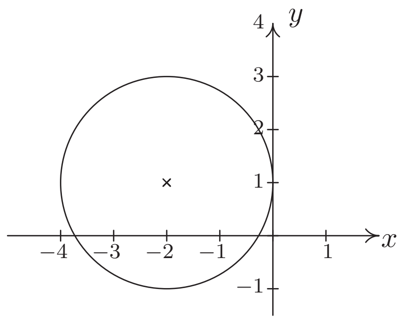
Solution. Here, $(h, k) = (-2, 3)$ and $r = 5$, so we get

$$\begin{aligned} (x - (-2))^2 + (y-3)^2 &= (5)^2 \\ (x+2)^2 + (y-3)^2 &= 25 \end{aligned}$$

□

Example 7.2.2. Graph $(x+2)^2 + (y-1)^2 = 4$. Find the center and radius.

Solution. From the standard form of a circle, Equation 7.1, we have that $x + 2$ is $x - h$, so $h = -2$ and $y - 1$ is $y - k$ so $k = 1$. This tells us that our center is $(-2, 1)$. Furthermore, $r^2 = 4$, so $r = 2$. Thus we have a circle centered at $(-2, 1)$ with a radius of 2. Graphing gives us



□

If we were to expand the equation in the previous example and gather up like terms, instead of the easily recognizable $(x+2)^2+(y-1)^2=4$, we'd be contending with $x^2+4x+y^2-2y+1=0$. If we're given such an equation, we can complete the square in each of the variables to see if it fits the form given in Equation 7.1 by following the steps given below.

To Write the Equation of a Circle in Standard Form

1. Group the same variables together on one side of the equation and position the constant on the other side.
2. Complete the square on both variables as needed.
3. Divide both sides by the coefficient of the squares. (For circles, they will be the same.)

Example 7.2.3. Complete the square to find the center and radius of $3x^2-6x+3y^2+4y-4=0$.

Solution.

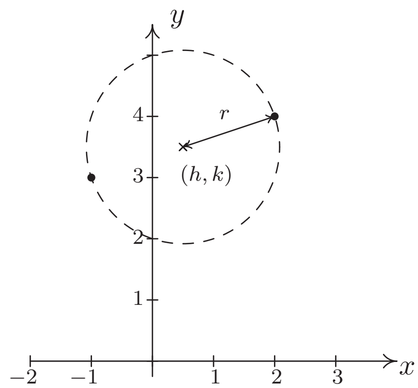
$$\begin{aligned}
3x^2-6x+3y^2+4y-4 &= 0 \\
3x^2-6x+3y^2+4y &= 4 && \text{add 4 to both sides} \\
3\left(x^2-2x\right)+3\left(y^2+\frac{4}{3}y\right) &= 4 && \text{factor out leading coefficients} \\
3\left(x^2-2x+\underline{1}\right)+3\left(y^2+\frac{4}{3}y+\underline{\underline{\frac{4}{9}}}\right) &= 4+3\underline{(1)}+3\underline{\underline{\left(\frac{4}{9}\right)}} && \text{complete the square in } x,\ y \\
3(x-1)^2+3\left(y+\frac{2}{3}\right)^2 &= \frac{25}{3} && \text{factor} \\
(x-1)^2+\left(y+\frac{2}{3}\right)^2 &= \frac{25}{9} && \text{divide both sides by 3}
\end{aligned}$$

From Equation 7.1, we identify $x-1$ as $x-h$, so $h=1$, and $y+\frac{2}{3}$ as $y-k$, so $k=-\frac{2}{3}$. Hence, the center is $(h,k)=\left(1,-\frac{2}{3}\right)$. Furthermore, we see that $r^2=\frac{25}{9}$ so the radius is $r=\frac{5}{3}$. □

It is possible to obtain equations like $(x-3)^2+(y+1)^2=0$ or $(x-3)^2+(y+1)^2=-1$, neither of which describes a circle. (Do you see why not?) The reader is encouraged to think about what, if any, points lie on the graphs of these two equations. The next example uses the Midpoint Formula, Equation 1.2, in conjunction with the ideas presented so far in this section.

Example 7.2.4. Write the standard equation of the circle which has $(-1,3)$ and $(2,4)$ as the endpoints of a diameter.

Solution. We recall that a diameter of a circle is a line segment containing the center and two points on the circle. Plotting the given data yields



Since the given points are endpoints of a diameter, we know their midpoint (h,k) is the center of the circle. Equation 1.2 gives us

$$\begin{aligned}(h,k) &= \left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)\\ &= \left(\frac{-1+2}{2},\frac{3+4}{2}\right)\\ &= \left(\frac{1}{2},\frac{7}{2}\right)\end{aligned}$$

The diameter of the circle is the distance between the given points, so we know that half of the distance is the radius. Thus,

$$\begin{aligned}r &= \frac{1}{2}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\\ &= \frac{1}{2}\sqrt{(2-(-1))^2+(4-3)^2}\\ &= \frac{1}{2}\sqrt{3^2+1^2}\\ &= \frac{\sqrt{10}}{2}\end{aligned}$$

Finally, since $\left(\frac{\sqrt{10}}{2}\right)^2=\frac{10}{4}$, our answer becomes $\left(x-\frac{1}{2}\right)^2+\left(y-\frac{7}{2}\right)^2=\frac{10}{4}$ □

We close this section with the most important[1] circle in all of mathematics: the **Unit Circle**.

Definition 7.2. The **Unit Circle** is the circle centered at $(0, 0)$ with a radius of 1. The standard equation of the Unit Circle is $x^2 + y^2 = 1$.

Example 7.2.5. Find the points on the unit circle with y-coordinate $\dfrac{\sqrt{3}}{2}$.

Solution. We replace y with $\dfrac{\sqrt{3}}{2}$ in the equation $x^2 + y^2 = 1$ to get

$$\begin{aligned} x^2 + y^2 &= 1 \\ x^2 + \left(\frac{\sqrt{3}}{2}\right)^2 &= 1 \\ \frac{3}{4} + x^2 &= 1 \\ x^2 &= \frac{1}{4} \\ x &= \pm\sqrt{\frac{1}{4}} \\ x &= \pm\frac{1}{2} \end{aligned}$$

Our final answers are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. □

[1] While this may seem like an opinion, it is indeed a fact. See Chapters 10 and 11 for details.

7.2.1 Exercises

In Exercises 1 - 6, find the standard equation of the circle and then graph it.

1. Center $(-1,-5)$, radius 10
2. Center $(4,-2)$, radius 3
3. Center $\left(-3,\frac{7}{13}\right)$, radius $\frac{1}{2}$
4. Center $(5,-9)$, radius $\ln(8)$
5. Center $\left(-e,\sqrt{2}\right)$, radius π
6. Center (π,e^2), radius $\sqrt[3]{91}$

In Exercises 7 - 12, complete the square in order to put the equation into standard form. Identify the center and the radius or explain why the equation does not represent a circle.

7. $x^2-4x+y^2+10y=-25$
8. $-2x^2-36x-2y^2-112=0$
9. $x^2+y^2+8x-10y-1=0$
10. $x^2+y^2+5x-y-1=0$
11. $4x^2+4y^2-24y+36=0$
12. $x^2+x+y^2-\frac{6}{5}y=1$

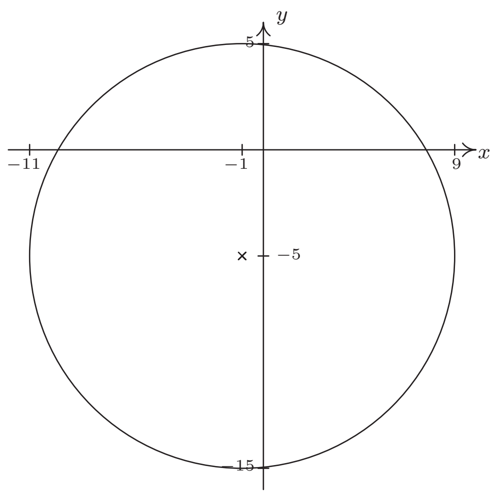
In Exercises 13 - 16, find the standard equation of the circle which satisfies the given criteria.

13. center $(3,5)$, passes through $(-1,-2)$
14. center $(3,6)$, passes through $(-1,4)$
15. endpoints of a diameter: $(3,6)$ and $(-1,4)$
16. endpoints of a diameter: $\left(\frac{1}{2},4\right)$, $\left(\frac{3}{2},-1\right)$
17. The Giant Wheel at Cedar Point is a circle with diameter 128 feet which sits on an 8 foot tall platform making its overall height is 136 feet.[2] Find an equation for the wheel assuming that its center lies on the y-axis and that the ground is the x-axis.
18. Verify that the following points lie on the Unit Circle: $(\pm 1,0)$, $(0,\pm 1)$, $\left(\pm\frac{\sqrt{2}}{2},\pm\frac{\sqrt{2}}{2}\right)$, $\left(\pm\frac{1}{2},\pm\frac{\sqrt{3}}{2}\right)$ and $\left(\pm\frac{\sqrt{3}}{2},\pm\frac{1}{2}\right)$
19. Discuss with your classmates how to obtain the standard equation of a circle, Equation 7.1, from the equation of the Unit Circle, $x^2+y^2=1$ using the transformations discussed in Section 1.7. (Thus every circle is just a few transformations away from the Unit Circle.)
20. Find an equation for the function represented graphically by the top half of the Unit Circle. Explain how the transformations is Section 1.7 can be used to produce a function whose graph is either the top or bottom of an arbitrary circle.
21. Find a one-to-one function whose graph is half of a circle. (Hint: Think piecewise.)

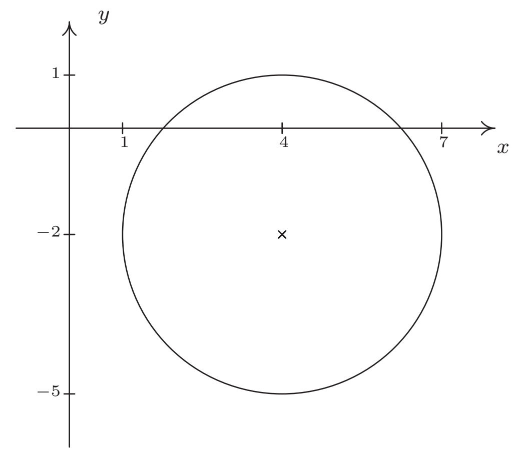
[2]Source: Cedar Point's webpage.

7.2.2 Answers

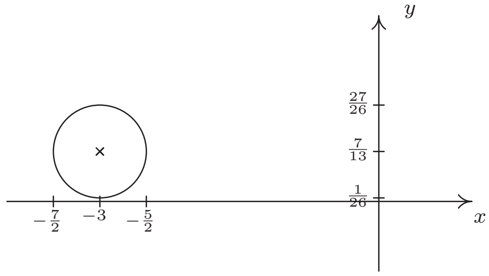
1. $(x+1)^2 + (y+5)^2 = 100$



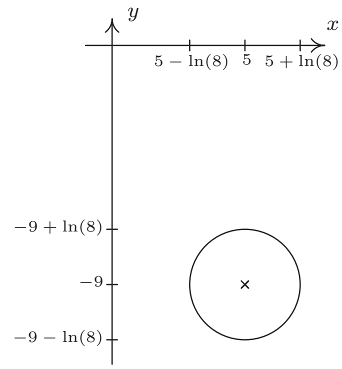
2. $(x-4)^2 + (y+2)^2 = 9$



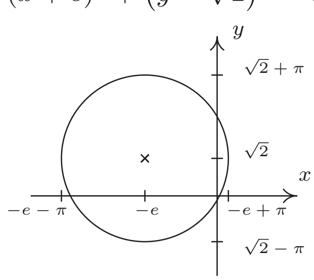
3. $(x+3)^2 + \left(y - \frac{7}{13}\right)^2 = \frac{1}{4}$



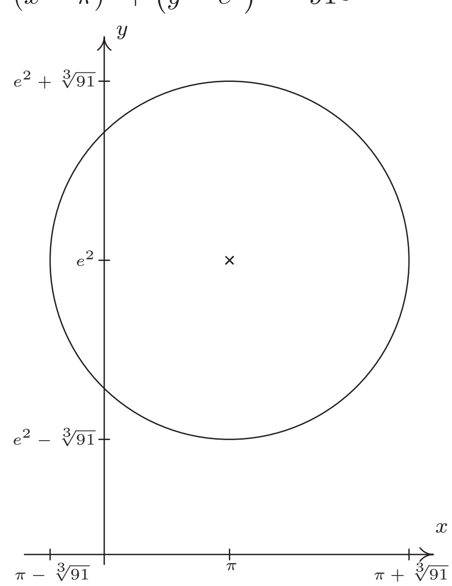
4. $(x-5)^2 + (y+9)^2 = (\ln(8))^2$



5. $(x+e)^2 + \left(y - \sqrt{2}\right)^2 = \pi^2$



6. $(x-\pi)^2 + \left(y - e^2\right)^2 = 91^{\frac{2}{3}}$



7. $(x-2)^2+(y+5)^2=4$
Center $(2,-5)$, radius $r=2$

8. $(x+9)^2+y^2=25$
Center $(-9,0)$, radius $r=5$

9. $(x+4)^2+(y-5)^2=42$
Center $(-4,5)$, radius $r=\sqrt{42}$

10. $\left(x+\frac{5}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{30}{4}$
Center $\left(-\frac{5}{2},\frac{1}{2}\right)$, radius $r=\frac{\sqrt{30}}{2}$

11. $x^2+(y-3)^2=0$
This is not a circle.

12. $\left(x+\frac{1}{2}\right)^2+\left(y-\frac{3}{5}\right)^2=\frac{161}{100}$
Center $\left(-\frac{1}{2},\frac{3}{5}\right)$, radius $r=\frac{\sqrt{161}}{10}$

13. $(x-3)^2+(y-5)^2=65$

14. $(x-3)^2+(y-6)^2=20$

15. $(x-1)^2+(y-5)^2=5$

16. $(x-1)^2+\left(y-\frac{3}{2}\right)^2=\frac{13}{2}$

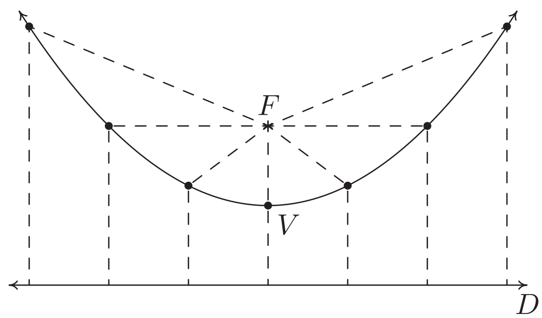
17. $x^2+(y-72)^2=4096$

7.3 PARABOLAS

We have already learned that the graph of a quadratic function $f(x) = ax^2 + bx + c$ $(a \neq 0)$ is called a **parabola**. To our surprise and delight, we may also define parabolas in terms of distance.

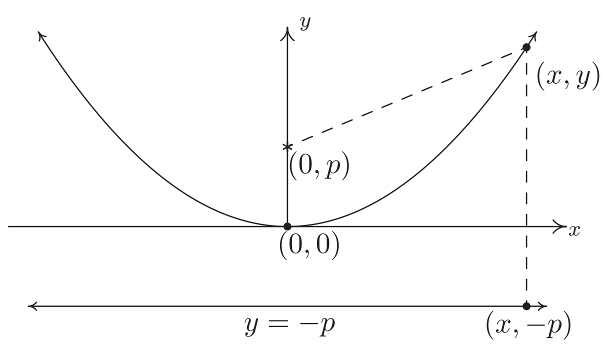
Definition 7.3. Let F be a point in the plane and D be a line not containing F. A **parabola** is the set of all points equidistant from F and D. The point F is called the **focus** of the parabola and the line D is called the **directrix** of the parabola.

Schematically, we have the following.



Each dashed line from the point F to a point on the curve has the same length as the dashed line from the point on the curve to the line D. The point suggestively labeled V is, as you should expect, the **vertex**. The vertex is the point on the parabola closest to the focus.

We want to use only the distance definition of parabola to derive the equation of a parabola and, if all is right with the universe, we should get an expression much like those studied in Section 2.3. Let p denote the directed[1] distance from the vertex to the focus, which by definition is the same as the distance from the vertex to the directrix. For simplicity, assume that the vertex is $(0,0)$ and that the parabola opens upwards. Hence, the focus is $(0,p)$ and the directrix is the line $y = -p$. Our picture becomes



From the definition of parabola, we know the distance from $(0,p)$ to (x,y) is the same as the distance from $(x,-p)$ to (x,y). Using the Distance Formula, Equation 1.1, we get

[1]We'll talk more about what 'directed' means later.

$$\begin{aligned}
\sqrt{(x-0)^2+(y-p)^2} &= \sqrt{(x-x)^2+(y-(-p))^2} & \\
\sqrt{x^2+(y-p)^2} &= \sqrt{(y+p)^2} & \\
x^2+(y-p)^2 &= (y+p)^2 & \text{square both sides} \\
x^2+y^2-2py+p^2 &= y^2+2py+p^2 & \text{expand quantities} \\
x^2 &= 4py & \text{gather like terms}
\end{aligned}$$

Solving for y yields $y = \frac{x^2}{4p}$, which is a quadratic function of the form found in Equation 2.4 with $a = \frac{1}{4p}$ and vertex $(0,0)$.

We know from previous experience that if the coefficient of x^2 is negative, the parabola opens downwards. In the equation $y = \frac{x^2}{4p}$ this happens when $p < 0$. In our formulation, we say that p is a 'directed distance' from the vertex to the focus: if $p > 0$, the focus is above the vertex; if $p < 0$, the focus is below the vertex. The **focal length** of a parabola is $|p|$.

If we choose to place the vertex at an arbitrary point (h,k), we arrive at the following formula using either transformations from Section 1.7 or re-deriving the formula from Definition 7.3.

Equation 7.2. The Standard Equation of a Vertical[a] Parabola: The equation of a (vertical) parabola with vertex (h,k) and focal length $|p|$ is

$$(x-h)^2 = 4p(y-k)$$

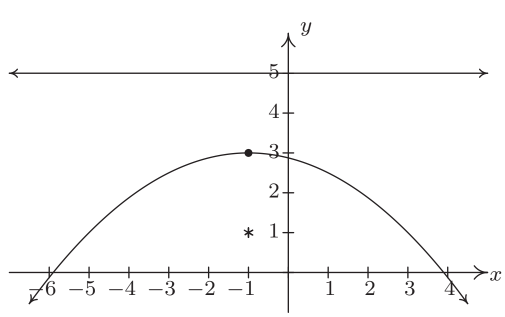
If $p > 0$, the parabola opens upwards; if $p < 0$, it opens downwards.

[a]That is, a parabola which opens either upwards or downwards.

Notice that in the standard equation of the parabola above, only one of the variables, x, is squared. This is a quick way to distinguish an equation of a parabola from that of a circle because in the equation of a circle, both variables are squared.

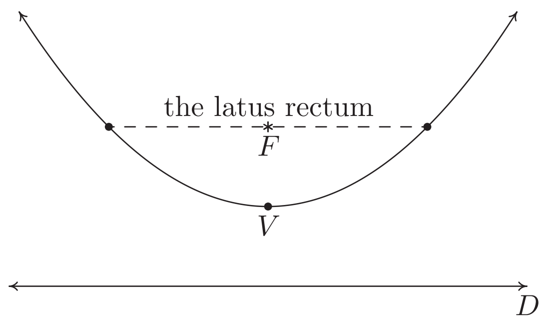
Example 7.3.1. Graph $(x+1)^2 = -8(y-3)$. Find the vertex, focus, and directrix.

Solution. We recognize this as the form given in Equation 7.2. Here, $x-h$ is $x+1$ so $h=-1$, and $y-k$ is $y-3$ so $k=3$. Hence, the vertex is $(-1,3)$. We also see that $4p=-8$ so $p=-2$. Since $p<0$, the focus will be below the vertex and the parabola will open downwards.



The distance from the vertex to the focus is $|p| = 2$, which means the focus is 2 units below the vertex. From $(-1, 3)$, we move down 2 units and find the focus at $(-1, 1)$. The directrix, then, is 2 units above the vertex, so it is the line $y = 5$. □

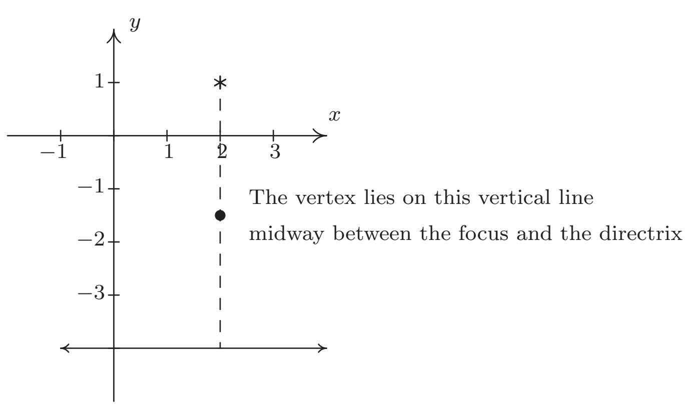
Of all of the information requested in the previous example, only the vertex is part of the graph of the parabola. So in order to get a sense of the actual shape of the graph, we need some more information. While we could plot a few points randomly, a more useful measure of how wide a parabola opens is the length of the parabola's latus rectum.[2] The **latus rectum** of a parabola is the line segment parallel to the directrix which contains the focus. The endpoints of the latus rectum are, then, two points on 'opposite' sides of the parabola. Graphically, we have the following.



It turns out[3] that the length of the latus rectum, called the **focal diameter** of the parabola is $|4p|$, which, in light of Equation 7.2, is easy to find. In our last example, for instance, when graphing $(x + 1)^2 = -8(y - 3)$, we can use the fact that the focal diameter is $|-8| = 8$, which means the parabola is 8 units wide at the focus, to help generate a more accurate graph by plotting points 4 units to the left and right of the focus.

Example 7.3.2. Find the standard form of the parabola with focus $(2, 1)$ and directrix $y = -4$.

Solution. Sketching the data yields,



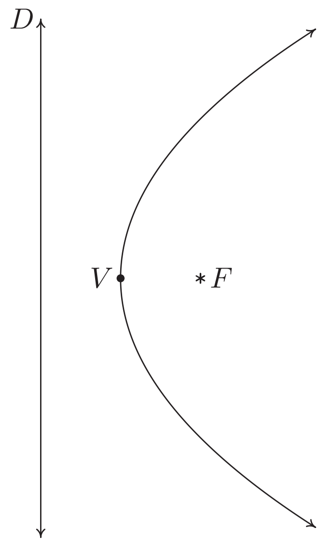
[2] No, I'm not making this up.
[3] Consider this an exercise to show what follows.

From the diagram, we see the parabola opens upwards. (Take a moment to think about it if you don't see that immediately.) Hence, the vertex lies below the focus and has an x-coordinate of 2. To find the y-coordinate, we note that the distance from the focus to the directrix is $1 - (-4) = 5$, which means the vertex lies $\frac{5}{2}$ units (halfway) below the focus. Starting at $(2, 1)$ and moving down $5/2$ units leaves us at $(2, -3/2)$, which is our vertex. Since the parabola opens upwards, we know p is positive. Thus $p = 5/2$. Plugging all of this data into Equation 7.2 give us

$$\begin{aligned} (x-2)^2 &= 4\left(\frac{5}{2}\right)\left(y - \left(-\frac{3}{2}\right)\right) \\ (x-2)^2 &= 10\left(y + \frac{3}{2}\right) \end{aligned}$$

□

If we interchange the roles of x and y, we can produce 'horizontal' parabolas: parabolas which open to the left or to the right. The directrices[4] of such animals would be vertical lines and the focus would either lie to the left or to the right of the vertex, as seen below.



Equation 7.3. The Standard Equation of a Horizontal Parabola: The equation of a (horizontal) parabola with vertex (h, k) and focal length $|p|$ is

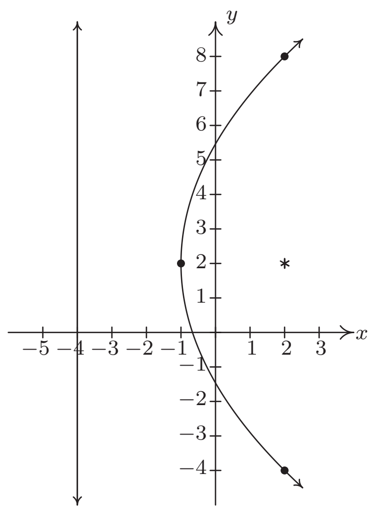
$$(y-k)^2 = 4p(x-h)$$

If $p > 0$, the parabola opens to the right; if $p < 0$, it opens to the left.

[4]plural of 'directrix'

Example 7.3.3. Graph $(y-2)^2 = 12(x+1)$. Find the vertex, focus, and directrix.

Solution. We recognize this as the form given in Equation 7.3. Here, $x-h$ is $x+1$ so $h=-1$, and $y-k$ is $y-2$ so $k=2$. Hence, the vertex is $(-1,2)$. We also see that $4p=12$ so $p=3$. Since $p>0$, the focus will be the right of the vertex and the parabola will open to the right. The distance from the vertex to the focus is $|p|=3$, which means the focus is 3 units to the right. If we start at $(-1,2)$ and move right 3 units, we arrive at the focus $(2,2)$. The directrix, then, is 3 units to the left of the vertex and if we move left 3 units from $(-1,2)$, we'd be on the vertical line $x=-4$. Since the focal diameter is $|4p|=12$, the parabola is 12 units wide at the focus, and thus there are points 6 units above and below the focus on the parabola.



□

As with circles, not all parabolas will come to us in the forms in Equations 7.2 or 7.3. If we encounter an equation with two variables in which exactly one variable is squared, we can attempt to put the equation into a standard form using the following steps.

To Write the Equation of a Parabola in Standard Form

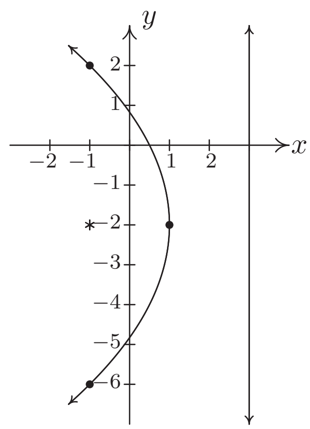
1. Group the variable which is squared on one side of the equation and position the non-squared variable and the constant on the other side.
2. Complete the square if necessary and divide by the coefficient of the perfect square.
3. Factor out the coefficient of the non-squared variable from it and the constant.

Example 7.3.4. Consider the equation $y^2+4y+8x=4$. Put this equation into standard form and graph the parabola. Find the vertex, focus, and directrix.

Solution. We need a perfect square (in this case, using y) on the left-hand side of the equation and factor out the coefficient of the non-squared variable (in this case, the x) on the other.

$$\begin{aligned} y^2 + 4y + 8x &= 4 \\ y^2 + 4y &= -8x + 4 \\ y^2 + 4y + 4 &= -8x + 4 + 4 \quad \text{complete the square in } y \text{ only} \\ (y+2)^2 &= -8x + 8 \quad \text{factor} \\ (y+2)^2 &= -8(x-1) \end{aligned}$$

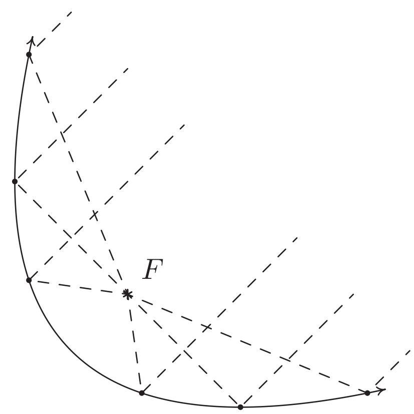
Now that the equation is in the form given in Equation 7.3, we see that $x - h$ is $x - 1$ so $h = 1$, and $y - k$ is $y + 2$ so $k = -2$. Hence, the vertex is $(1, -2)$. We also see that $4p = -8$ so that $p = -2$. Since $p < 0$, the focus will be the left of the vertex and the parabola will open to the left. The distance from the vertex to the focus is $|p| = 2$, which means the focus is 2 units to the left of 1, so if we start at $(1, -2)$ and move left 2 units, we arrive at the focus $(-1, -2)$. The directrix, then, is 2 units to the right of the vertex, so if we move right 2 units from $(1, -2)$, we'd be on the vertical line $x = 3$. Since the focal diameter is $|4p|$ is 8, the parabola is 8 units wide at the focus, so there are points 4 units above and below the focus on the parabola.



□

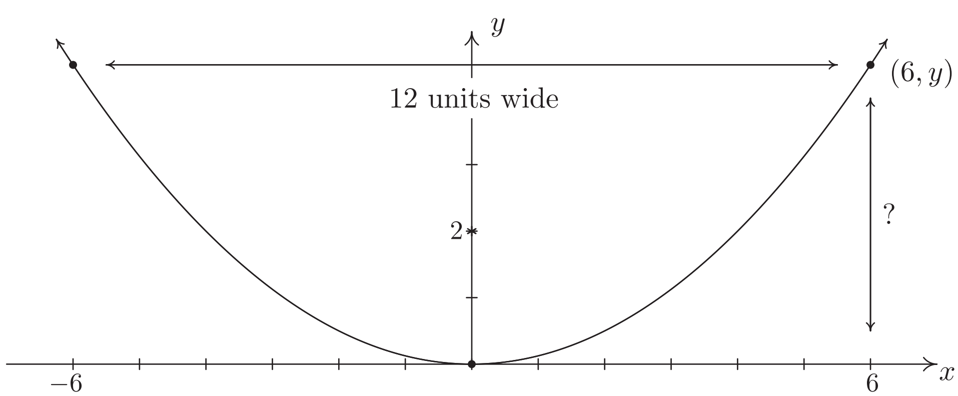
In studying quadratic functions, we have seen parabolas used to model physical phenomena such as the trajectories of projectiles. Other applications of the parabola concern its 'reflective property' which necessitates knowing about the focus of a parabola. For example, many satellite dishes are formed in the shape of a **paraboloid of revolution** as depicted below.

Every cross section through the vertex of the paraboloid is a parabola with the same focus. To see why this is important, imagine the dashed lines below as electromagnetic waves heading towards a parabolic dish. It turns out that the waves reflect off the parabola and concentrate at the focus which then becomes the optimal place for the receiver. If, on the other hand, we imagine the dashed lines as emanating from the focus, we see that the waves are reflected off the parabola in a coherent fashion as in the case in a flashlight. Here, the bulb is placed at the focus and the light rays are reflected off a parabolic mirror to give directional light.



Example 7.3.5. A satellite dish is to be constructed in the shape of a paraboloid of revolution. If the receiver placed at the focus is located 2 ft above the vertex of the dish, and the dish is to be 12 feet wide, how deep will the dish be?

Solution. One way to approach this problem is to determine the equation of the parabola suggested to us by this data. For simplicity, we'll assume the vertex is $(0,0)$ and the parabola opens upwards. Our standard form for such a parabola is $x^2 = 4py$. Since the focus is 2 units above the vertex, we know $p = 2$, so we have $x^2 = 8y$. Visually,



Since the parabola is 12 feet wide, we know the edge is 6 feet from the vertex. To find the depth, we are looking for the y value when $x = 6$. Substituting $x = 6$ into the equation of the parabola yields $6^2 = 8y$ or $y = \frac{36}{8} = \frac{9}{2} = 4.5$. Hence, the dish will be 4.5 feet deep. □

7.3.1 Exercises

In Exercises 1 - 8, sketch the graph of the given parabola. Find the vertex, focus and directrix. Include the endpoints of the latus rectum in your sketch.

1. $(x-3)^2 = -16y$
2. $\left(x+\frac{7}{3}\right)^2 = 2\left(y+\frac{5}{2}\right)$
3. $(y-2)^2 = -12(x+3)$
4. $(y+4)^2 = 4x$
5. $(x-1)^2 = 4(y+3)$
6. $(x+2)^2 = -20(y-5)$
7. $(y-4)^2 = 18(x-2)$
8. $\left(y+\frac{3}{2}\right)^2 = -7\left(x+\frac{9}{2}\right)$

In Exercises 9 - 14, put the equation into standard form and identify the vertex, focus and directrix.

9. $y^2 - 10y - 27x + 133 = 0$
10. $25x^2 + 20x + 5y - 1 = 0$
11. $x^2 + 2x - 8y + 49 = 0$
12. $2y^2 + 4y + x - 8 = 0$
13. $x^2 - 10x + 12y + 1 = 0$
14. $3y^2 - 27y + 4x + \frac{211}{4} = 0$

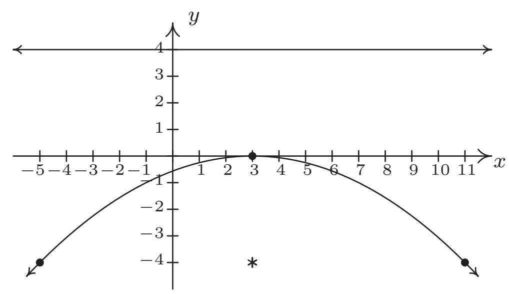
In Exercises 15 - 18, find an equation for the parabola which fits the given criteria.

15. Vertex $(7,0)$, focus $(0,0)$
16. Focus $(10,1)$, directrix $x=5$
17. Vertex $(-8,-9)$; $(0,0)$ and $(-16,0)$ are points on the curve
18. The endpoints of latus rectum are $(-2,-7)$ and $(4,-7)$
19. The mirror in Carl's flashlight is a paraboloid of revolution. If the mirror is 5 centimeters in diameter and 2.5 centimeters deep, where should the light bulb be placed so it is at the focus of the mirror?
20. A parabolic Wi-Fi antenna is constructed by taking a flat sheet of metal and bending it into a parabolic shape.[5] If the cross section of the antenna is a parabola which is 45 centimeters wide and 25 centimeters deep, where should the receiver be placed to maximize reception?
21. A parabolic arch is constructed which is 6 feet wide at the base and 9 feet tall in the middle. Find the height of the arch exactly 1 foot in from the base of the arch.
22. A popular novelty item is the 'mirage bowl.' Follow this link to see another startling application of the reflective property of the parabola.
23. With the help of your classmates, research spinning liquid mirrors. To get you started, check out this website.

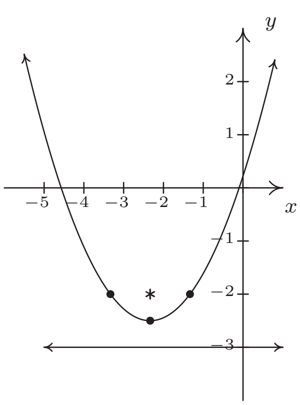
[5] This shape is called a 'parabolic cylinder.'

7.3.2 ANSWERS

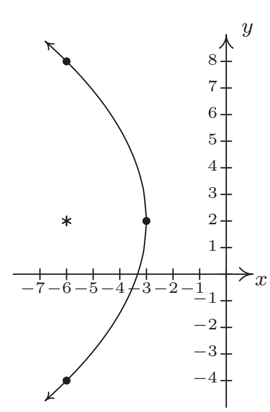
1. $(x-3)^2 = -16y$
   Vertex $(3, 0)$
   Focus $(3, -4)$
   Directrix $y = 4$
   Endpoints of latus rectum $(-5, -4)$, $(11, -4)$



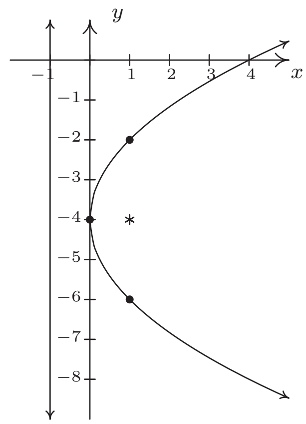
2. $\left(x + \frac{7}{3}\right)^2 = 2\left(y + \frac{5}{2}\right)$
   Vertex $\left(-\frac{7}{3}, -\frac{5}{2}\right)$
   Focus $\left(-\frac{7}{3}, -2\right)$
   Directrix $y = -3$
   Endpoints of latus rectum $\left(-\frac{10}{3}, -2\right)$, $\left(-\frac{4}{3}, -2\right)$



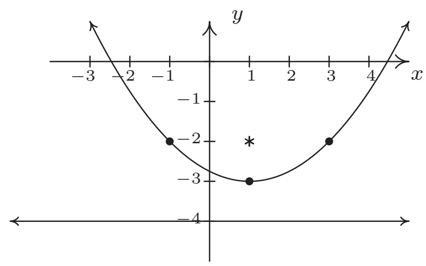
3. $(y-2)^2 = -12(x+3)$
   Vertex $(-3, 2)$
   Focus $(-6, 2)$
   Directrix $x = 0$
   Endpoints of latus rectum $(-6, 8)$, $(-6, -4)$



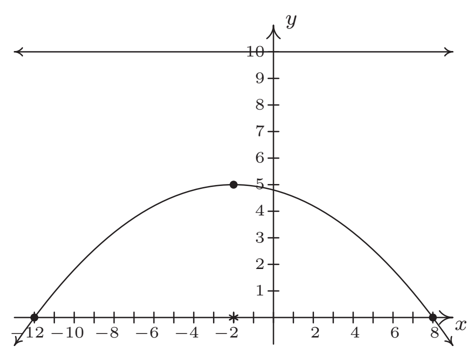
4. $(y+4)^2 = 4x$
Vertex $(0,-4)$
Focus $(1,-4)$
Directrix $x=-1$
Endpoints of latus rectum $(1,-2)$, $(1,-6)$



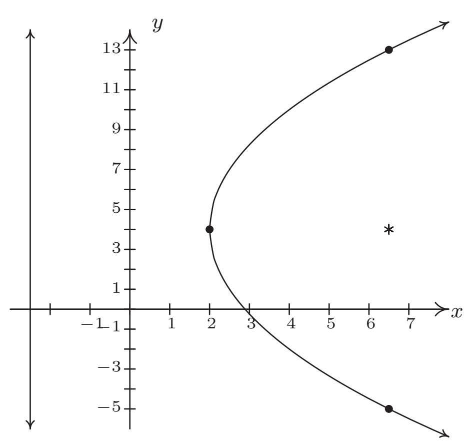
5. $(x-1)^2 = 4(y+3)$
Vertex $(1,-3)$
Focus $(1,-2)$
Directrix $y=-4$
Endpoints of latus rectum $(3,-2)$, $(-1,-2)$



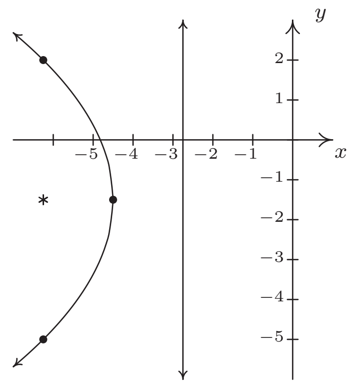
6. $(x+2)^2 = -20(y-5)$
Vertex $(-2,5)$
Focus $(-2,0)$
Directrix $y=10$
Endpoints of latus rectum $(-12,0)$, $(8,0)$



7. $(y-4)^2 = 18(x-2)$
Vertex $(2,4)$
Focus $\left(\frac{13}{2},4\right)$
Directrix $x=-\frac{5}{2}$
Endpoints of latus rectum $\left(\frac{13}{2},-5\right)$, $\left(\frac{13}{2},13\right)$



8. $\left(y+\frac{3}{2}\right)^2=-7\left(x+\frac{9}{2}\right)$
Vertex $\left(-\frac{9}{2},-\frac{3}{2}\right)$
Focus $\left(-\frac{25}{4},-\frac{3}{2}\right)$
Directrix $x=-\frac{11}{4}$
Endpoints of latus rectum $\left(-\frac{25}{4},2\right)$, $\left(-\frac{25}{4},-5\right)$



9. $(y-5)^2=27(x-4)$
Vertex $(4,5)$
Focus $\left(\frac{43}{4},5\right)$
Directrix $x=-\frac{11}{4}$

10. $\left(x+\frac{2}{5}\right)^2=-\frac{1}{5}(y-1)$
Vertex $\left(-\frac{2}{5},1\right)$
Focus $\left(-\frac{2}{5},\frac{19}{20}\right)$
Directrix $y=\frac{21}{20}$

11. $(x+1)^2=8(y-6)$
Vertex $(-1,6)$
Focus $(-1,8)$
Directrix $y=4$

12. $(y+1)^2=-\frac{1}{2}(x-10)$
Vertex $(10,-1)$
Focus $\left(\frac{79}{8},-1\right)$
Directrix $x=\frac{81}{8}$

13. $(x-5)^2=-12(y-2)$
Vertex $(5,2)$
Focus $(5,-1)$
Directrix $y=5$

14. $\left(y-\frac{9}{2}\right)^2=-\frac{4}{3}(x-2)$
Vertex $\left(2,\frac{9}{2}\right)$
Focus $\left(\frac{5}{3},\frac{9}{2}\right)$
Directrix $x=\frac{7}{3}$

15. $y^2=-28(x-7)$

16. $(y-1)^2=10\left(x-\frac{15}{2}\right)$

17. $(x+8)^2=\frac{64}{9}(y+9)$

18. $(x-1)^2=6\left(y+\frac{17}{2}\right)$ or
$(x-1)^2=-6\left(y+\frac{11}{2}\right)$

19. The bulb should be placed 0.625 centimeters above the vertex of the mirror. (As verified by Carl himself!)

20. The receiver should be placed 5.0625 centimeters from the vertex of the cross section of the antenna.

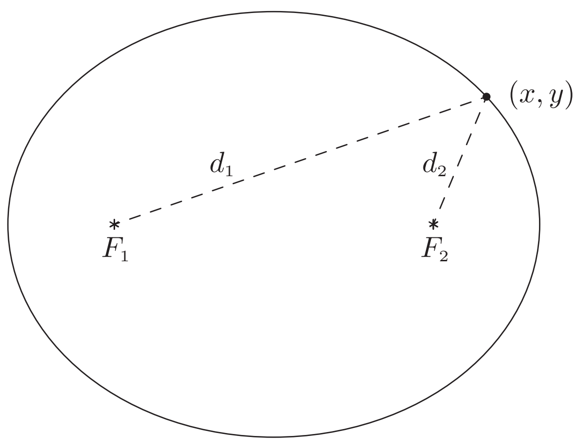
21. The arch can be modeled by $x^2=-(y-9)$ or $y=9-x^2$. One foot in from the base of the arch corresponds to either $x=\pm 2$, so the height is $y=9-(\pm 2)^2=5$ feet.

7.4 Ellipses

In the definition of a circle, Definition 7.1, we fixed a point called the **center** and considered all of the points which were a fixed distance r from that one point. For our next conic section, the ellipse, we fix two distinct points and a distance d to use in our definition.

Definition 7.4. Given two distinct points F_1 and F_2 in the plane and a fixed distance d, an **ellipse** is the set of all points (x, y) in the plane such that the sum of each of the distances from F_1 and F_2 to (x, y) is d. The points F_1 and F_2 are called the **foci**[a] of the ellipse.

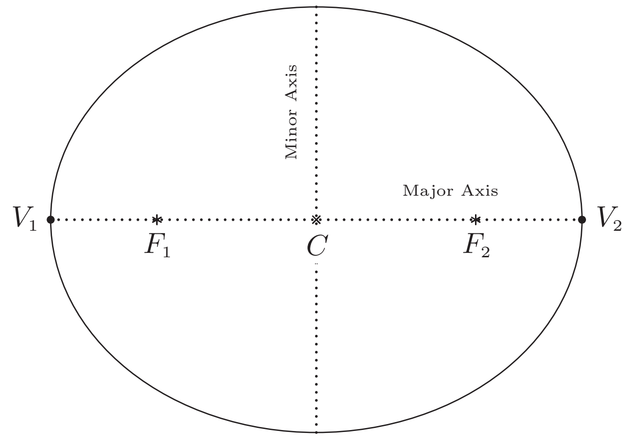
[a]the plural of 'focus'



$d_1 + d_2 = d$ for all (x, y) on the ellipse

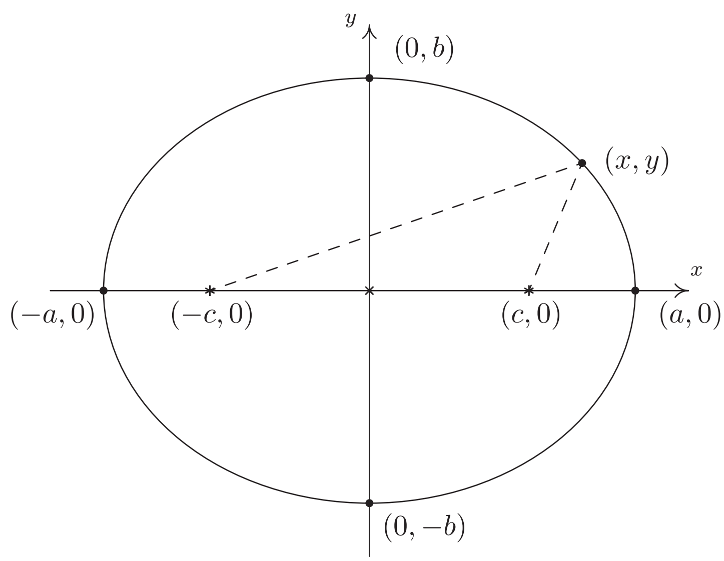
We may imagine taking a length of string and anchoring it to two points on a piece of paper. The curve traced out by taking a pencil and moving it so the string is always taut is an ellipse.

The **center** of the ellipse is the midpoint of the line segment connecting the two foci. The **major axis** of the ellipse is the line segment connecting two opposite ends of the ellipse which also contains the center and foci. The **minor axis** of the ellipse is the line segment connecting two opposite ends of the ellipse which contains the center but is perpendicular to the major axis. The **vertices** of an ellipse are the points of the ellipse which lie on the major axis. Notice that the center is also the midpoint of the major axis, hence it is the midpoint of the vertices. In pictures we have,



An ellipse with center C; foci F_1, F_2; and vertices V_1, V_2

Note that the major axis is the longer of the two axes through the center, and likewise, the minor axis is the shorter of the two. In order to derive the standard equation of an ellipse, we assume that the ellipse has its center at $(0,0)$, its major axis along the x-axis, and has foci $(c,0)$ and $(-c,0)$ and vertices $(-a,0)$ and $(a,0)$. We will label the y-intercepts of the ellipse as $(0,b)$ and $(0,-b)$ (We assume a, b, and c are all positive numbers.) Schematically,



Note that since $(a,0)$ is on the ellipse, it must satisfy the conditions of Definition 7.4. That is, the distance from $(-c,0)$ to $(a,0)$ plus the distance from $(c,0)$ to $(a,0)$ must equal the fixed distance d. Since all of these points lie on the x-axis, we get

$$\begin{aligned} \text{distance from } (-c,0) \text{ to } (a,0) + \text{distance from } (c,0) \text{ to } (a,0) &= d \\ (a+c)+(a-c) &= d \\ 2a &= d \end{aligned}$$

In other words, the fixed distance d mentioned in the definition of the ellipse is none other than the length of the major axis. We now use that fact $(0, b)$ is on the ellipse, along with the fact that $d = 2a$ to get

$$\begin{aligned}
\text{distance from } (-c,0) \text{ to } (0,b) + \text{distance from } (c,0) \text{ to } (0,b) &= 2a \\
\sqrt{(0-(-c))^2 + (b-0)^2} + \sqrt{(0-c)^2 + (b-0)^2} &= 2a \\
\sqrt{b^2+c^2} + \sqrt{b^2+c^2} &= 2a \\
2\sqrt{b^2+c^2} &= 2a \\
\sqrt{b^2+c^2} &= a
\end{aligned}$$

From this, we get $a^2 = b^2 + c^2$, or $b^2 = a^2 - c^2$, which will prove useful later. Now consider a point (x, y) on the ellipse. Applying Definition 7.4, we get

$$\begin{aligned}
\text{distance from } (-c,0) \text{ to } (x,y) + \text{distance from } (c,0) \text{ to } (x,y) &= 2a \\
\sqrt{(x-(-c))^2 + (y-0)^2} + \sqrt{(x-c)^2 + (y-0)^2} &= 2a \\
\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} &= 2a
\end{aligned}$$

In order to make sense of this situation, we need to make good use of Intermediate Algebra.

$$\begin{aligned}
\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} &= 2a \\
\sqrt{(x+c)^2 + y^2} &= 2a - \sqrt{(x-c)^2 + y^2} \\
\left(\sqrt{(x+c)^2 + y^2}\right)^2 &= \left(2a - \sqrt{(x-c)^2 + y^2}\right)^2 \\
(x+c)^2 + y^2 &= 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2 \\
4a\sqrt{(x-c)^2 + y^2} &= 4a^2 + (x-c)^2 - (x+c)^2 \\
4a\sqrt{(x-c)^2 + y^2} &= 4a^2 - 4cx \\
a\sqrt{(x-c)^2 + y^2} &= a^2 - cx \\
\left(a\sqrt{(x-c)^2 + y^2}\right)^2 &= \left(a^2 - cx\right)^2 \\
a^2\left((x-c)^2 + y^2\right) &= a^4 - 2a^2cx + c^2x^2 \\
a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2 &= a^4 - 2a^2cx + c^2x^2 \\
a^2x^2 - c^2x^2 + a^2y^2 &= a^4 - a^2c^2 \\
\left(a^2 - c^2\right)x^2 + a^2y^2 &= a^2\left(a^2 - c^2\right)
\end{aligned}$$

We are nearly finished. Recall that $b^2 = a^2 - c^2$ so that

$$\begin{aligned}
\left(a^2 - c^2\right)x^2 + a^2y^2 &= a^2\left(a^2 - c^2\right) \\
b^2x^2 + a^2y^2 &= a^2b^2 \\
\frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1
\end{aligned}$$

This equation is for an ellipse centered at the origin. To get the formula for the ellipse centered at (h,k), we could use the transformations from Section 1.7 or re-derive the equation using Definition 7.4 and the distance formula to obtain the formula below.

Equation 7.4. The Standard Equation of an Ellipse: For positive unequal numbers a and b, the equation of an ellipse with center (h,k) is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

Some remarks about Equation 7.4 are in order. First note that the values a and b determine how far in the x and y directions, respectively, one counts from the center to arrive at points on the ellipse. Also take note that if $a>b$, then we have an ellipse whose major axis is horizontal, and hence, the foci lie to the left and right of the center. In this case, as we've seen in the derivation, the distance from the center to the focus, c, can be found by $c=\sqrt{a^2-b^2}$. If $b>a$, the roles of the major and minor axes are reversed, and the foci lie above and below the center. In this case, $c=\sqrt{b^2-a^2}$. In either case, c is the distance from the center to each focus, and $c=\sqrt{\text{bigger denominator}-\text{smaller denominator}}$. Finally, it is worth mentioning that if we take the standard equation of a circle, Equation 7.1, and divide both sides by r^2, we get

Equation 7.5. The Alternate Standard Equation of a Circle: The equation of a circle with center (h,k) and radius $r>0$ is

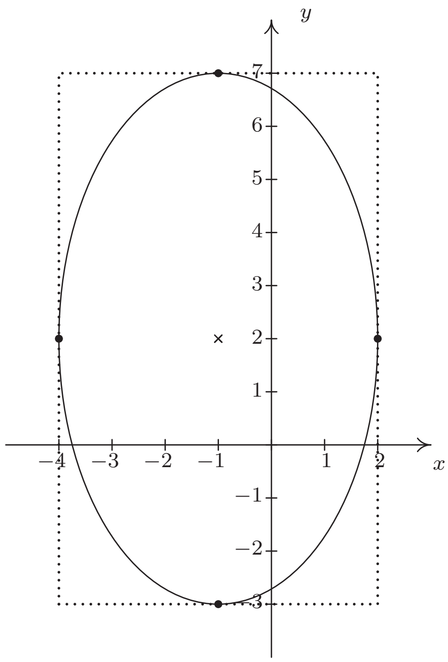
$$\frac{(x-h)^2}{r^2}+\frac{(y-k)^2}{r^2}=1$$

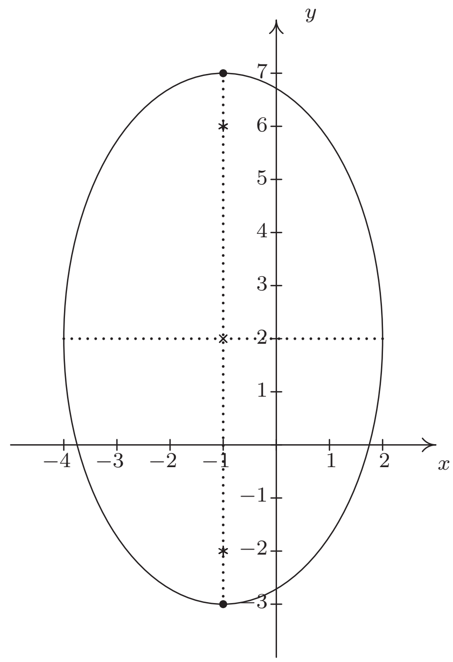
Notice the similarity between Equation 7.4 and Equation 7.5. Both equations involve a sum of squares equal to 1; the difference is that with a circle, the denominators are the same, and with an ellipse, they are different. If we take a transformational approach, we can consider both Equations 7.4 and 7.5 as shifts and stretches of the Unit Circle $x^2+y^2=1$ in Definition 7.2. Replacing x with $(x-h)$ and y with $(y-k)$ causes the usual horizontal and vertical shifts. Replacing x with $\frac{x}{a}$ and y with $\frac{y}{b}$ causes the usual vertical and horizontal stretches. In other words, it is perfectly fine to think of an ellipse as the deformation of a circle in which the circle is stretched farther in one direction than the other.[1]

Example 7.4.1. Graph $\frac{(x+1)^2}{9}+\frac{(y-2)^2}{25}=1$. Find the center, the lines which contain the major and minor axes, the vertices, the endpoints of the minor axis, and the foci.

Solution. We see that this equation is in the standard form of Equation 7.4. Here $x-h$ is $x+1$ so $h=-1$, and $y-k$ is $y-2$ so $k=2$. Hence, our ellipse is centered at $(-1,2)$. We see that $a^2=9$ so $a=3$, and $b^2=25$ so $b=5$. This means that we move 3 units left and right from the center and 5 units up and down from the center to arrive at points on the ellipse. As an aid to sketching, we draw a rectangle matching this description, called a **guide rectangle**, and sketch the ellipse inside this rectangle as seen below on the left.

[1] This was foreshadowed in Exercise 19 in Section 7.2.

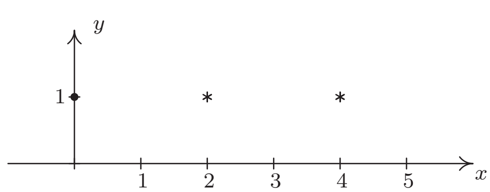




Since we moved farther in the y direction than in the x direction, the major axis will lie along the vertical line $x = -1$, which means the minor axis lies along the horizontal line, $y = 2$. The vertices are the points on the ellipse which lie along the major axis so in this case, they are the points $(-1, 7)$ and $(-1, -3)$, and the endpoints of the minor axis are $(-4, 2)$ and $(2, 2)$. (Notice these points are the four points we used to draw the guide rectangle.) To find the foci, we find $c = \sqrt{25 - 9} = \sqrt{16} = 4$, which means the foci lie 4 units from the center. Since the major axis is vertical, the foci lie 4 units above and below the center, at $(-1, -2)$ and $(-1, 6)$. Plotting all this information gives the graph seen above on the right. □

Example 7.4.2. Find the equation of the ellipse with foci $(2, 1)$ and $(4, 1)$ and vertex $(0, 1)$.

Solution. Plotting the data given to us, we have



From this sketch, we know that the major axis is horizontal, meaning $a > b$. Since the center is the midpoint of the foci, we know it is $(3, 1)$. Since one vertex is $(0, 1)$ we have that $a = 3$, so $a^2 = 9$. All that remains is to find b^2. Since the foci are 1 unit away from the center, we know $c = 1$. Since $a > b$, we have $c = \sqrt{a^2 - b^2}$, or $1 = \sqrt{9 - b^2}$, so $b^2 = 8$. Substituting all of our findings into the equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, we get our final answer to be $\frac{(x-3)^2}{9} + \frac{(y-1)^2}{8} = 1$. □

As with circles and parabolas, an equation may be given which is an ellipse, but isn't in the standard form of Equation 7.4. In those cases, as with circles and parabolas before, we will need to massage the given equation into the standard form.

To Write the Equation of an Ellipse in Standard Form

1. Group the same variables together on one side of the equation and position the constant on the other side.
2. Complete the square in both variables as needed.
3. Divide both sides by the constant term so that the constant on the other side of the equation becomes 1.

Example 7.4.3. Graph $x^2 + 4y^2 - 2x + 24y + 33 = 0$. Find the center, the lines which contain the major and minor axes, the vertices, the endpoints of the minor axis, and the foci.

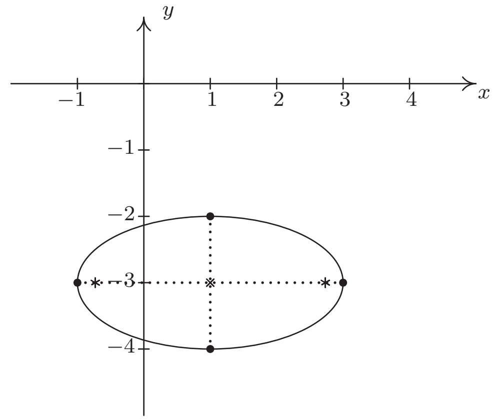
Solution. Since we have a sum of squares and the squared terms have unequal coefficients, it's a good bet we have an ellipse on our hands.[2] We need to complete both squares, and then divide, if necessary, to get the right-hand side equal to 1.

$$\begin{aligned}
x^2 + 4y^2 - 2x + 24y + 33 &= 0 \\
x^2 - 2x + 4y^2 + 24y &= -33 \\
x^2 - 2x + 4\left(y^2 + 6y\right) &= -33 \\
\left(x^2 - 2x + 1\right) + 4\left(y^2 + 6y + 9\right) &= -33 + 1 + 4(9) \\
(x-1)^2 + 4(y+3)^2 &= 4 \\
\frac{(x-1)^2 + 4(y+3)^2}{4} &= \frac{4}{4} \\
\frac{(x-1)^2}{4} + (y+3)^2 &= 1 \\
\frac{(x-1)^2}{4} + \frac{(y+3)^2}{1} &= 1
\end{aligned}$$

Now that this equation is in the standard form of Equation 7.4, we see that $x - h$ is $x - 1$ so $h = 1$, and $y - k$ is $y + 3$ so $k = -3$. Hence, our ellipse is centered at $(1, -3)$. We see that $a^2 = 4$ so $a = 2$, and $b^2 = 1$ so $b = 1$. This means we move 2 units left and right from the center and 1 unit up and down from the center to arrive at points on the ellipse. Since we moved farther in the x direction than in the y direction, the major axis will lie along the horizontal line $y = -3$, which means the minor axis lies along the vertical line $x = 1$. The vertices are the points on the ellipse which lie along the major axis so in this case, they are the points $(-1, -3)$ and $(3, -3)$, and the endpoints of the minor axis are $(1, -2)$ and $(1, -4)$. To find the foci, we find $c = \sqrt{4-1} = \sqrt{3}$, which means

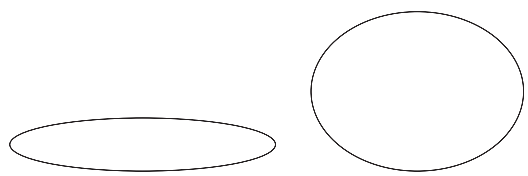
[2] The equation of a parabola has only one squared variable and the equation of a circle has two squared variables with *identical* coefficients.

the foci lie $\sqrt{3}$ units from the center. Since the major axis is horizontal, the foci lie $\sqrt{3}$ units to the left and right of the center, at $(1-\sqrt{3},-3)$ and $(1+\sqrt{3},-3)$. Plotting all of this information gives



□

As you come across ellipses in the homework exercises and in the wild, you'll notice they come in all shapes in sizes. Compare the two ellipses below.



Certainly, one ellipse is more round than the other. This notion of 'roundness' is quantified below.

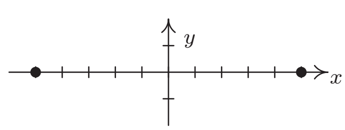
Definition 7.5. The **eccentricity** of an ellipse, denoted e, is the following ratio:

$$e = \frac{\text{distance from the center to a focus}}{\text{distance from the center to a vertex}}$$

In an ellipse, the foci are closer to the center than the vertices, so $0 < e < 1$. The ellipse above on the left has eccentricity $e \approx 0.98$; for the ellipse above on the right, $e \approx 0.66$. In general, the closer the eccentricity is to 0, the more 'circular' the ellipse; the closer the eccentricity is to 1, the more 'eccentric' the ellipse.

Example 7.4.4. Find the equation of the ellipse whose vertices are $(\pm 5, 0)$ with eccentricity $e = \frac{1}{4}$.

Solution. As before, we plot the data given to us

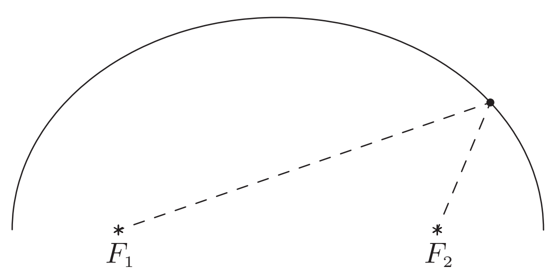


From this sketch, we know that the major axis is horizontal, meaning $a > b$. With the vertices located at $(\pm 5, 0)$, we get $a = 5$ so $a^2 = 25$. We also know that the center is $(0, 0)$ because the center is the midpoint of the vertices. All that remains is to find b^2. To that end, we use the fact that the eccentricity $e = \frac{1}{4}$ which means

$$e = \frac{\text{distance from the center to a focus}}{\text{distance from the center to a vertex}} = \frac{c}{a} = \frac{c}{5} = \frac{1}{4}$$

from which we get $c = \frac{5}{4}$. To get b^2, we use the fact that $c = \sqrt{a^2 - b^2}$, so $\frac{5}{4} = \sqrt{25 - b^2}$ from which we get $b^2 = \frac{375}{16}$. Substituting all of our findings into the equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, yields our final answer $\frac{x^2}{25} + \frac{16y^2}{375} = 1$. □

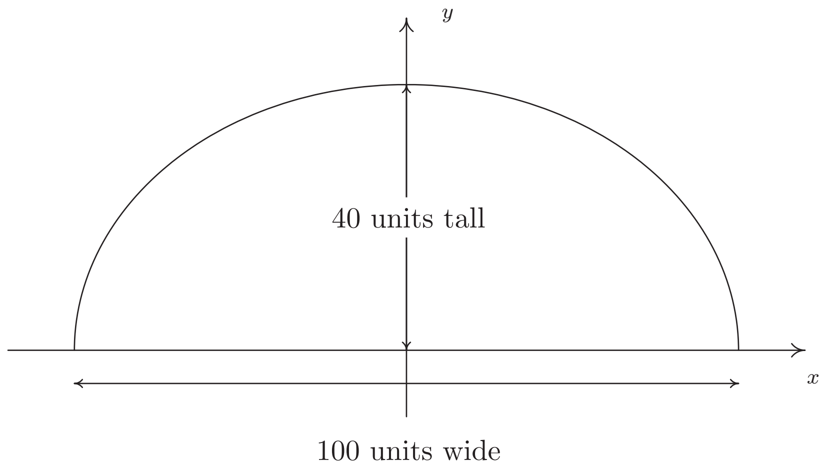
As with parabolas, ellipses have a reflective property. If we imagine the dashed lines below representing sound waves, then the waves emanating from one focus reflect off the top of the ellipse and head towards the other focus.



Such geometry is exploited in the construction of so-called 'Whispering Galleries'. If a person whispers at one focus, a person standing at the other focus will hear the first person as if they were standing right next to them. We explore the Whispering Galleries in our last example.

Example 7.4.5. Jamie and Jason want to exchange secrets (terrible secrets) from across a crowded whispering gallery. Recall that a whispering gallery is a room which, in cross section, is half of an ellipse. If the room is 40 feet high at the center and 100 feet wide at the floor, how far from the outer wall should each of them stand so that they will be positioned at the foci of the ellipse?

Solution. Graphing the data yields



It's most convenient to imagine this ellipse centered at $(0, 0)$. Since the ellipse is 100 units wide and 40 units tall, we get $a = 50$ and $b = 40$. Hence, our ellipse has the equation $\frac{x^2}{50^2} + \frac{y^2}{40^2} = 1$. We're looking for the foci, and we get $c = \sqrt{50^2 - 40^2} = \sqrt{900} = 30$, so that the foci are 30 units from the center. That means they are $50 - 30 = 20$ units from the vertices. Hence, Jason and Jamie should stand 20 feet from opposite ends of the gallery. $\square$

7.4.1 Exercises

In Exercises 1 - 8, graph the ellipse. Find the center, the lines which contain the major and minor axes, the vertices, the endpoints of the minor axis, the foci and the eccentricity.

1. $\dfrac{x^2}{169} + \dfrac{y^2}{25} = 1$
2. $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$
3. $\dfrac{(x-2)^2}{4} + \dfrac{(y+3)^2}{9} = 1$
4. $\dfrac{(x+5)^2}{16} + \dfrac{(y-4)^2}{1} = 1$
5. $\dfrac{(x-1)^2}{10} + \dfrac{(y-3)^2}{11} = 1$
6. $\dfrac{(x-1)^2}{9} + \dfrac{(y+3)^2}{4} = 1$
7. $\dfrac{(x+2)^2}{16} + \dfrac{(y-5)^2}{20} = 1$
8. $\dfrac{(x-4)^2}{8} + \dfrac{(y-2)^2}{18} = 1$

In Exercises 9 - 14, put the equation in standard form. Find the center, the lines which contain the major and minor axes, the vertices, the endpoints of the minor axis, the foci and the eccentricity.

9. $9x^2 + 25y^2 - 54x - 50y - 119 = 0$
10. $12x^2 + 3y^2 - 30y + 39 = 0$
11. $5x^2 + 18y^2 - 30x + 72y + 27 = 0$
12. $x^2 - 2x + 2y^2 - 12y + 3 = 0$
13. $9x^2 + 4y^2 - 4y - 8 = 0$
14. $6x^2 + 5y^2 - 24x + 20y + 14 = 0$

In Exercises 15 - 20, find the standard form of the equation of the ellipse which has the given properties.

15. Center $(3, 7)$, Vertex $(3, 2)$, Focus $(3, 3)$
16. Foci $(0, \pm 5)$, Vertices $(0, \pm 8)$.
17. Foci $(\pm 3, 0)$, length of the Minor Axis 10
18. Vertices $(3, 2)$, $(13, 2)$; Endpoints of the Minor Axis $(8, 4)$, $(8, 0)$
19. Center $(5, 2)$, Vertex $(0, 2)$, eccentricity $\frac{1}{2}$
20. All points on the ellipse are in Quadrant IV except $(0, -9)$ and $(8, 0)$. (One might also say that the ellipse is "tangent to the axes" at those two points.)
21. Repeat Example 7.4.5 for a whispering gallery 200 feet wide and 75 feet tall.
22. An elliptical arch is constructed which is 6 feet wide at the base and 9 feet tall in the middle. Find the height of the arch exactly 1 foot in from the base of the arch. Compare your result with your answer to Exercise 21 in Section 7.3.

23. The Earth's orbit around the sun is an ellipse with the sun at one focus and eccentricity $e \approx 0.0167$. The length of the semimajor axis (that is, half of the major axis) is defined to be 1 astronomical unit (AU). The vertices of the elliptical orbit are given special names: 'aphelion' is the vertex farthest from the sun, and 'perihelion' is the vertex closest to the sun. Find the distance in AU between the sun and aphelion and the distance in AU between the sun and perihelion.

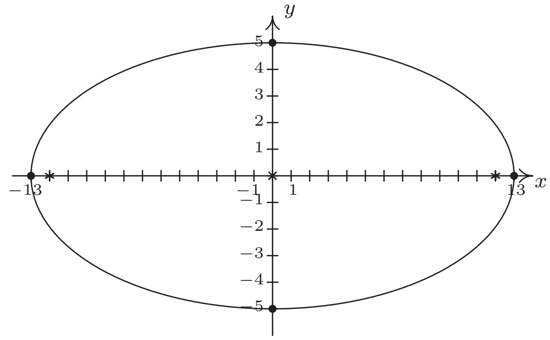
24. The graph of an ellipse clearly fails the Vertical Line Test, Theorem 1.1, so the equation of an ellipse does not define y as a function of x. However, much like with circles and horizontal parabolas, we can split an ellipse into a top half and a bottom half, each of which would indeed represent y as a function of x. With the help of your classmates, use your calculator to graph the ellipses given in Exercises 1 - 8 above. What difficulties arise when you plot them on the calculator?

25. Some famous examples of whispering galleries include St. Paul's Cathedral in London, England, National Statuary Hall in Washington, D.C., and The Cincinnati Museum Center. With the help of your classmates, research these whispering galleries. How does the whispering effect compare and contrast with the scenario in Example 7.4.5?

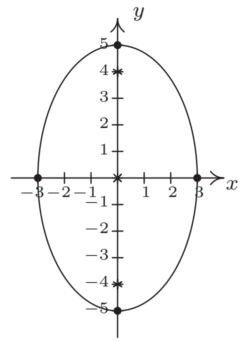
26. With the help of your classmates, research "extracorporeal shock-wave lithotripsy". It uses the reflective property of the ellipsoid to dissolve kidney stones.

7.4.2 Answers

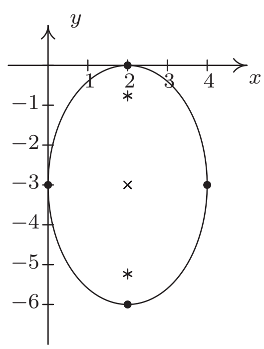
1. $\dfrac{x^2}{169} + \dfrac{y^2}{25} = 1$
   Center $(0,0)$
   Major axis along $y = 0$
   Minor axis along $x = 0$
   Vertices $(13,0)$, $(-13,0)$
   Endpoints of Minor Axis $(0,-5)$, $(0,5)$
   Foci $(12,0)$, $(-12,0)$
   $e = \frac{12}{13}$



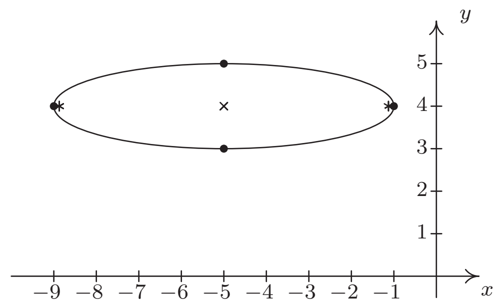
2. $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$
   Center $(0,0)$
   Major axis along $x = 0$
   Minor axis along $y = 0$
   Vertices $(0,5)$, $(0,-5)$
   Endpoints of Minor Axis $(-3,0)$, $(3,0)$
   Foci $(0,-4)$, $(0,4)$
   $e = \frac{4}{5}$



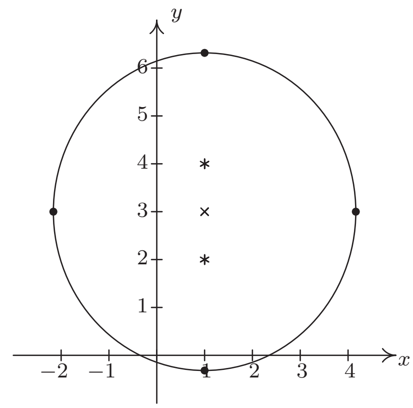
3. $\dfrac{(x-2)^2}{4} + \dfrac{(y+3)^2}{9} = 1$
   Center $(2,-3)$
   Major axis along $x = 2$
   Minor axis along $y = -3$
   Vertices $(2,0)$, $(2,-6)$
   Endpoints of Minor Axis $(0,-3)$, $(4,-3)$
   Foci $(2,-3+\sqrt{5})$, $(2,-3-\sqrt{5})$
   $e = \frac{\sqrt{5}}{3}$



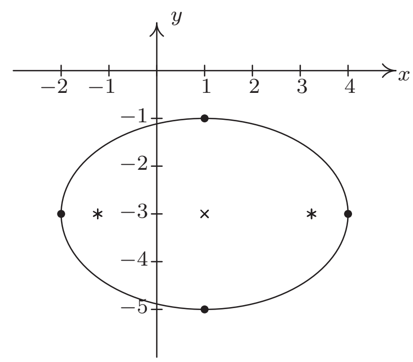
4. $\dfrac{(x+5)^2}{16} + \dfrac{(y-4)^2}{1} = 1$
   Center $(-5,4)$
   Major axis along $y = 4$
   Minor axis along $x = -5$
   Vertices $(-9,4)$, $(-1,4)$
   Endpoints of Minor Axis $(-5,3)$, $(-5,5)$
   Foci $(-5+\sqrt{15},4)$, $(-5-\sqrt{15},4)$
   $e = \frac{\sqrt{15}}{4}$



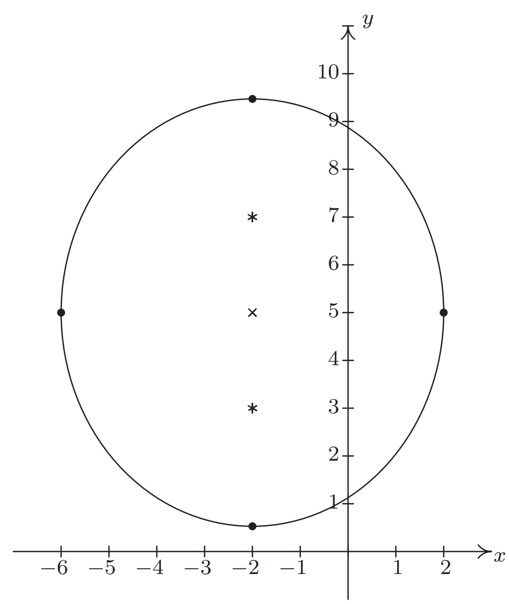
5. $\dfrac{(x-1)^2}{10} + \dfrac{(y-3)^2}{11} = 1$
Center $(1,3)$
Major axis along $x = 1$
Minor axis along $y = 3$
Vertices $(1, 3+\sqrt{11})$, $(1, 3-\sqrt{11})$
Endpoints of the Minor Axis
$(1-\sqrt{10}, 3)$, $(1+\sqrt{10}, 3)$
Foci $(1,2)$, $(1,4)$
$e = \frac{\sqrt{11}}{11}$



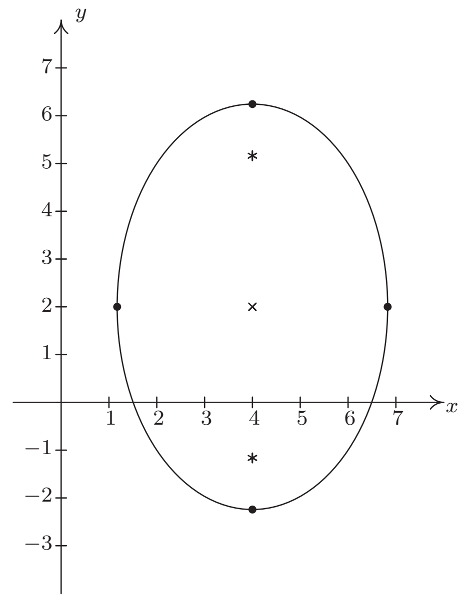
6. $\dfrac{(x-1)^2}{9} + \dfrac{(y+3)^2}{4} = 1$
Center $(1,-3)$
Major axis along $y = -3$
Minor axis along $x = 1$
Vertices $(4,-3)$, $(-2,-3)$
Endpoints of the Minor Axis $(1,-1)$, $(1,-5)$
Foci $(1+\sqrt{5}, -3)$, $(1-\sqrt{5}, -3)$
$e = \frac{\sqrt{5}}{3}$



7. $\dfrac{(x+2)^2}{16} + \dfrac{(y-5)^2}{20} = 1$
Center $(-2,5)$
Major axis along $x = -2$
Minor axis along $y = 5$
Vertices $(-2, 5+2\sqrt{5})$, $(-2, 5-2\sqrt{5})$
Endpoints of the Minor Axis $(-6,5)$, $(2,5)$
Foci $(-2,7)$, $(-2,3)$
$e = \frac{\sqrt{5}}{5}$



8. $\dfrac{(x-4)^2}{8}+\dfrac{(y-2)^2}{18}=1$
Center $(4,2)$
Major axis along $x=4$
Minor axis along $y=2$
Vertices $(4,2+3\sqrt{2})$, $(4,2-3\sqrt{2})$
Endpoints of the Minor Axis
$(4-2\sqrt{2},2)$, $(4+2\sqrt{2},2)$
Foci $(4,2+\sqrt{10})$, $(4,2-\sqrt{10})$
$e=\frac{\sqrt{5}}{3}$



9. $\dfrac{(x-3)^2}{25}+\dfrac{(y-1)^2}{9}=1$
Center $(3,1)$
Major Axis along $y=1$
Minor Axis along $x=3$
Vertices $(8,1)$, $(-2,1)$
Endpoints of Minor Axis $(3,4)$, $(3,-2)$
Foci $(7,1)$, $(-1,1)$
$e=\frac{4}{5}$

10. $\dfrac{x^2}{3}+\dfrac{(y-5)^2}{12}=1$
Center $(0,5)$
Major axis along $x=0$
Minor axis along $y=5$
Vertices $(0,5-2\sqrt{3}),(0,5+2\sqrt{3})$
Endpoints of Minor Axis $(-\sqrt{3},5)$, $(\sqrt{3},5)$
Foci $(0,2),(0,8)$
$e=\frac{\sqrt{3}}{2}$

11. $\dfrac{(x-3)^2}{18}+\dfrac{(y+2)^2}{5}=1$
Center $(3,-2)$
Major axis along $y=-2$
Minor axis along $x=3$
Vertices $(3-3\sqrt{2},-2),(3+3\sqrt{2},-2)$
Endpoints of Minor Axis $(3,-2+\sqrt{5})$, $(3,-2-\sqrt{5})$
Foci $(3-\sqrt{13},-2),(3+\sqrt{13},-2)$
$e=\frac{\sqrt{26}}{6}$

12. $\dfrac{(x-1)^2}{16}+\dfrac{(y-3)^2}{8}=1$
Center $(1,3)$
Major Axis along $y=3$
Minor Axis along $x=1$
Vertices $(5,3)$, $(-3,3)$
Endpoints of Minor Axis $(1,3+2\sqrt{2})$, $(1,3-2\sqrt{2})$
Foci $(1+2\sqrt{2},3)$, $(1-2\sqrt{2},3)$
$e=\frac{\sqrt{2}}{2}$

13. $\dfrac{x^2}{1} + \dfrac{4\left(y - \frac{1}{2}\right)^2}{9} = 1$
Center $\left(0, \frac{1}{2}\right)$
Major Axis along $x = 0$ (the y-axis)
Minor Axis along $y = \frac{1}{2}$
Vertices $(0, 2)$, $(0, -1)$
Endpoints of Minor Axis $\left(-1, \frac{1}{2}\right)$, $\left(1, \frac{1}{2}\right)$
Foci $\left(0, \frac{1+\sqrt{5}}{2}\right)$, $\left(0, \frac{1-\sqrt{5}}{2}\right)$
$e = \frac{\sqrt{5}}{3}$

14. $\dfrac{(x-2)^2}{5} + \dfrac{(y+2)^2}{6} = 1$
Center $(2, -2)$
Major Axis along $x = 2$
Minor Axis along $y = -2$
Vertices $\left(2, -2 + \sqrt{6}\right)$, $\left(2, -2 - \sqrt{6}\right)$
Endpoints of Minor Axis $\left(2 - \sqrt{5}, -2\right)$, $\left(2 + \sqrt{5}, -2\right)$
Foci $(2, -1)$, $(2, -3)$
$e = \frac{\sqrt{6}}{6}$

15. $\dfrac{(x-3)^2}{9} + \dfrac{(y-7)^2}{25} = 1$

16. $\dfrac{x^2}{39} + \dfrac{y^2}{64} = 1$

17. $\dfrac{x^2}{34} + \dfrac{y^2}{25} = 1$

18. $\dfrac{(x-8)^2}{25} + \dfrac{(y-2)^2}{4} = 1$

19. $\dfrac{(x-5)^2}{25} + \dfrac{4(y-2)^2}{75} = 1$

20. $\dfrac{(x-8)^2}{64} + \dfrac{(y+9)^2}{81} = 1$

21. Jamie and Jason should stand $100 - 25\sqrt{7} \approx 33.86$ feet from opposite ends of the gallery.

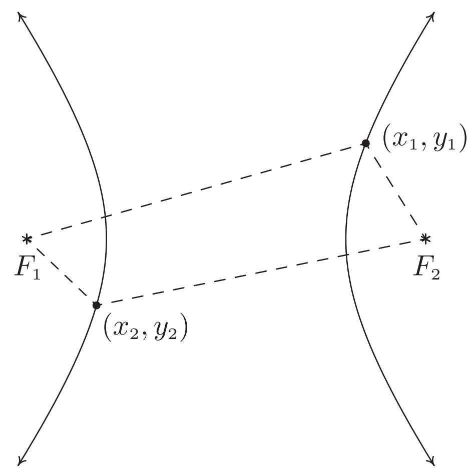
22. The arch can be modeled by the top half of $\frac{x^2}{9} + \frac{y^2}{81} = 1$. One foot in from the base of the arch corresponds to either $x = \pm 2$. Plugging in $x = \pm 2$ gives $y = \pm 3\sqrt{5}$ and since y represents a height, we choose $y = 3\sqrt{5} \approx 6.71$ feet.

23. Distance from the sun to aphelion ≈ 1.0167 AU.
Distance from the sun to perihelion ≈ 0.9833 AU.

7.5 Hyperbolas

In the definition of an ellipse, Definition 7.4, we fixed two points called foci and looked at points whose distances to the foci always **added** to a constant distance d. Those prone to syntactical tinkering may wonder what, if any, curve we'd generate if we replaced **added** with **subtracted**. The answer is a hyperbola.

Definition 7.6. Given two distinct points F_1 and F_2 in the plane and a fixed distance d, a **hyperbola** is the set of all points (x, y) in the plane such that the absolute value of the difference of each of the distances from F_1 and F_2 to (x, y) is d. The points F_1 and F_2 are called the **foci** of the hyperbola.



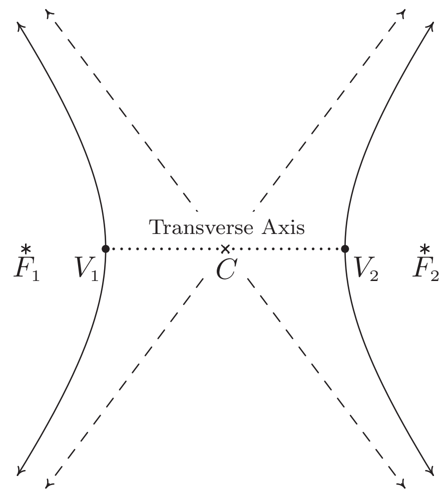
In the figure above:

$$\text{the distance from } F_1 \text{ to } (x_1, y_1) - \text{the distance from } F_2 \text{ to } (x_1, y_1) \quad = \quad d$$

and

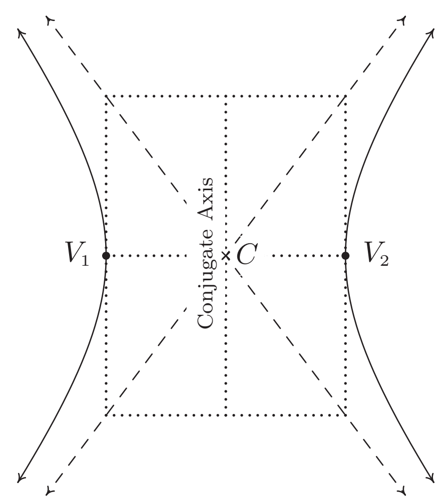
$$\text{the distance from } F_2 \text{ to } (x_2, y_2) - \text{the distance from } F_1 \text{ to } (x_2, y_2) \quad = \quad d$$

Note that the hyperbola has two parts, called **branches**. The **center** of the hyperbola is the midpoint of the line segment connecting the two foci. The **transverse axis** of the hyperbola is the line segment connecting two opposite ends of the hyperbola which also contains the center and foci. The **vertices** of a hyperbola are the points of the hyperbola which lie on the transverse axis. In addition, we will show momentarily that there are lines called **asymptotes** which the branches of the hyperbola approach for large x and y values. They serve as guides to the graph. In pictures,



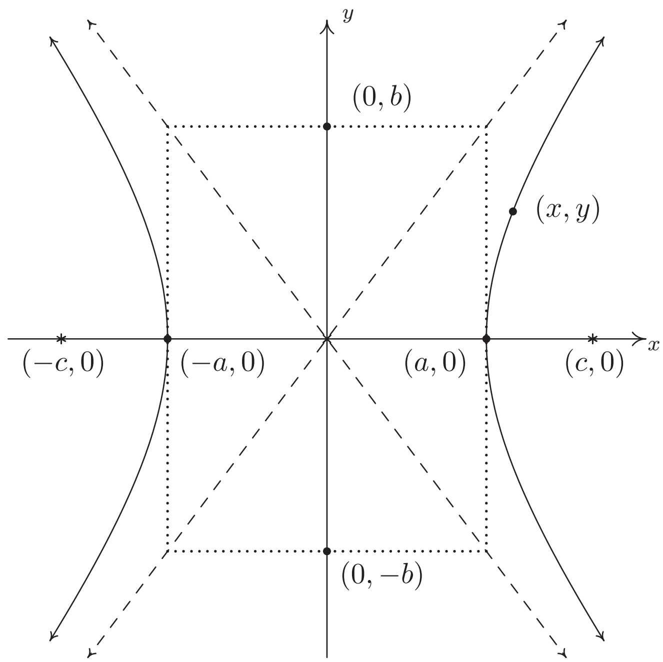
A hyperbola with center C; foci F_1, F_2; and vertices V_1, V_2 and asymptotes (dashed)

Before we derive the standard equation of the hyperbola, we need to discuss one further parameter, the **conjugate axis** of the hyperbola. The conjugate axis of a hyperbola is the line segment through the center which is perpendicular to the transverse axis and has the same length as the line segment through a vertex which connects the asymptotes. In pictures we have



Note that in the diagram, we can construct a rectangle using line segments with lengths equal to the lengths of the transverse and conjugate axes whose center is the center of the hyperbola and whose diagonals are contained in the asymptotes. This **guide rectangle**, much akin to the one we saw Section 7.4 to help us graph ellipses, will aid us in graphing hyperbolas.

Suppose we wish to derive the equation of a hyperbola. For simplicity, we shall assume that the center is $(0, 0)$, the vertices are $(a, 0)$ and $(-a, 0)$ and the foci are $(c, 0)$ and $(-c, 0)$. We label the endpoints of the conjugate axis $(0, b)$ and $(0, -b)$. (Although b does not enter into our derivation, we will have to justify this choice as you shall see later.) As before, we assume a, b, and c are all positive numbers. Schematically we have



Since $(a, 0)$ is on the hyperbola, it must satisfy the conditions of Definition 7.6. That is, the distance from $(-c, 0)$ to $(a, 0)$ minus the distance from $(c, 0)$ to $(a, 0)$ must equal the fixed distance d. Since all these points lie on the x-axis, we get

$$\begin{aligned} \text{distance from } (-c,0) \text{ to } (a,0) - \text{distance from } (c,0) \text{ to } (a,0) &= d \\ (a+c)-(c-a) &= d \\ 2a &= d \end{aligned}$$

In other words, the fixed distance d from the definition of the hyperbola is actually the length of the transverse axis! (Where have we seen that type of coincidence before?) Now consider a point (x, y) on the hyperbola. Applying Definition 7.6, we get

$$\begin{aligned} \text{distance from } (-c,0) \text{ to } (x,y) - \text{distance from } (c,0) \text{ to } (x,y) &= 2a \\ \sqrt{(x-(-c))^2+(y-0)^2} - \sqrt{(x-c)^2+(y-0)^2} &= 2a \\ \sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2} &= 2a \end{aligned}$$

Using the same arsenal of Intermediate Algebra weaponry we used in deriving the standard formula of an ellipse, Equation 7.4, we arrive at the following.[1]

[1] It is a good exercise to actually work this out.

$$\left(a^2 - c^2\right) x^2 + a^2 y^2 = a^2 \left(a^2 - c^2\right)$$

What remains is to determine the relationship between a, b and c. To that end, we note that since a and c are both positive numbers with $a < c$, we get $a^2 < c^2$ so that $a^2 - c^2$ is a negative number. Hence, $c^2 - a^2$ is a positive number. For reasons which will become clear soon, we re-write the equation by solving for y^2/x^2 to get

$$\begin{aligned}
\left(a^2 - c^2\right) x^2 + a^2 y^2 &= a^2 \left(a^2 - c^2\right) \\
-\left(c^2 - a^2\right) x^2 + a^2 y^2 &= -a^2 \left(c^2 - a^2\right) \\
a^2 y^2 &= \left(c^2 - a^2\right) x^2 - a^2 \left(c^2 - a^2\right) \\
\frac{y^2}{x^2} &= \frac{\left(c^2 - a^2\right)}{a^2} - \frac{\left(c^2 - a^2\right)}{x^2}
\end{aligned}$$

As x and y attain very large values, the quantity $\frac{\left(c^2 - a^2\right)}{x^2} \to 0$ so that $\frac{y^2}{x^2} \to \frac{\left(c^2 - a^2\right)}{a^2}$. By setting $b^2 = c^2 - a^2$ we get $\frac{y^2}{x^2} \to \frac{b^2}{a^2}$. This shows that $y \to \pm\frac{b}{a}x$ as $|x|$ grows large. Thus $y = \pm\frac{b}{a}x$ are the asymptotes to the graph as predicted and our choice of labels for the endpoints of the conjugate axis is justified. In our equation of the hyperbola we can substitute $a^2 - c^2 = -b^2$ which yields

$$\begin{aligned}
\left(a^2 - c^2\right) x^2 + a^2 y^2 &= a^2 \left(a^2 - c^2\right) \\
-b^2 x^2 + a^2 y^2 &= -a^2 b^2 \\
\frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1
\end{aligned}$$

The equation above is for a hyperbola whose center is the origin and which opens to the left and right. If the hyperbola were centered at a point (h, k), we would get the following.

Equation 7.6. The Standard Equation of a Horizontal[a] Hyperbola For positive numbers a and b, the equation of a horizontal hyperbola with center (h, k) is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

[a] That is, a hyperbola whose branches open to the left and right

If the roles of x and y were interchanged, then the hyperbola's branches would open upwards and downwards and we would get a 'vertical' hyperbola.

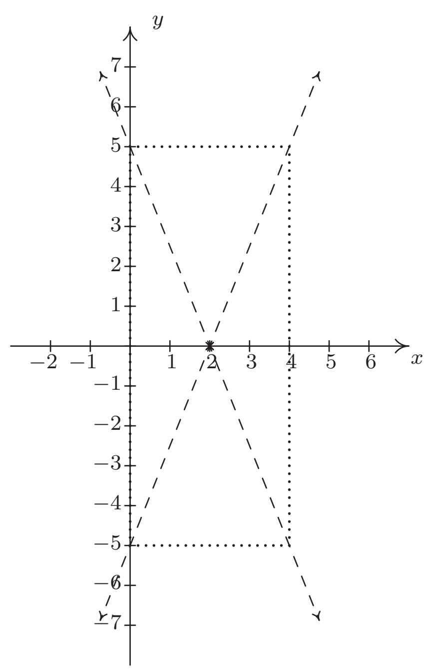
Equation 7.7. The Standard Equation of a Vertical Hyperbola For positive numbers a and b, the equation of a vertical hyperbola with center (h, k) is:

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

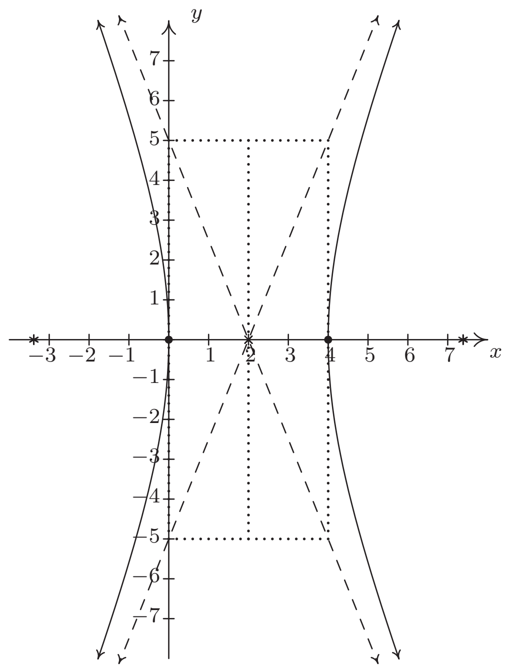
The values of a and b determine how far in the x and y directions, respectively, one counts from the center to determine the rectangle through which the asymptotes pass. In both cases, the distance from the center to the foci, c, as seen in the derivation, can be found by the formula $c = \sqrt{a^2 + b^2}$. Lastly, note that we can quickly distinguish the equation of a hyperbola from that of a circle or ellipse because the hyperbola formula involves a **difference** of squares where the circle and ellipse formulas both involve the **sum** of squares.

Example 7.5.1. Graph the equation $\frac{(x-2)^2}{4} - \frac{y^2}{25} = 1$. Find the center, the lines which contain the transverse and conjugate axes, the vertices, the foci and the equations of the asymptotes.

Solution. We first see that this equation is given to us in the standard form of Equation 7.6. Here $x - h$ is $x - 2$ so $h = 2$, and $y - k$ is y so $k = 0$. Hence, our hyperbola is centered at $(2, 0)$. We see that $a^2 = 4$ so $a = 2$, and $b^2 = 25$ so $b = 5$. This means we move 2 units to the left and right of the center and 5 units up and down from the center to arrive at points on the guide rectangle. The asymptotes pass through the center of the hyperbola as well as the corners of the rectangle. This yields the following set up.



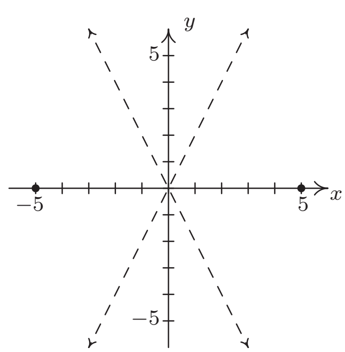
Since the y^2 term is being subtracted from the x^2 term, we know that the branches of the hyperbola open to the left and right. This means that the transverse axis lies along the x-axis. Hence, the conjugate axis lies along the vertical line $x = 2$. Since the vertices of the hyperbola are where the hyperbola intersects the transverse axis, we get that the vertices are 2 units to the left and right of $(2, 0)$ at $(0, 0)$ and $(4, 0)$. To find the foci, we need $c = \sqrt{a^2 + b^2} = \sqrt{4 + 25} = \sqrt{29}$. Since the foci lie on the transverse axis, we move $\sqrt{29}$ units to the left and right of $(2, 0)$ to arrive at $(2 - \sqrt{29}, 0)$ (approximately $(-3.39, 0)$) and $(2 + \sqrt{29}, 0)$ (approximately $(7.39, 0)$). To determine the equations of the asymptotes, recall that the asymptotes go through the center of the hyperbola, $(2, 0)$, as well as the corners of guide rectangle, so they have slopes of $\pm\frac{b}{a} = \pm\frac{5}{2}$. Using the point-slope equation of a line, Equation 2.2, yields $y - 0 = \pm\frac{5}{2}(x-2)$, so we get $y = \frac{5}{2}x - 5$ and $y = -\frac{5}{2}x + 5$. Putting it all together, we get



□

Example 7.5.2. Find the equation of the hyperbola with asymptotes $y = \pm 2x$ and vertices $(\pm 5, 0)$.

Solution. Plotting the data given to us, we have



This graph not only tells us that the branches of the hyperbola open to the left and to the right, it also tells us that the center is $(0,0)$. Hence, our standard form is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Since the vertices are $(\pm 5, 0)$, we have $a = 5$ so $a^2 = 25$. In order to determine b^2, we recall that the slopes of the asymptotes are $\pm\frac{b}{a}$. Since $a = 5$ and the slope of the line $y = 2x$ is 2, we have that $\frac{b}{5} = 2$, so $b = 10$. Hence, $b^2 = 100$ and our final answer is $\frac{x^2}{25} - \frac{y^2}{100} = 1$. □

As with the other conic sections, an equation whose graph is a hyperbola may not be given in either of the standard forms. To rectify that, we have the following.

To Write the Equation of a Hyperbola in Standard Form

1. Group the same variables together on one side of the equation and position the constant on the other side
2. Complete the square in both variables as needed
3. Divide both sides by the constant term so that the constant on the other side of the equation becomes 1

Example 7.5.3. Consider the equation $9y^2 - x^2 - 6x = 10$. Put this equation in to standard form and graph. Find the center, the lines which contain the transverse and conjugate axes, the vertices, the foci, and the equations of the asymptotes.

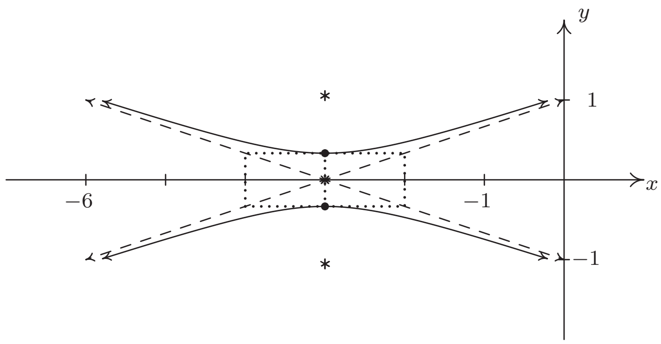
Solution. We need only complete the square on x:

$$\begin{aligned} 9y^2 - x^2 - 6x &= 10 \\ 9y^2 - 1\left(x^2 + 6x\right) &= 10 \\ 9y^2 - \left(x^2 + 6x + 9\right) &= 10 - 1(9) \\ 9y^2 - (x+3)^2 &= 1 \\ \frac{y^2}{\frac{1}{9}} - \frac{(x+3)^2}{1} &= 1 \end{aligned}$$

Now that this equation is in the standard form of Equation 7.7, we see that $x - h$ is $x + 3$ so $h = -3$, and $y - k$ is y so $k = 0$. Hence, our hyperbola is centered at $(-3, 0)$. We find that $a^2 = 1$ so $a = 1$, and $b^2 = \frac{1}{9}$ so $b = \frac{1}{3}$. This means that we move 1 unit to the left and right of the center and $\frac{1}{3}$ units up and down from the center to arrive at points on the guide rectangle. Since the x^2 term is being subtracted from the y^2 term, we know the branches of the hyperbola open upwards and downwards. This means the transverse axis lies along the vertical line $x = -3$ and the conjugate axis lies along the x-axis. Since the vertices of the hyperbola are where the hyperbola intersects the transverse axis, we get that the vertices are $\frac{1}{3}$ of a unit above and below $(-3, 0)$ at $\left(-3, \frac{1}{3}\right)$ and $\left(-3, -\frac{1}{3}\right)$. To find the foci, we use

$$c = \sqrt{a^2 + b^2} = \sqrt{\frac{1}{9} + 1} = \frac{\sqrt{10}}{3}$$

Since the foci lie on the transverse axis, we move $\frac{\sqrt{10}}{3}$ units above and below $(-3, 0)$ to arrive at $\left(-3, \frac{\sqrt{10}}{3}\right)$ and $\left(-3, -\frac{\sqrt{10}}{3}\right)$. To determine the asymptotes, recall that the asymptotes go through the center of the hyperbola, $(-3, 0)$, as well as the corners of guide rectangle, so they have slopes of $\pm\frac{b}{a} = \pm\frac{1}{3}$. Using the point-slope equation of a line, Equation 2.2, we get $y = \frac{1}{3}x + 1$ and $y = -\frac{1}{3}x - 1$. Putting it all together, we get



□

Hyperbolas can be used in so-called 'trilateration,' or 'positioning' problems. The procedure outlined in the next example is the basis of the (now virtually defunct) LOng Range Aid to Navigation (LORAN for short) system.[2]

Example 7.5.4. Jeff is stationed 10 miles due west of Carl in an otherwise empty forest in an attempt to locate an elusive Sasquatch. At the stroke of midnight, Jeff records a Sasquatch call 9 seconds earlier than Carl. If the speed of sound that night is 760 miles per hour, determine a hyperbolic path along which Sasquatch must be located.

Solution. Since Jeff hears Sasquatch sooner, it is closer to Jeff than it is to Carl. Since the speed of sound is 760 miles per hour, we can determine how much closer Sasquatch is to Jeff by multiplying

$$760\,\frac{\text{miles}}{\text{hour}} \times \frac{1\,\text{hour}}{3600\,\text{seconds}} \times 9\,\text{seconds} = 1.9\,\text{miles}$$

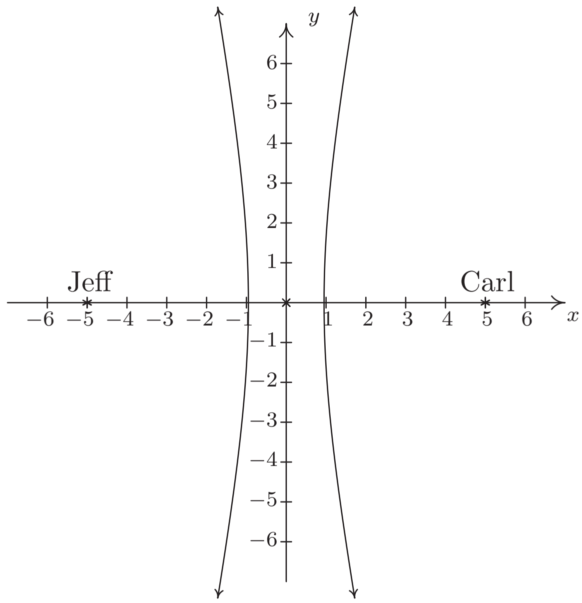
This means that Sasquatch is 1.9 miles closer to Jeff than it is to Carl. In other words, Sasquatch must lie on a path where

$$(\text{the distance to Carl}) - (\text{the distance to Jeff}) = 1.9$$

This is exactly the situation in the definition of a hyperbola, Definition 7.6. In this case, Jeff and Carl are located at the foci,[3] and our fixed distance d is 1.9. For simplicity, we assume the hyperbola is centered at $(0,0)$ with its foci at $(-5,0)$ and $(5,0)$. Schematically, we have

[2]GPS now rules the positioning kingdom. Is there still a place for LORAN and other land-based systems? Do satellites ever malfunction?

[3]We usually like to be the *center* of attention, but being the *focus* of attention works equally well.



We are seeking a curve of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in which the distance from the center to each focus is $c = 5$. As we saw in the derivation of the standard equation of the hyperbola, Equation 7.6, $d = 2a$, so that $2a = 1.9$, or $a = 0.95$ and $a^2 = 0.9025$. All that remains is to find b^2. To that end, we recall that $a^2 + b^2 = c^2$ so $b^2 = c^2 - a^2 = 25 - 0.9025 = 24.0975$. Since Sasquatch is closer to Jeff than it is to Carl, it must be on the western (left hand) branch of $\frac{x^2}{0.9025} - \frac{y^2}{24.0975} = 1$. □

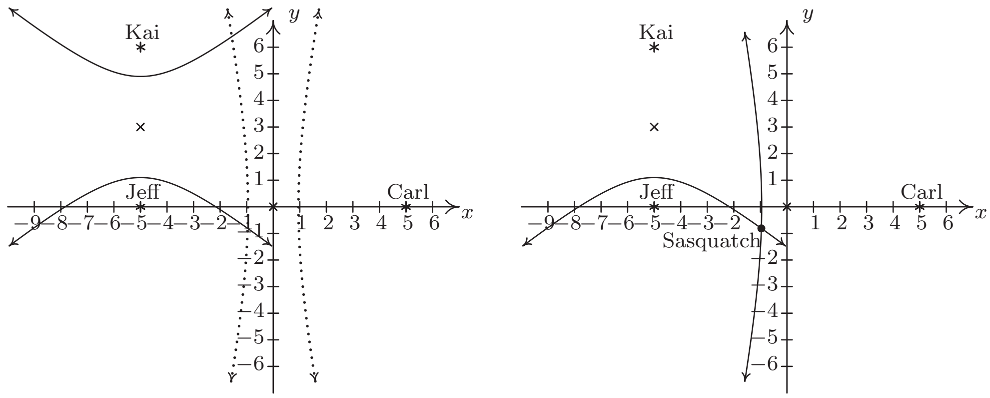
In our previous example, we did not have enough information to pin down the exact location of Sasquatch. To accomplish this, we would need a third observer.

Example 7.5.5. By a stroke of luck, Kai was also camping in the woods during the events of the previous example. He was located 6 miles due north of Jeff and heard the Sasquatch call 18 seconds after Jeff did. Use this added information to locate Sasquatch.

Solution. Kai and Jeff are now the foci of a second hyperbola where the fixed distance d can be determined as before

$$760\,\frac{\text{miles}}{\text{hour}} \times \frac{1\,\text{hour}}{3600\,\text{seconds}} \times 18\,\text{seconds} = 3.8\,\text{miles}$$

Since Jeff was positioned at $(-5, 0)$, we place Kai at $(-5, 6)$. This puts the center of the new hyperbola at $(-5, 3)$. Plotting Kai's position and the new center gives us the diagram below on the left. The second hyperbola is vertical, so it must be of the form $\frac{(y-3)^2}{b^2} - \frac{(x+5)^2}{a^2} = 1$. As before, the distance d is the length of the major axis, which in this case is $2b$. We get $2b = 3.8$ so that $b = 1.9$ and $b^2 = 3.61$. With Kai 6 miles due North of Jeff, we have that the distance from the center to the focus is $c = 3$. Since $a^2 + b^2 = c^2$, we get $a^2 = c^2 - b^2 = 9 - 3.61 = 5.39$. Kai heard the Sasquatch call after Jeff, so Kai is farther from Sasquatch than Jeff. Thus Sasquatch must lie on the southern branch of the hyperbola $\frac{(y-3)^2}{3.61} - \frac{(x+5)^2}{5.39} = 1$. Looking at the western branch of the hyperbola determined by Jeff and Carl along with the southern branch of the hyperbola determined by Kai and Jeff, we see that there is exactly one point in common, and this is where Sasquatch must have been when it called.



To determine the coordinates of this point of intersection exactly, we would need techniques for solving systems of non-linear equations (which we won't see until Section 8.7), so we use the calculator[4] Doing so, we get Sasquatch is approximately at $(-0.9629, -0.8113)$. □

Each of the conic sections we have studied in this chapter result from graphing equations of the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$ for different choices of A, C, D, E, and[5] F. While we've seen examples[6] demonstrate *how* to convert an equation from this general form to one of the standard forms, we close this chapter with some advice about *which* standard form to choose.[7]

Strategies for Identifying Conic Sections

Suppose the graph of equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$ is a non-degenerate conic section.[a]

- If just *one* variable is squared, the graph is a parabola. Put the equation in the form of Equation 7.2 (if x is squared) or Equation 7.3 (if y is squared).

If *both* variables are squared, look at the coefficients of x^2 and y^2, A and B.

- If $A = B$, the graph is a circle. Put the equation in the form of Equation 7.1.
- If $A \neq B$ but A and B have the *same sign*, the graph is an ellipse. Put the equation in the form of Equation 7.4.
- If A and B have the *different signs*, the graph is a hyperbola. Put the equation in the form of either Equation 7.6 or Equation 7.7.

[a]That is, a parabola, circle, ellipse, or hyperbola – see Section 7.1.

[4]First solve each hyperbola for y, and choose the correct equation (branch) before proceeding.
[5]See Section 11.6 to see why we skip B.
[6]Examples 7.2.3, 7.3.4, 7.4.3, and 7.5.3, in particular.
[7]We formalize this in Exercise 34.

7.5.1 Exercises

In Exercises 1 - 8, graph the hyperbola. Find the center, the lines which contain the transverse and conjugate axes, the vertices, the foci and the equations of the asymptotes.

1. $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$
2. $\dfrac{y^2}{9} - \dfrac{x^2}{16} = 1$
3. $\dfrac{(x-2)^2}{4} - \dfrac{(y+3)^2}{9} = 1$
4. $\dfrac{(y-3)^2}{11} - \dfrac{(x-1)^2}{10} = 1$
5. $\dfrac{(x+4)^2}{16} - \dfrac{(y-4)^2}{1} = 1$
6. $\dfrac{(x+1)^2}{9} - \dfrac{(y-3)^2}{4} = 1$
7. $\dfrac{(y+2)^2}{16} - \dfrac{(x-5)^2}{20} = 1$
8. $\dfrac{(x-4)^2}{8} - \dfrac{(y-2)^2}{18} = 1$

In Exercises 9 - 12, put the equation in standard form. Find the center, the lines which contain the transverse and conjugate axes, the vertices, the foci and the equations of the asymptotes.

9. $12x^2 - 3y^2 + 30y - 111 = 0$
10. $18y^2 - 5x^2 + 72y + 30x - 63 = 0$
11. $9x^2 - 25y^2 - 54x - 50y - 169 = 0$
12. $-6x^2 + 5y^2 - 24x + 40y + 26 = 0$

In Exercises 13 - 18, find the standard form of the equation of the hyperbola which has the given properties.

13. Center $(3,7)$, Vertex $(3,3)$, Focus $(3,2)$
14. Vertex $(0,1)$, Vertex $(8,1)$, Focus $(-3,1)$
15. Foci $(0,\pm 8)$, Vertices $(0,\pm 5)$.
16. Foci $(\pm 5,0)$, length of the Conjugate Axis 6
17. Vertices $(3,2)$, $(13,2)$; Endpoints of the Conjugate Axis $(8,4)$, $(8,0)$
18. Vertex $(-10,5)$, Asymptotes $y = \pm\frac{1}{2}(x-6)+5$

In Exercises 19 - 28, find the standard form of the equation using the guidelines on page 540 and then graph the conic section.

19. $x^2 - 2x - 4y - 11 = 0$
20. $x^2 + y^2 - 8x + 4y + 11 = 0$
21. $9x^2 + 4y^2 - 36x + 24y + 36 = 0$
22. $9x^2 - 4y^2 - 36x - 24y - 36 = 0$

23. $y^2 + 8y - 4x + 16 = 0$

24. $4x^2 + y^2 - 8x + 4 = 0$

25. $4x^2 + 9y^2 - 8x + 54y + 49 = 0$

26. $x^2 + y^2 - 6x + 4y + 14 = 0$

27. $2x^2 + 4y^2 + 12x - 8y + 25 = 0$

28. $4x^2 - 5y^2 - 40x - 20y + 160 = 0$

29. The graph of a vertical or horizontal hyperbola clearly fails the Vertical Line Test, Theorem 1.1, so the equation of a vertical of horizontal hyperbola does not define y as a function of x.[8] However, much like with circles, horizontal parabolas and ellipses, we can split a hyperbola into pieces, each of which would indeed represent y as a function of x. With the help of your classmates, use your calculator to graph the hyperbolas given in Exercises 1 - 8 above. How many pieces do you need for a vertical hyperbola? How many for a horizontal hyperbola?

30. The location of an earthquake's epicenter − the point on the surface of the Earth directly above where the earthquake actually occurred − can be determined by a process similar to how we located Sasquatch in Example 7.5.5. (As we said back in Exercise 75 in Section 6.1, earthquakes are complicated events and it is not our intent to provide a complete discussion of the science involved in them. Instead, we refer the interested reader to a course in Geology or the U.S. Geological Survey's Earthquake Hazards Program found here.) Our technique works only for relatively small distances because we need to assume that the Earth is flat in order to use hyperbolas in the plane.[9] The P-waves ("P" stands for Primary) of an earthquake in Sasquatchia travel at 6 kilometers per second.[10] Station A records the waves first. Then Station B, which is 100 kilometers due north of Station A, records the waves 2 seconds later. Station C, which is 150 kilometers due west of Station A records the waves 3 seconds after that (a total of 5 seconds after Station A). Where is the epicenter?

31. The notion of eccentricity introduced for ellipses in Definition 7.5 in Section 7.4 is the same for hyperbolas in that we can define the eccentricity e of a hyperbola as

$$e = \frac{\text{distance from the center to a focus}}{\text{distance from the center to a vertex}}$$

   (a) With the help of your classmates, explain why $e > 1$ for any hyperbola.

   (b) Find the equation of the hyperbola with vertices $(\pm 3, 0)$ and eccentricity $e = 2$.

   (c) With the help of your classmates, find the eccentricity of each of the hyperbolas in Exercises 1 - 8. What role does eccentricity play in the shape of the graphs?

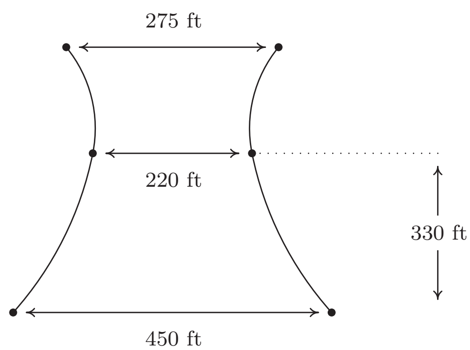
32. On page 510 in Section 7.3, we discussed paraboloids of revolution when studying the design of satellite dishes and parabolic mirrors. In much the same way, 'natural draft' cooling towers are often shaped as **hyperboloids of revolution**. Each vertical cross section of these towers

[8] We will see later in the text that the graphs of certain rotated hyperbolas pass the Vertical Line Test.

[9] Back in the Exercises in Section 1.1 you were asked to research people who believe the world is flat. What did you discover?

[10] Depending on the composition of the crust at a specific location, P-waves can travel between 5 kps and 8 kps.

is a hyperbola. Suppose the a natural draft cooling tower has the cross section below. Suppose the tower is 450 feet wide at the base, 275 feet wide at the top, and 220 feet at its narrowest point (which occurs 330 feet above the ground.) Determine the height of the tower to the nearest foot.



33. With the help of your classmates, research the Cassegrain Telescope. It uses the reflective property of the hyperbola as well as that of the parabola to make an ingenious telescope.

34. With the help of your classmates show that if $Ax^2 + Cy^2 + Dx + Ey + F = 0$ determines a non-degenerate conic[11] then

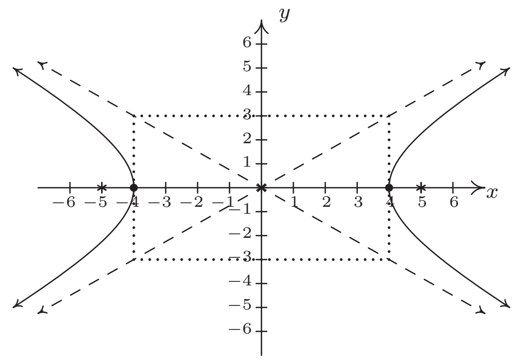
    - $AC < 0$ means that the graph is a hyperbola
    - $AC = 0$ means that the graph is a parabola
    - $AC > 0$ means that the graph is an ellipse or circle

    NOTE: This result will be generalized in Theorem 11.11 in Section 11.6.1.

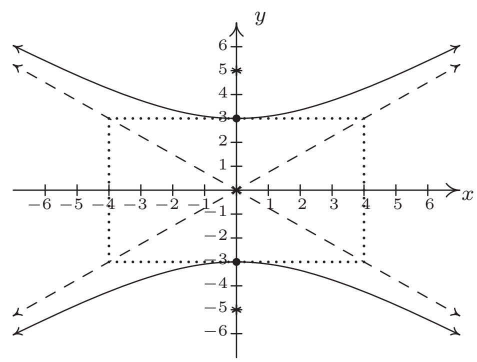
[11] Recall that this means its graph is either a circle, parabola, ellipse or hyperbola.

7.5.2 ANSWERS

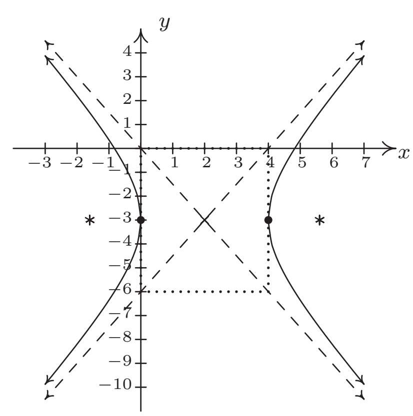
1. $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$
   Center $(0,0)$
   Transverse axis on $y=0$
   Conjugate axis on $x=0$
   Vertices $(4,0),(-4,0)$
   Foci $(5,0),(-5,0)$
   Asymptotes $y = \pm\frac{3}{4}x$



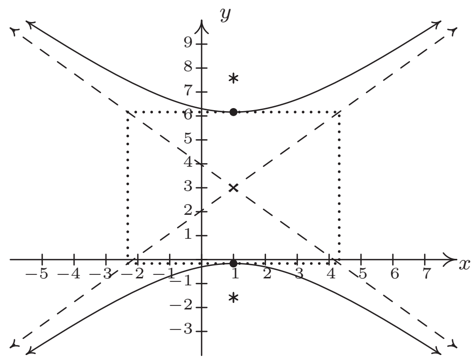
2. $\dfrac{y^2}{9} - \dfrac{x^2}{16} = 1$
   Center $(0,0)$
   Transverse axis on $x=0$
   Conjugate axis on $y=0$
   Vertices $(0,3),(0,-3)$
   Foci $(0,5),(0,-5)$
   Asymptotes $y = \pm\frac{3}{4}x$



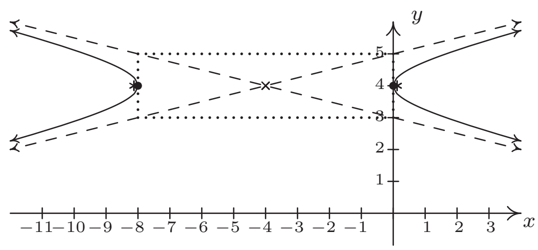
3. $\dfrac{(x-2)^2}{4} - \dfrac{(y+3)^2}{9} = 1$
   Center $(2,-3)$
   Transverse axis on $y=-3$
   Conjugate axis on $x=2$
   Vertices $(0,-3),(4,-3)$
   Foci $(2+\sqrt{13},-3),(2-\sqrt{13},-3)$
   Asymptotes $y = \pm\frac{3}{2}(x-2) - 3$



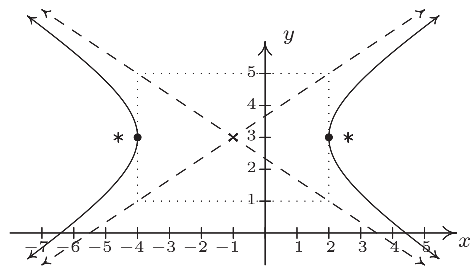
4. $\dfrac{(y-3)^2}{11} - \dfrac{(x-1)^2}{10} = 1$
   Center $(1, 3)$
   Transverse axis on $x = 1$
   Conjugate axis on $y = 3$
   Vertices $(1, 3 + \sqrt{11}), (1, 3 - \sqrt{11})$
   Foci $(1, 3 + \sqrt{21}), (1, 3 - \sqrt{21})$
   Asymptotes $y = \pm \frac{\sqrt{110}}{10}(x - 1) + 3$



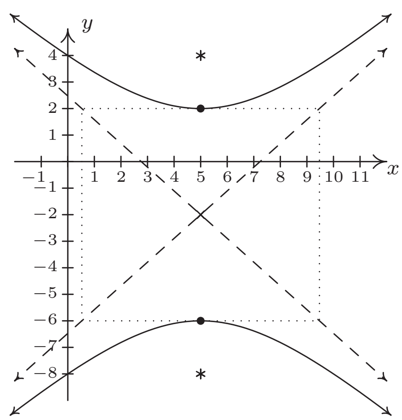
5. $\dfrac{(x+4)^2}{16} - \dfrac{(y-4)^2}{1} = 1$
   Center $(-4, 4)$
   Transverse axis on $y = 4$
   Conjugate axis on $x = -4$
   Vertices $(-8, 4), (0, 4)$
   Foci $(-4 + \sqrt{17}, 4), (-4 - \sqrt{17}, 4)$
   Asymptotes $y = \pm \frac{1}{4}(x + 4) + 4$



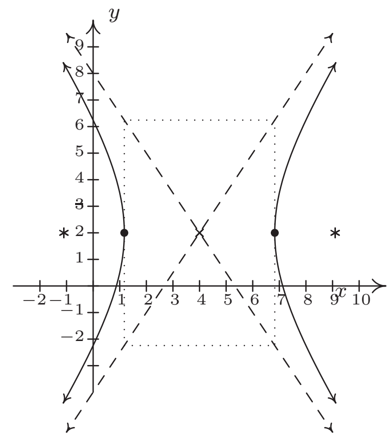
6. $\dfrac{(x+1)^2}{9} - \dfrac{(y-3)^2}{4} = 1$
   Center $(-1, 3)$
   Transverse axis on $y = 3$
   Conjugate axis on $x = -1$
   Vertices $(2, 3), (-4, 3)$
   Foci $\left(-1 + \sqrt{13}, 3\right), \left(-1 - \sqrt{13}, 3\right)$
   Asymptotes $y = \pm \frac{2}{3}(x + 1) + 3$



7. $\dfrac{(y+2)^2}{16} - \dfrac{(x-5)^2}{20} = 1$
   Center $(5, -2)$
   Transverse axis on $x = 5$
   Conjugate axis on $y = -2$
   Vertices $(5, 2), (5, -6)$
   Foci $(5, 4), (5, -8)$
   Asymptotes $y = \pm \frac{2\sqrt{5}}{5}(x - 5) - 2$



8. $\dfrac{(x-4)^2}{8} - \dfrac{(y-2)^2}{18} = 1$
Center $(4,2)$
Transverse axis on $y=2$
Conjugate axis on $x=4$
Vertices $\left(4+2\sqrt{2},2\right), \left(4-2\sqrt{2},2\right)$
Foci $\left(4+\sqrt{26},2\right), \left(4-\sqrt{26},2\right)$
Asymptotes $y=\pm\frac{3}{2}(x-4)+2$



9. $\dfrac{x^2}{3} - \dfrac{(y-5)^2}{12} = 1$
Center $(0,5)$
Transverse axis on $y=5$
Conjugate axis on $x=0$
Vertices $(\sqrt{3},5), (-\sqrt{3},5)$
Foci $(\sqrt{15},5), (-\sqrt{15},5)$
Asymptotes $y=\pm 2x+5$

10. $\dfrac{(y+2)^2}{5} - \dfrac{(x-3)^2}{18} = 1$
Center $(3,-2)$
Transverse axis on $x=3$
Conjugate axis on $y=-2$
Vertices $(3,-2+\sqrt{5}), (3,-2-\sqrt{5})$
Foci $(3,-2+\sqrt{23}), (3,-2-\sqrt{23})$
Asymptotes $y=\pm\frac{\sqrt{10}}{6}(x-3)-2$

11. $\dfrac{(x-3)^2}{25} - \dfrac{(y+1)^2}{9} = 1$
Center $(3,-1)$
Transverse axis on $y=-1$
Conjugate axis on $x=3$
Vertices $(8,-1), (-2,-1)$
Foci $\left(3+\sqrt{34},-1\right), \left(3-\sqrt{34},-1\right)$
Asymptotes $y=\pm\frac{3}{5}(x-3)-1$

12. $\dfrac{(y+4)^2}{6} - \dfrac{(x+2)^2}{5} = 1$
Center $(-2,-4)$
Transverse axis on $x=-2$
Conjugate axis on $y=-4$
Vertices $\left(-2,-4+\sqrt{6}\right), \left(-2,-4-\sqrt{6}\right)$
Foci $\left(-2,-4+\sqrt{11}\right), \left(-2,-4-\sqrt{11}\right)$
Asymptotes $y=\pm\frac{\sqrt{30}}{5}(x+2)-4$

13. $\dfrac{(y-7)^2}{16} - \dfrac{(x-3)^2}{9} = 1$

14. $\dfrac{(x-4)^2}{16} - \dfrac{(y-1)^2}{33} = 1$

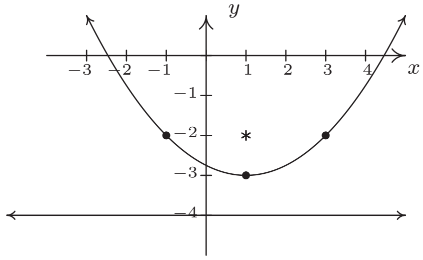
15. $\dfrac{y^2}{25} - \dfrac{x^2}{39} = 1$

16. $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$

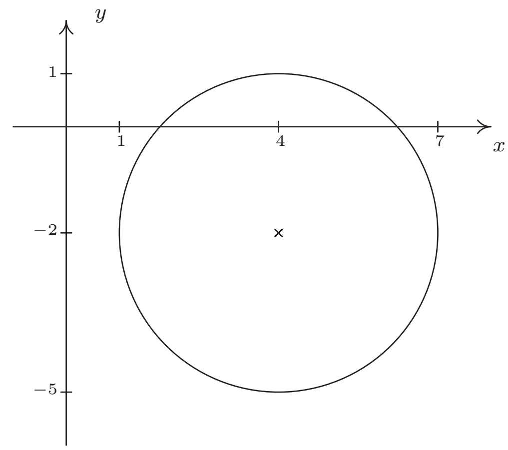
17. $\dfrac{(x-8)^2}{25} - \dfrac{(y-2)^2}{4} = 1$

18. $\dfrac{(x-6)^2}{256} - \dfrac{(y-5)^2}{64} = 1$

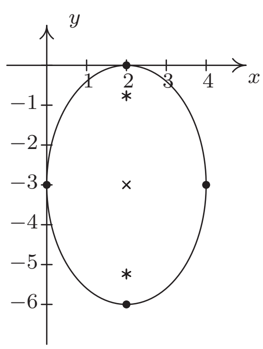
19. $(x-1)^2 = 4(y+3)$



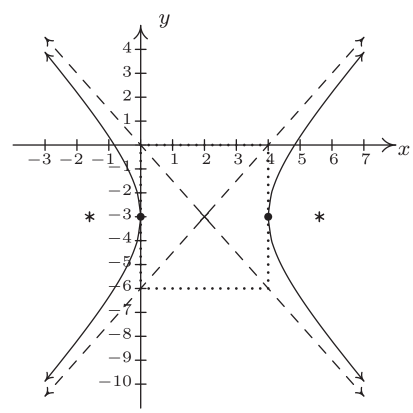
20. $(x-4)^2 + (y+2)^2 = 9$



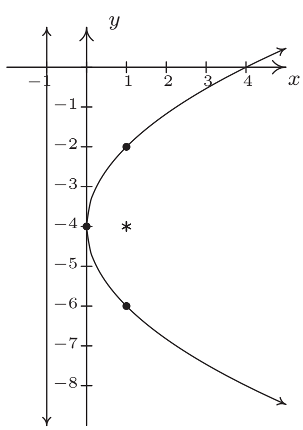
21. $\dfrac{(x-2)^2}{4} + \dfrac{(y+3)^2}{9} = 1$



22. $\dfrac{(x-2)^2}{4} - \dfrac{(y+3)^2}{9} = 1$

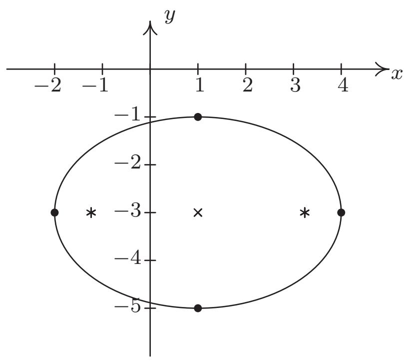


23. $(y+4)^2 = 4x$



24. $\dfrac{(x-1)^2}{1} + \dfrac{y^2}{4} = 0$
The graph is the point $(1,0)$ only.

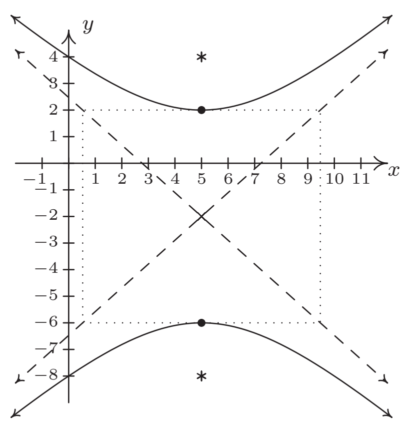
25. $\dfrac{(x-1)^2}{9}+\dfrac{(y+3)^2}{4}=1$



26. $(x-3)^2+(y+2)^2=-1$
There is no graph.

27. $\dfrac{(x+3)^2}{2}+\dfrac{(y-1)^2}{1}=-\dfrac{3}{4}$
There is no graph.

28. $\dfrac{(y+2)^2}{16}-\dfrac{(x-5)^2}{20}=1$



30. By placing Station A at $(0,-50)$ and Station B at $(0,50)$, the two second time difference yields the hyperbola $\frac{y^2}{36}-\frac{x^2}{2464}=1$ with foci A and B and center $(0,0)$. Placing Station C at $(-150,-50)$ and using foci A and C gives us a center of $(-75,-50)$ and the hyperbola $\frac{(x+75)^2}{225}-\frac{(y+50)^2}{5400}=1$. The point of intersection of these two hyperbolas which is closer to A than B and closer to A than C is $(-57.8444,-9.21336)$ so that is the epicenter.

31. (b) $\dfrac{x^2}{9}-\dfrac{y^2}{27}=1$.

32. The tower may be modeled (approximately)[12] by $\frac{x^2}{12100}-\frac{(y-330)^2}{34203}=1$. To find the height, we plug in $x=137.5$ which yields $y\approx 191$ or $y\approx 469$. Since the top of the tower is above the narrowest point, we get the tower is approximately 469 feet tall.

[12]The exact value underneath $(y-330)^2$ is $\frac{52707600}{1541}$ in case you need more precision.